Assignment 6

Exercises on lecture 6/chapter 6

22 October 2024

We will work on the following exercises during the next exercise class. You can choose among the problems of sheet 4 – 9 to work one out and submit it individually for feedback.

Exercise 6.1 — Let P and Q be posets. Prove that the pointwise order on maps $P \to Q$ is a partial order.

Exercise 6.2 — Show that the map f_c defined in eq. (6.1) is monotone.

Exercise 6.3 — Prove that any lfp is a fixed point (lemma 6.6).

Exercise 6.4 — Show that the join is monotone and the meet antitone for inclusion: If $U, V \subseteq P$ in a poset $(P, \leq)$ with $U \subseteq V$, then $\sup U \leq \sup V$ and $\inf V \leq \inf U$.

Exercise 6.5 — Prove that $\bigvee \emptyset = \bot_P$ if it exists in P.

Exercise 6.6 — Given a monotone map $f\colon (P, \leq_P) \to (Q, \leq_Q)$ and a set $U \subseteq P$, we write $f^{\to}U$ for the of U under f, given by $f^{\to}U = \{f(x) \mid x \in U\}$. Prove that if the joins $\bigvee U$ and $\bigvee f^{\to}U$ exist, then $f(\bigvee U) \leq_Q \bigvee(f^{\to}U)$. Formulate and prove the dual results for meets.

Exercise 6.7 — Prove that a map $c\colon \mathbb{N} \to P$ is a chain if and only if $c(n) \leq_P c(n+1)$ for all $n \in \mathbb{N}$.

Exercise 6.8 — Prove lemma 6.2, this is, prove that the pushforward of chains along a monotone map and the derivative map are monotone for the point-wise order.

Exercise 6.9 — A chain c in a poset P is called if there a $k \in \mathbb{N}$, such that $c_n = c_{n+1}$ for all $n \geq k$. Show that any eventually constant chain c has a supremum.

Exercise 6.10 — Complete the proof of lemma 6.11. (Note that monotonicity of sup in this case is is different from exercise 6.4!)

Exercise 6.11 — Prove that the identity map is continuous, and that continuous maps are closed under composition.

Exercise 6.12 — Show that the map f_c defined in eq. (6.1) is continuous. Assume that $[\Sigma, \Sigma_\bot]$ is a based ωCPO, which we will prove in theorem 7.3, and use theorem 6.12 to conclude that $w_\infty = \mu f_c$.