Assignment 12

Exercises on lecture 12/chapter 12

26 November 2024

We will work on the following exercises during the next exercise class.

Exercise 12.1 — Prove that the unit and associativity axioms hold for the composition of the classifying category $\mathcal{C}_{\lambda\mathrm{Y}}$. This requires you to prove first a substitution lemma: $t[\tau][\sigma] = t[\tau \circ \sigma]$ holds for all terms t, and all substitutions σ and τ.

Exercise 12.2 — Prove that $\mathcal{C}_{\lambda\mathrm{Y}}$ has functorial products (lemma 12.2).

Exercise 12.3 — Contextual equivalence can observe that terms treat their arguments differently, even though their result may be the same for all terminating inputs. Let $t_1 = \lambda x.\, 0$ and $t_2 = \lambda x.\, \mathbf{if}_0\ x\ \mathbf{then}\ 0\ \mathbf{else}\ 0$. Provide a $(\bullet, \mathbf{N} \to \mathbf{N})$-closing context C, such that $C[t_1]$ and $C[t_2]$ have different operational semantics and are thus not contextually equivalent.

Exercise 12.4 — Complete the proof of lemma 12.7 by showing that contextual equivalence is symmetric and transitive.

Exercise 12.5 — Let C be a context with $\Delta \vdash C[t] : B$ for all $\Gamma \vdash t : A$ and C' a (Δ, B)-closing context. Show that there is a (Γ, A)-closing context $C' \odot C$, such that $(C' \odot C)[t] = C'[C[t]]$ for all terms $\Gamma \vdash t : A$.

Exercise 12.6 — Prove that fully faithful functors are respectively closed under composition, and prove that the identity is fully faithful. Thus fully faithful functors form a subcategory of the functor category

Exercise 12.7 — We know from exercise 2.8 that functor preserve isomorphisms. Show that fully faithful functors also reflect isomorphisms: Given a fully faithful functor $F \colon \mathcal{C} \to \mathcal{D}$ and two objects A and B in $\mathcal{C}$, such that $FA \cong FB$ in $\mathcal{D}$, show that $A \cong B$ in $\mathcal{C}$.

Exercise 12.8 — Prove if $F \colon \mathcal{C} \to \mathcal{D}$ is part of an equivalence, then F is fully faithful.

Exercise 12.9 — Prove that if $F \colon \mathcal{C} \to \mathcal{D}$ is an equivalence, then $\mathcal{D}$ has all products whenever $\mathcal{C}$ has all products and F preserves them.

Problem 12.10 — A $A \rightharpoonup B$ between sets A and B is given by a subset $\operatorname{dom} f \subseteq A$ and a map $f \colon \operatorname{dom} f \to B$. We call $\operatorname{dom} f$ the of f.

1. Show that there is a category $\mathbf{Pfn}$ with sets as objects and partial maps as morphisms.

2. Show that $\mathbf{Pfn} \simeq \mathbf{Set}_\bullet$.

Problem 12.11 — We call a functor $F\colon \mathcal{C} \to \mathcal{D}$ a embedding if it is fully faithful and injective on objects, that is, if $FA = FB$ implies $A = B$ for all objects A and B in $\mathcal{C}$. Show that the flat poset functor restricts to an embedding $\mathrm{Flat}_\bullet\colon \mathbf{Set}_\bullet \to \mathbf{Pos}_s$, cf. item 4 of example 5.5.