Introduction to Logic (Spring 2020)

Example Exam for Student s1234567

Monday 15th June 2020

Instructions

- It is expected that you work out this exam yourself and hand in your own solutions
- Each student receives a personalised exam
- The exam is distributed Monday 15th June 2020 at 10am
- You have **24 hours** to submit your solution from then on
- Indicate all the sources that you used (except the material provided in this course) and the persons that you have worked with at the beginning of your submission
- After the exam, we will select a number of students for a short oral interview to talk about the provided submission
- All admissible rules and results that were proved in the lecture or the exercises can be used. Results from examples or from exercises need to be properly referenced: "Example 7.5" or "Exercise 8.1" or "Homework 8.1".
- The solution for each exercise has to start on a new page.

Maximally 100 points can be obtained.

Handing in your answers Submit your solution through **Blackboard** as a single PDF file named `submission-exam-sN.pdf`, where N is your student number. The document **has to be created using LaTeX** (or variants like XeLaTeX). You have to **use the LaTeX-template** that is provided to you together with your copy of the exam. Please **use the proper logic connectives and proof rules**, as they were introduced in the course. Their use is demonstrated in the template. If you do not have a working LaTeX installation, then you can use Overleaf (`https://www.overleaf.com/`). Make sure that your **name, student number and studies** are **clearly written on the document**. All students have to prepare and submit their own solution. Answers have to be provided in **Dutch or English**.

Submissions that fail to meet these requirements are not considered.

Special requests Special requests, like the possibility of handing in a hand-written submission, have to be made **at least one week prior to the start of the exam!**

Deadline You have to upload your submission before **Tuesday 16th June 2020 10am**. This deadline is strict.

Propositional Logic

This part of the examination will be concerned with the below image.

Exercise 1

Give a Horn clause ψ_1 in propositional logic that uses **exactly 3 different variables** and that expresses an aspect in or associated to the above picture. Indicate clearly the meaning of each propositional variable that your formula uses. ___/ 10 p.

Exercise 2

Give the truth table of your formula ψ_1. Conclude whether your formula ψ_1 is a tautology, satisfiable or unsatisfiable. ___/ 10 p.

You have to use the order of values as shown in the example truth table below, where the first three columns are for your variables.

			ψ_1
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

Exercise 3

_____/ 10 p.

Give another two Horn clauses ψ_2 and ψ_3 that describe another aspect of the picture and that **adhere to the following rules**. The formulas use

- at least one variable a from the previous exercise,
- exactly two new variables b and c, and
- at least once $\bot$.

Use the algorithm `HORN` to decide whether the Horn formula

$$\psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \neg a \wedge b \wedge c$$

is satisfiable.

First-Order Logic

Let $\mathcal{L}$ be the signature with $\mathcal{F} = \{a, f, k, \}$, $\mathcal{R} = \{I, P, R\}$, and the following arities.

$$\begin{array}{ccc} \mathrm{ar}(a) = 0 & \mathrm{ar}(f) = 1 & \mathrm{ar}(k) = 1 \\ \mathrm{ar}(I) = 0 & \mathrm{ar}(P) = 1 & \mathrm{ar}(R) = 2 \end{array}$$

Exercise 4

_____/ 20 p.

Give formula $\varphi_1 \in \mathrm{Form}^{=}(\mathcal{L})$ and prove $\vdash \varphi_1$ in $\mathbf{ND}_1^{=}$ using **Fitch-style**. The formula φ_1 must use **at least 4 out of the 6 symbols** in $\mathcal{L}$ **and equality**. The proof must use **at least 5 different proof rules**, not counting (Assum), and should be **around half a page long but no longer than a 3/4 page**. All proof steps have to be annotated with the labels of the corresponding proof rules. **Points will be subtracted for missing rule labels.**

Exercise 5

_____/ 10 p.

Provide an $\mathcal{L}^{=}$-model $\mathcal{M}_1$ with universe $\mathbb{N}$, that is, the universe $|\mathcal{M}_1|$ of $\mathcal{M}_1$ must be the natural numbers $\mathbb{N}$ (including 0). The interpretation of the function and predicate symbols can be freely chosen.

Exercise 6

______/5 p.

Show that your model satisfies your formula φ_1.

Exercise 7

______/ 20 p.

Find another formula φ_2 that your model satisfies but is not a tautology. The formula φ_2 must use **at least 4 out of the 6 symbols** in $\mathcal{L}$.

