Assignment 10

Exercises on lecture 10/chapter 10

12 November 2024

We will work on the following exercises during the next exercise class.

Exercise 10.1 — Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $G\colon \mathcal{D} \to \mathcal{C}$. Suppose that there is a object map $F_0\colon |\mathcal{C}| \to |\mathcal{D}|$ together with a family of morphisms $\eta_A\colon A \to G_0(F_0 A)$ indexed by objects A in $\mathcal{C}$, such that $(F_0 A, \eta_A)$ is a reflection of A along G for all A in $\mathcal{C}$. Show that F_0 can be extended to a functor F, such that the family $\{\eta_A\}_{A \in |\mathcal{C}|}$ assembles into a natural transformation and $F \dashv G$.

Exercise 10.2 — Let P and Q be posets, and consider them as small categories $\mathcal{P}$ and $\mathcal{Q}$ like in example 2.4.4. Prove that the functor category $[\mathcal{P}, \mathcal{Q}]$ is isomorphic to the poset $[P, Q]$ of monotone maps considered as small category.

Exercise 10.3 — Prove the interchange law from lemma 10.7.

Exercise 10.4 — Let $\mathcal{C}$ be a locally small category. Prove that the mapping $\mathcal{C}(-, +)$ defined in definition 10.3 is indeed a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$. Show moreover that $f\colon A \to B$ is an isomorphism in $\mathcal{C}$ if and only if $\mathcal{C}(f, +)\colon \mathcal{C}(B, +) \to \mathcal{C}(A, +)$ is an isomorphism of functors.

Exercise 10.5 — Prove the Yoneda lemma 10.4.

Exercise 10.6 — Finish the proof of theorem 10.9, by showing that in the first step $\lambda_{A,B} \circ \rho_{A,B} = \mathrm{id}$, and that **??** 10.3 commutes in the second step.

Exercise 10.7 — Let $I\colon \boldsymbol{\omega}\mathbf{CPO}_s \to \boldsymbol{\omega}\mathbf{CPO}_\perp$ be the inclusion of strict continuous maps into all continuous maps. Is the composed functor $I \circ \mathrm{Flat}\colon \mathbf{Set} \to \boldsymbol{\omega}\mathbf{CPO}_\perp$ left adjoint to the forgetful functor $\boldsymbol{\omega}\mathbf{CPO}_\perp \to \mathbf{Set}$? If not, provide a counterexample.

Exercise 10.8 — Prove the essential uniqueness of adjoints in theorem 10.14 and formulate its dual.

Exercise 10.9 — Show that terminal objects can be characterised as adjunctions. (Hint: Consider the category $\mathbb{1}$ with one object 0 and only the identity morphism id_0.)

Exercise 10.10 — Show that the categories $\mathbf{Set}$, $\mathbf{Pos}$, $\boldsymbol{\omega}\mathbf{CPO}$ and $\boldsymbol{\omega}\mathbf{CPO}_\perp$ have terminal objects.

Exercise 10.11 — Prove that $\mathbf{Pos}$ and $\boldsymbol{\omega}\mathbf{CPO}$ are Cartesian closed. (Hint: Use definition 5.6, lemma 8.1, and theorems 7.5, 8.4 and 8.6 and the Cartesian closure of $\mathbf{Set}$.)

Exercise 10.12 — Show that the category $\mathbf{Set}_\bullet$ of based sets and maps is not Cartesian closed. Hint: The issue is the base point of the product.

Problem 10.13 — Recall from example 2.4 that a preorder $\leq$ on a set P is a reflexive and transitive relation, not necessarily anti-symmetric. Let $\mathbf{Preord}$ be the category of pairs $(P, \leq)$ of a set P with preorder on it and monotone maps between them (as for posets: $p \leq q \implies f(p) \leq f(q)$). There is an inclusion functor $I\colon \mathbf{Pos} \to \mathbf{Preord}$. Show that I has a left adjoint $F\colon \mathbf{Preord} \to \mathbf{Pos}$.

Problem 10.14 — Denote by $I\colon \boldsymbol{\omega}\mathbf{CPO} \to \mathbf{Pos}$ the inclusion of ωCPOs and continuous maps into poset and monotone maps. Show that I has a left adjoint.