Assignment 2

Exercises on lecture 2/chapter 2

9 September 2025

We will work on the following exercises during the tutorial session. Make sure that you understand the solution that we work out together and to solve the remaining exercises yourself.

Exercise 2.1 — Give detailed proofs that $\mathbf{Set}$ and $\mathcal{R}$ for a preorder relation R from example 2.4 are categories.

Exercise 2.2 — Let us denote by $[1\rangle$ the singleton set $\{0\}$.

a) Show that an element of a set X is the same as a map $[1\rangle \to X$.

b) Prove that two morphisms $f, g\colon X \to Y$ in $\mathbf{Set}$ are equal if and only if $f \circ a = g \circ a$ for all generalised elements $[1\rangle \to X$.

The singleton set does not always conform to our intuition and care has to be taken in other categories than $\mathbf{Set}$.

c) Show that for every based set (X, x_0) there is a unique based map $([1\rangle, 0)$.

d) Let $\mathbb{2} = ([2\rangle, 0)$ be the based set with $[2\rangle = \{0, 1\}$. Show that an element of a based set (X, x_0) is the same as a based map $\mathbb{2} \to (X, x_0)$.

Exercise 2.3 — Constant morphisms are often what we expect, but not always.

a) Show that constant morphism in $\mathbf{Set}$ are precisely the constant maps, that is, maps $f\colon X \to Y$ with $f(u) = f(v)$ for all $u, v \in X$.

b) Show that if Y has at least one element, then $f\colon X \to Y$ is constant if and only if there is an element $a \in Y$ with $f(u) = a$ for all $u \in X$. (Note that this uses classical logic.)

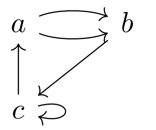
c) Show that there is exactly one constant morphisms between objects in $\mathbf{Set}_\bullet$.

A more useful concept in $\mathbf{Set}_\bullet$ is the following. Given a based set (X, x_0), define a based map $r\colon (X, x_0) \to \mathbb{2}$ by $r(x_0) = 0$ and $r(u) = 1$ for all $u \neq x_0$. We call a morphism $s\colon \mathbb{2} \to (X, x_0)$ a of r if $r \circ s = \mathrm{id}_{\mathbb{2}}$, and a based map $f\colon (X, x_0) \to (Y, y_0)$ if $f \circ r = f \circ r'$ for all sections r, r' of s.

d) Show that f is constant relative to s if and only if f is constant as map of sets when it is restricted to $X \setminus \{x_0\}$.

Exercise 2.4 — A (directed) graph G is a tuple (N, E, s, t) of a set N of nodes, a set E of edges and maps $s\colon E \to N$ and $t\colon E \to N$ that return for an edge, respectively, its source and its target. By a graph homomorphism $f\colon G_1 \to G_2$ is a pair (f_0, f_1) of maps $f_0\colon N_1 \to N_2$ and $f_1\colon E_1 \to E_2$, such that $s_2 \circ f_1 = f_0 \circ s_1$ and $t_2 \circ f_1 = f_0 \circ t_1$.

a) Express the graph in the following diagram in this form.



b) Give graph homomorphism from the graph above to the graph in the following diagram.

c) Prove that there is a category **DiGr** of directed graphs and their homomorphisms.

Exercise 2.5 — Let $\mathcal{C}$ be a category and f a morphism $A \to B$. We say for morphisms $g\colon B \to A$ and $h\colon B \to A$ that g is a if $f \circ g = \mathrm{id}_B$ and h a if $h \circ f = \mathrm{id}_A$. Prove that the following are equivalent in $\mathcal{C}$.

1. f is an isomorphism.
2. f admits a preinverse g and a postinverse h.
3. f admits a preinverse g and g a preinverse h.
4. f admits a postinverse h and h a postinverse g.

Exercise 2.6 —

a) Prove that the inverse of an isomorphism in a category $\mathcal{C}$ is unique.

b) Prove that id_A is an isomorphism for all objects A in a category $\mathcal{C}$.

c) Prove that the inverse of an isomorphism is also an isomorphism.

d) Prove that if $A \cong B$ and $B \cong C$, then also $A \cong C$.

e) Conclude from 2 - 4 that “being isomorphic” is an equivalence relation.

Exercise 2.7 — Provide the details for example 2.8.

Exercise 2.8 — Prove that functors preserve isomorphisms: Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ and an isomorphism $f\colon A \to B$ in $\mathcal{C}$, prove that Ff is an isomorphism in $\mathcal{D}$.

Exercise 2.9 — Prove that $(D, \mathrm{err}) \cong E(\mathbb{Q}_{\geq 0})$ in $\mathbf{Set}_\bullet$ only by using the UMP of E.