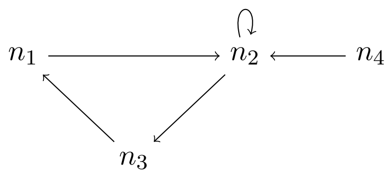
a) Show that $\mathcal{M}_1 \vDash \varphi_2$.

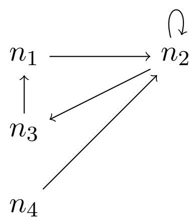
b) Provide a model $\mathcal{M}_2$ and a valuation $v\colon \mathrm{Var} \to |\mathcal{M}_2|$ in that model, and show that $[\![\varphi_2]\!]_v^{\mathcal{M}_2} = 0$.

Automatic Deduction

In this exercise, we describe a graph and find a path from one node to another by using uniform proofs.

Let G be the graph given by the following diagram.





Exercise 8

______/5 p.

Formalise the edges as binary predicate E by giving a logic program Γ_0. Each Horn clause in Γ_0 will be of the form $E(n_i, n_k)$.

Consider the following first-order Horn clauses

$$\varphi_r = \forall x.\, P(x, x, e)$$
$$\varphi_t = \forall x.\, \forall y.\, \forall z.\, \forall p.\, E(x, z) \land P(z, y, p) \to P(x, y, s(z, p))$$

that describe how paths in a graph look like. The clauses use two kinds of symbols: e for the empty path, and s for extending a path by one step. Let $\Gamma = \Gamma_0, \varphi_r, \varphi_t$ be the logic program consisting out of your description the graph G and the description of paths.

Exercise 9

Derive $\Gamma \vdash_u \exists p.\, P(n_4, n_3, p)$ using uniform proofs in Fitch-style. You may use the following proof layout ______/ 10 p.

$$
\begin{array}{r|l}
1 & \Gamma_0 \\
2 & \varphi_r \\
3 & \varphi_t \\
\hline
4 & \vdots \\
5 & \exists p.\, P(n_4, n_3, p)
\end{array}
$$

and refer to line 1 whenever you use any formula in Γ_0.

Solutions to the Exercises

Solution 1

10 p.

An example of a formula could be as follows.

Variables:

- B – person with backpack walks forward
- N – person with Nike shirt walks forward
- A – an accident between the two persons will happen

Formula ψ_1:

$$N \wedge B \to A$$

Solution 2

10 p.

Truth table:

N	B	A	$N \wedge B \to A$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

The formula ψ_1 is not a tautology because of the second to last line in the truth table. However, ψ_1 is satisfiable by, for example, the first line of the truth table.

Solution 3

10 p.

New variables:

- S – person with backpack walks stops
- U – person with Nike shirt walks backwards

Formulas:

$$\psi_2 = \neg(B \wedge U \wedge A)$$
$$\psi_3 = S \wedge U \to A$$

Thus, we have to check

$$(N \wedge B \to A) \wedge \neg(B \wedge U \wedge A) \wedge (S \wedge U \to A) \wedge \neg A \wedge (\top \to S) \wedge (\top \to U)$$

Step	$\top$	B	N	U	S	A	$\bot$	Used clause
0	X							$\top$
1	X				X			$\top \to S$
2	X			X	X			$\top \to U$
3	X			X	X	X		$S \wedge U \to A$
3	X			X	X	X	X	$\neg A$

Since $\bot$ is marked at the end, the Horn formula is not satisfiable

Solution 4

20 p.

We use the following formula.

$$\begin{aligned}
\psi_0 &= \exists x.\, R(f(x), x) \\
\psi_1 &= \exists x.\, P(x) \wedge R(k(x), x) \\
\psi_2 &= \forall x.\, P(x) \to k(x) \doteq f(x) \\
\varphi_1 &= \psi_1 \wedge \psi_2 \to \psi_0
\end{aligned}$$

A possible proof of $\vdash \varphi_1$ in **ND$_1$** goes as follows.

1	$\psi_1 \wedge \psi_2$	
2	ψ_1	$\wedge$E, 1
3	ψ_2	$\wedge$E, 1
4	x $P(x) \wedge R(k(x), x)$	
5	$P(x)$	$\wedge$E, 4
6	$R(k(x), x)$	$\wedge$E, 4
7	$P(x) \to k(x) \doteq f(x)$	$\forall$E, 3
8	$k(x) \doteq f(x)$	$\to$E, 7, 5
9	$R(f(x), x)$	Repl, 6, 8
10	$\exists x.\, R(f(x), x)$	$\exists$I, 9
11	ψ_0	$\exists$E, 2, 4–10
12	$\psi_1 \wedge \psi_2 \to \psi_0$	$\to$I, 1–11

Solution 5

10 p.

