# Assignment 4

**Exercises on lecture 4/chapter 4**

24 September 2023

We will work on the following exercises during the next exercise class. You can choose among the problems of sheet 4 – 9 to work one out and submit it individually for feedback.

**Exercise 4.1** — Write a program in `Imp` that computes in a location $y$ the Fibonacci number corresponding to the value in location $x$, if that value is non-negative and otherwise diverges.

**Exercise 4.2** — Prove lemma 4.10.

**Exercise 4.3** — Prove that the or-rule in lemma 4.8 is admissible.

**Exercise 4.4** — Let $c$ be the command from example 4.3 for the greatest common divisor and $\sigma = [x \mapsto 2][y \mapsto 3]$. Compute $(c \lhd \sigma) \Downarrow \tau$.

**Exercise 4.5** — Extend the syntax of `Imp` with division and devise the corresponding rule for the big-step operational semantics.

**Problem 4.6** — The goal of this problem is to devise small-step operational semantics for `Imp` that agree with the big-step operational semantics.

**a)** Provide small-step semantics for `Imp` in form of rules for relations

- $\longrightarrow_A \subseteq (\text{AExp} \times \Sigma) \times (\text{AExp} \times \Sigma)$,
- $\longrightarrow_B \subseteq (\text{BExp} \times \Sigma) \times (\text{BExp} \times \Sigma)$, and
- $\longrightarrow_C \subseteq (\text{Com} \times \Sigma) \times (\text{Com} \times \Sigma)$.

We will denote membership in the relation $\longrightarrow_C$ by $(c \lhd \sigma) \longrightarrow_C (c' \lhd \sigma')$ and similarly for $\longrightarrow_A$ and $\longrightarrow_B$. (Hint: To give rules for $\longrightarrow_A$ and $\longrightarrow_B$ you can take inspiration from the rules for `Arith` in section 1.4, while the rules for sequential composition and loops can be informed by exercise 4.2. Finally, your rules should ensure that $(\textbf{noop} \lhd \sigma)$ has no transition to any $(c \lhd \sigma)$, and that the relations $\longrightarrow_A$ and $\longrightarrow_B$ do not change the memory.)

**b)** Prove the following progress lemmas.

1. For all $a \in \text{AExp}$ and $\sigma \in \Sigma$, either $a$ is a value or $(a \lhd \sigma) \longrightarrow (a' \lhd \sigma)$ for some $a' \in \text{AExp}$.
2. For all $b \in \text{BExp}$ and $\sigma \in \Sigma$, either $b$ is a value or $(b \lhd \sigma) \longrightarrow (b' \lhd \sigma)$ for some $b' \in \text{AExp}$.

3. For all $c \in \text{Com}$ and $\sigma \in \Sigma$, either $c = \textbf{noop}$ or $(c \lhd \sigma) \longrightarrow (c' \lhd \sigma')$ for some $c' \in \text{Com}$ and $\sigma' \in \Sigma$.

Given a relation $R \subseteq X \times X$ on some set $X$, we let the $\text{rt}(R)$ of $R$ be the least reflexive and transitive preorder relation that contains $R$. More explicitly, it is inductively defined by the following two rules, in which $x, y, z \in X$.

$$\frac{}{x \ \text{rt}(R) \ x} \ \text{Refl} \qquad \frac{x \ R \ y \qquad y \ \text{rt}(R) \ z}{x \ \text{rt}(R) \ z} \ \text{Step}$$

**c)** Prove that $R \subseteq \text{rt}(R)$ and that $\text{rt}(R)$ is transitive.

In what follows, we will drop the indices on the transition relations and just write $\longrightarrow$ instead of $\longrightarrow_A$ etc. Moreover, we will denote $\text{rt}(\longrightarrow)$ by $\twoheadrightarrow$.

**d)** Prove that if $(c \lhd \sigma) \Downarrow \sigma'$, then $(c \lhd \sigma) \twoheadrightarrow (\textbf{noop} \lhd \sigma')$.