Assignment 5

Exercises on lecture 5/chapter 5

30 September 2025

We will work on the following exercises during the tutorial session. Make sure that you understand the solution that we work out together and to solve the remaining exercises yourself.

Exercise 5.1 — Prove that the program $\Omega = \mathbf{while}\ \mathtt{tt}\ \mathbf{do}\ \mathbf{noop}$ diverges. You will have to derive a contradiction by using induction on derivations for $(\Omega \lhd \sigma) \Downarrow \sigma'$ and all $\sigma, \sigma' \in \Sigma$.

Exercise 5.2 — Prove lemma 5.2 (the big-step operational semantics of `Imp` is deterministic).

Exercise 5.3 — Prove that $\mathbf{Pos}$ from definition 5.3 is a category by showing that monotone maps are closed under composition and that the identity map is monotone. (If you want to be modular, then prove that a map is monotone if and only if it is a functor between the categories induced by the order, see example 2.4, item 4.)

Exercise 5.4 — Show in detail that the divisibility relation and the flat order from example 5.4 are partial orders.

Exercise 5.5 — Show in detail that the maps m_a, i_V, $f^{\leftarrow}$ and a_f from example 5.5 are monotone maps $m_a \colon (\mathbb{N}, |) \to (\mathbb{N}, |)$, $i_V \colon (\mathcal{P}(X), \subseteq) \to (\mathcal{P}(X), \subseteq)$, $f^{\leftarrow} \colon (\mathcal{P}(Y), \subseteq) \to (\mathcal{P}(Y), \subseteq)$, and $a_f \colon (\mathbb{R}^X, \sqsubseteq) \to (\mathbb{R}^X, \sqsubseteq)$ respectively.

Exercise 5.6 — Prove lemma 5.8 (existence of forgetful, discrete and flat functor).

Exercise 5.7 — Show that being an order isomorphisms is a stronger condition than being a monotone bijection. Specifically:

a) Show that the map underlying an order isomorphism is a bijection (Hint: Use exercise 2.8).

b) Give an example of two posets $(P, \leq_1)$ and $(Q, \leq_2)$ and a monotone, bijective map $f \colon P \to Q$, such that the inverse of f is not monotone. (Hint: Understand monotone maps $\operatorname{Disc} X \to (Q, \leq_2)$. The smallest example two elements in P and Q.)

Exercise 5.8 — Prove lemma 5.9 and show that the opposite order yields a functor $D \colon \mathbf{Pos} \to \mathbf{Pos}$ and that $D \circ D = \mathrm{Id}$.

Exercise 5.9 — Prove that $\mathbf{Pos}_s$ from definition 5.11 is a category by showing that strict maps are closed under composition and that the identity map is strict.