Assignment 12

Exercises on lecture 12/chapter 12

18 November 2025

We will work on the following exercises during the tutorial session. Make sure that you understand the solution that we work out together and to solve the remaining exercises yourself.

Exercise 12.1 — Prove that the unit and associativity axioms hold for the composition of the classifying category $\mathcal{C}_{\lambda Y}$. This requires you to prove first a substitution lemma: $t[\tau][\sigma] = t[\tau \circ \sigma]$ holds for all terms t, and all substitutions σ and τ.

Exercise 12.2 — Prove that $\mathcal{C}_{\lambda Y}$ has functorial products (lemma 12.2).

Exercise 12.3 — Contextual equivalence can observe that terms treat their arguments differently, even though their result may be the same for all terminating inputs. Let $t_1 = \lambda x.\, 0$ and $t_2 = \lambda x.\, \mathbf{if}_0\ x\ \mathbf{then}\ 0\ \mathbf{else}\ 0$. Provide a $(\bullet, \mathbf{N} \to \mathbf{N})$-closing context C, such that $C[t_1]$ and $C[t_2]$ have different operational semantics and are thus not contextually equivalent.

Exercise 12.4 — Complete the proof of lemma 12.7 by showing that contextual equivalence is symmetric and transitive.

Exercise 12.5 — Let C be a context with $\Delta \vdash C[t] : B$ for all $\Gamma \vdash t : A$ and C' a (Δ, B)-closing context. Show that there is a (Γ, A)-closing context $C' \odot C$, such that $(C' \odot C)[t] = C'[C[t]]$ for all terms $\Gamma \vdash t : A$.

Exercise 12.6 — Prove that fully faithful functors are respectively closed under composition, and prove that the identity is fully faithful. Thus fully faithful functors form a subcategory of the functor category

Exercise 12.7 — We know from exercise 2.8 that functor preserve isomorphisms. Show that fully faithful functors also reflect isomorphisms: Given a fully faithful functor $F\colon \mathcal{C} \to \mathcal{D}$ and two objects A and B in $\mathcal{C}$, such that $FA \cong FB$ in $\mathcal{D}$, show that $A \cong B$ in $\mathcal{C}$.

Exercise 12.8 — Prove if $F\colon \mathcal{C} \to \mathcal{D}$ is part of an equivalence, then F is fully faithful.

Exercise 12.9 — Prove that if $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence, then $\mathcal{D}$ has all products whenever $\mathcal{C}$ has all products and F preserves them.