A possible model $\mathcal{M}_1$ with $|\mathcal{M}_1| = \mathbb{N}$ is the following, where we indicate the types of the interpretation maps and predicates on the left.

$$
\begin{array}{ll}
a^{\mathcal{M}_1} \colon \mathbb{1} \to \mathbb{N} & a^{\mathcal{M}_1}(*) = 0 \\
f^{\mathcal{M}_1} \colon \mathbb{N} \to \mathbb{N} & f^{\mathcal{M}_1}(n) = \begin{cases} k, & n = 2k \\ k, & n = 2k+1 \end{cases} \\
k^{\mathcal{M}_1} \colon \mathbb{N} \to \mathbb{N} & k^{\mathcal{M}_1}(n) = \begin{cases} k, & n = 2k \\ k+1, & n = 2k+1 \end{cases} \\
I^{\mathcal{M}_1} \subseteq \mathbb{1} & I^{\mathcal{M}_1} = \mathbb{1} \\
P^{\mathcal{M}_1} \subseteq \mathbb{N} & P^{\mathcal{M}_1} = \{n \in \mathbb{N} \mid n \text{ even}\} \\
R^{\mathcal{M}_1} \subseteq \mathbb{N} \times \mathbb{N} & R^{\mathcal{M}_1} = \{(m,n) \mid m \leq n\}
\end{array}
$$

Note that the model fulfils the premises ψ_1 and ψ_2 that appear in φ_1. This is not a requirement to solve this exercise, but comes in handy later

Solution 6

5 p.

To show $\mathcal{M}_1 \vDash \varphi_1$, let $v \colon \mathrm{Var} \to \mathbb{N}$ be an arbitrary valuation and prove $[\![\varphi_1]\!]_v = 1$:

$$
\begin{array}{ll}
[\![\varphi_1]\!]_v = 1 & \\
\text{iff } [\![\psi_1 \wedge \psi_2 \to \psi_0]\!]_v = 1 & \\
\text{iff } [\![\psi_1 \wedge \psi_2]\!]_v \leq [\![\psi_0]\!]_v & \\
\text{if } [\![\psi_0]\!]_v = 1 & \text{(note the simplification here)} \\
\text{iff } [\![\exists x.\, R(k(x), x)]\!]_v = 1 & \\
\text{iff } [\![R(k(x), x)]\!]_{v[x \mapsto n]} = 1 & \text{for some } n \in \mathbb{N} \\
\text{iff } (k^{\mathcal{M}_1}(n), n) \in R^{\mathcal{M}_1} & \text{for some } n \in \mathbb{N} \\
\text{iff } k^{\mathcal{M}_1}(n) \leq n & \text{for some } n \in \mathbb{N} \\
\text{iff } 0 = k^{\mathcal{M}_1}(0) \leq 0 &
\end{array}
$$

Since $k^{\mathcal{M}_1}(0) = 0 \leq 0$, we can choose 0 for n. Thus, $\mathcal{M}_1 \vDash \psi_0$ and $\mathcal{M}_1 \vDash \varphi_1$.

Solution 7

20 p.

Recall the formulas

$$
\begin{aligned}
\psi_0 &= \exists x.\, R(f(x), x) \\
\psi_2 &= \forall x.\, P(x) \to f(x) \doteq k(x)
\end{aligned}
$$

and put

$$\varphi_2 = \psi_0 \wedge \psi_2\,.$$

a) From the previous exercise, we know that $\mathcal{M}_1 \vDash \psi_0$. Moreover, we have $\mathcal{M}_1 \vDash \psi_2$ as follows. Let $v\colon \mathrm{Var} \to \mathbb{N}$ be a valuation. We then have

$$
\begin{aligned}
&\text{iff } [\![\psi_2]\!]_v = 1 \\
&\text{iff } [\![\forall x.\, P(x) \to f(x) \doteq k(x)]\!]_v = 1 \\
&\text{iff } [\![P(x) \to f(x) \doteq k(x)]\!]_{v[x \mapsto n]} = 1 && \text{for all } n \in \mathbb{N} \\
&\text{iff } [\![P(x)]\!]_{v[x \mapsto n]} = 1 \text{ implies } [\![f(x) \doteq k(x)]\!]_{v[x \mapsto n]} = 1 && \text{for all } n \in \mathbb{N} \\
&\text{iff, if } n \text{ even, then } f^{\mathcal{M}_1}(n) = k^{\mathcal{M}_1}(n) && \text{for all } n \in \mathbb{N}\,.
\end{aligned}
$$

