Assignment 11

Exercises on lecture 11/chapter 11

11 November 2025

We will work on the following exercises during the tutorial session. Make sure that you understand the solution that we work out together and to solve the remaining exercises yourself.

Exercise 11.1 — Give λ_Y-terms $\vdash * : \mathbf{N} \to \mathbf{N} \to \mathbf{N}$ and $\vdash \text{fib} : \mathbf{N} \to \mathbf{N} \to \mathbf{N}$ that implement, respectively, the computation of multiplication and Fibonacci numbers.

Exercise 11.2 — Recall that the class of primitive recursive functions on the natural numbers consists of constant 0 maps, successor, projections, composition and primitive recursion. Except the last two, all the others are already built into λ_Y, and composition is straightforward to implement in λ_Y. A map $h\colon \mathbb{N} \times Y \to Z$ is said to be given by primitive recursion of functions $f\colon Y \to Z$ and $g\colon \mathbb{N} \times Y \times Z \to Z$ if the following two equations hold.

$$h(0, y) = f(y)$$
$$h(n+1, y) = g(n, y, h(n, y))$$

Given terms $t\colon A \to B$ and $s\colon \mathbf{N} \to A \to B \to B$, define primitive recursion as a term $\text{PR}(t, s)\colon \mathbf{N} \to A \to B$ in λ_Y.

Exercise 11.3 — Define a λ_Y-term of type $\mathbf{N} \to \mathbf{N} \to \mathbf{N}$ that implements the Ackermann function. This is a function that cannot be implemented by just primitive recursion on natural numbers but requires you to use (primitive) recursion on function types.

Exercise 11.4 — The final piece to Turing-completeness, when combined with exercise 11.2, is the so-called minimisation operator or μ-recursion. A function $f\colon \mathbb{N} \to \mathbb{N}_\perp$ is said to be given by μ-recursion from a function $g\colon \mathbb{N} \to \mathbb{N}$, if the following holds.

$$f(n) = \begin{cases} \min\{k \in \mathbb{N} \mid k \geq n \text{ and } g(k) = 0\}, & \text{if there is a } k \geq n \text{ with } g(k) = 0 \\ \perp, & \text{otherwise} \end{cases}$$

For a term $t\colon \mathbf{N} \to \mathbf{N}$, give a term $\text{Min}(t)\colon \mathbf{N} \to \mathbf{N}$ that implements minimisation in λ_Y.

Exercise 11.5 — Pick a term $t\colon \mathbf{N} \to \mathbf{N}$ and evaluate the term $\text{Min}(t)$ that you constructed in exercise 11.4 on an input using the big-step semantics of λ_Y.