Assignment 3

Exercises on lecture 3/chapter 3

16 September 2025

We will work on the following exercises during the tutorial session. Make sure that you understand the solution that we work out together and to solve the remaining exercises yourself.

Exercise 3.1 — Prove that S, as defined in theorem 3.4 is indeed a functor. Discuss why the functor laws formalise compositionality.

Exercise 3.2 — Prove that O from example 3.5 fulfils the functor laws.

Exercise 3.3 — Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be functors. Prove that the equations

$$(G \circ F)X = G(FX) \quad \text{and} \quad (G \circ F)f = G(Ff)$$

define a functor $G \circ F\colon \mathcal{C} \to \mathcal{E}$.

Exercise 3.4 — Prove that there is for every category $\mathcal{C}$ an identity functor $\mathrm{Id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$, such that $\mathrm{Id}_{\mathcal{D}} \circ F = F$ and $F \circ \mathrm{Id}_{\mathcal{C}} = F$ for all functors $F\colon \mathcal{C} \to \mathcal{D}$.

Exercise 3.5 — Prove that the family of inclusions $i_X\colon X \to U(EX)$ from lemma 2.9 form a natural transformation $i\colon \mathrm{Id}_{\mathbf{Set}} \to U \circ E$.

Exercise 3.6 — Prove lemma 3.8. Hint: prove first item 3, and then 1 and 2 together by induction on the context c.

Exercise 3.7 — The opposite category of a category $\mathcal{C}$ is the category $\mathcal{C}^{\mathrm{op}}$ with object $|\mathcal{C}^{\mathrm{op}}| = |\mathcal{C}|$ and morphism $\mathcal{C}^{\mathrm{op}}(A, B) = \mathcal{C}(B, A)$, that is, $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$ but morphisms point in the reverse direction.
Let A be a fixed set.

1. Define a functor $H\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ given on objects by mapping a set X to the set of maps $X \to A$, that is, by $HX = A^X$.
2. Prove that there is a natural transformation $\mathrm{Id}_{\mathbf{Set}} \to H \circ H$.

Exercise 3.8 — Do this exercise if you know about vector spaces. Write $\mathbf{Vec}_f$ for the category with finite-dimensional vector spaces over the real numbers as objects and linear maps as morphisms.

1. Show that that there is a functor $H\colon \mathbf{Vec}_f \to \mathbf{Vec}_f$, which is given on objects by $HV = \{f\colon V \to \mathbb{R} \mid f \text{ linear}\}$.
2. Provide a natural transformation $\alpha\colon \mathrm{Id}_{\mathbf{Vec}_f} \to H \circ H$.
3. Prove that α_V is an isomorphism for every finite dimensional vector space V.
4. Prove that for every V there is a map $V \to HV$ but that there is no natural transformation $\mathrm{Id}_{\mathbf{Vec}_f} \to H$. (Hint: Each map depends on a choice of a basis, which cannot be made consistently.)