Assignment 10

Exercises on lecture 10/chapter 10

4 November 2025

We will work on the following exercises during the tutorial session. Make sure that you understand the solution that we work out together and to solve the remaining exercises yourself.

Exercise 10.1 — Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $G\colon \mathcal{D} \to \mathcal{C}$. Suppose that there is a object map $F_0\colon |\mathcal{C}| \to |\mathcal{D}|$ together with a family of morphisms $\eta_A\colon A \to G_0(F_0 A)$ indexed by objects A in $\mathcal{C}$, such that $(F_0 A, \eta_A)$ is a reflection of A along G for all A in $\mathcal{C}$. Show that F_0 can be extended to a functor F, such that the family $\{\eta_A\}_{A \in |\mathcal{C}|}$ assembles into a natural transformation and $F \dashv G$.

Exercise 10.2 — Let P and Q be posets, and consider them as small categories $\mathcal{P}$ and $\mathcal{Q}$ like in example 2.4.4. Prove that the functor category $[\mathcal{P}, \mathcal{Q}]$ is isomorphic to the poset $[P, Q]$ of monotone maps considered as small category.

Exercise 10.3 — Prove the interchange law from lemma 10.7.

Exercise 10.4 — Let $\mathcal{C}$ be a locally small category. Prove that the mapping $\mathcal{C}(-, +)$ defined in definition 10.3 is indeed a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$. Show moreover that $f\colon A \to B$ is an isomorphism in $\mathcal{C}$ if and only if $\mathcal{C}(f, +)\colon \mathcal{C}(B, +) \to \mathcal{C}(A, +)$ is an isomorphism of functors.

Exercise 10.5 — Prove the Yoneda lemma 10.4.

Exercise 10.6 — Finish the proof of theorem 10.9, by showing that in the first step $\lambda_{A,B} \circ \rho_{A,B} = \mathrm{id}$, and that **??** 10.3 commutes in the second step.

Exercise 10.7 — Let $I\colon \boldsymbol{\omega}\mathbf{CPO}_s \to \boldsymbol{\omega}\mathbf{CPO}_\bot$ be the inclusion of strict continuous maps into all continuous maps. Is the composed functor $I \circ \mathrm{Flat}\colon \mathbf{Set} \to \boldsymbol{\omega}\mathbf{CPO}_\bot$ left adjoint to the forgetful functor $\boldsymbol{\omega}\mathbf{CPO}_\bot \to \mathbf{Set}$? If not, provide a counterexample.

Exercise 10.8 — Prove the essential uniqueness of adjoints in theorem 10.14 and formulate its dual.

Exercise 10.9 — Show that terminal objects can be characterised as adjunctions. (Hint: Consider the category $\mathbb{1}$ with one object 0 and only the identity morphism id_0.)

Exercise 10.10 — Show that the categories $\mathbf{Set}$, $\mathbf{Pos}$, $\boldsymbol{\omega}\mathbf{CPO}$ and $\boldsymbol{\omega}\mathbf{CPO}_\bot$ have terminal objects.

Exercise 10.11 — Prove that $\mathbf{Pos}$ and $\boldsymbol{\omega}\mathbf{CPO}$ are Cartesian closed. (Hint: Use definition 5.6, lemma 8.1, and theorems 7.5, 8.4 and 8.6 and the Cartesian closure of $\mathbf{Set}$.)

Exercise 10.12 — Show that the category $\mathbf{Set}_\bullet$ of based sets and maps is not Cartesian closed. Hint: The issue is the base point of the product.