If n is even, then $n = 2k$ for some $k \in \mathbb{N}$ and thus $f^{\mathcal{M}_1}(n) = k = k^{\mathcal{M}_1}(n)$. By the above chain of reasoning, we thus obtain $\mathcal{M}_1 \vDash \psi_2$. As we have shown that $\mathcal{M}_1 \vDash \psi_0$ and $\mathcal{M}_1 \vDash \psi_2$, we obtain $\mathcal{M}_1 \vDash \psi_0 \wedge \psi_2$ and thus $\mathcal{M}_1 \vDash \varphi_2$.

b) For $\mathcal{M}_2$ we use again $|\mathcal{M}_2| = \mathbb{N}$ and interpret almost all symbols in the same way, except R:

$$
\begin{aligned}
&a^{\mathcal{M}_2}\colon \mathbb{1} \to \mathbb{N} && a^{\mathcal{M}_2}(*) = 0 \\
&f^{\mathcal{M}_2}\colon \mathbb{N} \to \mathbb{N} && f^{\mathcal{M}_2}(n) = \begin{cases} k, & n = 2k \\ k, & n = 2k+1 \end{cases} \\
&k^{\mathcal{M}_2}\colon \mathbb{N} \to \mathbb{N} && k^{\mathcal{M}_2}(n) = \begin{cases} k, & n = 2k \\ k+1, & n = 2k+1 \end{cases} \\
&I^{\mathcal{M}_2} \subseteq \mathbb{1} && I^{\mathcal{M}_2} = \mathbb{1} \\
&P^{\mathcal{M}_2} \subseteq \mathbb{N} && P^{\mathcal{M}_2} = \{n \in \mathbb{N} \mid n \text{ even}\} \\
&R^{\mathcal{M}_2} \subseteq \mathbb{N} \times \mathbb{N} && R^{\mathcal{M}_2} = \{(m, n) \mid 0 < n \text{ and } n \leq m\}
\end{aligned}
$$

We now have that $\mathcal{M}_2$ does not satisfy ψ_0: Let v be the valuation with $v(x) = 0$ for all $x \in \mathrm{Var}$. Then

$$
\begin{aligned}
[\![\psi_0]\!]_v &= [\![\exists x.\, R(f(x), x)]\!]_v \\
&= \max\{[\![R(f(x), x)]\!]_{v[x \mapsto n]} \mid n \in \mathbb{N}\} \\
&= \begin{cases} 1, & \text{there is } n \in \mathbb{N} \text{ with } (f^{\mathcal{M}_2}(n), n) \in R^{\mathcal{M}_2} \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} 1, & \text{there is } n \in \mathbb{N} \text{ with } 0 < n \text{ and } n \leq f^{\mathcal{M}_2}(n) \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

As $0 < n$ implies that $k < n$ whenever $n = 2k$ or $n = 2k + 1$, we have that $f^{\mathcal{M}_2}(n) < n$. Thus, there cannot be an n with $0 < n$ and $n \leq f^{\mathcal{M}_2}(n)$. Hence, $[\![\psi_0]\!]_v = 0$ in $\mathcal{M}_2$. Since $\varphi_2 = \psi_0 \wedge \psi_2$, we thus obtain that $\mathcal{M}_2$ does not satisfy φ_2. This, in turn, means that φ_2 is not a tautology.

Solution 8 5 p.

The edges of the graph a described by the following logic program.

$$\Gamma_0 = E(n_1, n_2), E(n_2, n_2), E(n_2, n_3), E(n_3, n_1), E(n_4, n_2)$$

Solution 9 10 p.

A path from n_4 to n_3 is described by the term $s(n_2, s(n_3, e))$ and we can derive the uniform sequent $\Gamma \vdash_u \exists p.\, P(n_4, n_3, p)$ as follows.

1	Γ_0	
2	φ_r	
3	φ_t	
4	$E(n_2, n_3)$	B, 1
5	$P(n_3, n_3, e)$	B, 2
6	$P(n_2, n_3, s(n_3, e))$	B, 3, 4, 5
7	$E(n_4, n_2)$	B, 1
8	$P(n_4, n_3, s(n_2, s(n_3, e)))$	B, 3, 7, 6
9	$\exists p.\, P(n_4, n_3, p)$	∃I, 8