Assignment 13

Exercises on lecture 13/chapter 13

03 December 2024

We will work on the following exercises during the next exercise class.

Exercise 13.1 — Calculate the denotational semantics of the terms $\Omega_{\mathbf{N}}$ from example 11.10, $\lambda x.\, \mathbf{if}_0\ x\ \mathbf{then}\ 0\ \mathbf{else}\ \Omega_{\mathbf{N}}$, and subtraction as given in example 11.7.

Exercise 13.2 — Let $\mathcal{C}$ be a category with binary products and a terminal object T. Show for all objects A that $A \times T \cong A \cong T \times A$. Use this to show that $[T, A] \cong A$ if $\mathcal{C}$ is Cartesian closed.

Exercise 13.3 — Give an explicit description of ite_0 in each of the characterisation of adjunctions in theorem 10.9.

Exercise 13.4 — Prove naturality of $\mathrm{fix}_{P,Q}$ in P, that is, prove lemma 13.1.

Exercise 13.5 — Complete the proof of lemma 13.2 by showing that $[\![(\lambda x.\, t)[\sigma]]\!] = [\![\lambda x.\, t]\!] \circ D\sigma$.

Exercise 13.6 — Prove lemma 13.3 by induction on contexts.

Exercise 13.7 — Let x, y and z be distinct variables. Prove that the contexts $(x : \mathbf{N}, y : \mathbf{N})$ and $(z : \mathbf{N} \times \mathbf{N})$ are isomorphic in $c\mathcal{C}^0_{\lambda\mathrm{Y}}$. Discuss why this does not happen in $\mathcal{C}^0_{\lambda\mathrm{Y}}$.

Exercise 13.8 — Prove lemma 13.9 by induction on types.

Problem 13.9 — We let, for this problem, $\lambda_\times$ be the language that arises from λ_Y by not allowing any fixed point terms. That is, the terms and contexts of $\lambda_\times$ are the terms and contexts of λ_Y that do not contain any context of the form $\mathbf{fix}\, x.\, C$. You may assume, in what follows, that contextual equivalence thus restricts to $\lambda_\times$ and is a congruence. This language is also called the with product types. We denote by $\mathcal{C}_{\lambda\times}$ the classifying category of $\lambda_\times$, which is given by restricting the classifying category $\mathcal{C}_{\lambda\mathrm{Y}}$ to $\lambda_\times$, and by the $I\colon \mathcal{C}_{\lambda\times} \to \mathcal{C}_{\lambda\mathrm{Y}}$ the inclusion functor. Similarly, we obtain a subcategory $i_\times : \mathcal{C}^0_{\lambda\times} \to \mathcal{C}_{\lambda\times}$ of terms at zero-order type. Finally, since contextual equivalence is a congruence, we also obtain quotient categories $Q_\times : \mathcal{C}_{\lambda\times} \to c\mathcal{C}_{\lambda\times}$ and $Q^0_\times : \mathcal{C}^0_{\lambda\times} \to c\mathcal{C}^0_{\lambda\times}$

a) Assume that we are given a Cartesian closed category $\mathcal{C}$. Show that there is a product-preserving functor $S\colon \mathcal{C}_{\lambda\times} \to \mathcal{C}$.

b) Consider **Set** as Cartesian closed category and show that $S\colon \mathcal{C}_{\lambda\times} \to \mathbf{Set}$ is part of a good semantics pair, in the sense that there is a faithful functor $S^0 : c\mathcal{C}^0_{\lambda\times} \to \mathbf{Set}$ with $S^0 \circ Q^0_\times = S \circ i_\times$. You will have to use a suitably defined logical relation, similar to the approach in section 13.2.

c) Conclude that for every term $\vdash t : A$ in $\lambda_\times$ there is a value v with $t \Downarrow_A v$.