Combinatorics

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(Enumerative) Combinatorics

- [Today: some principles of counting quickly.]
- [We have seen probabilities of events often take the form: number of possibilities in event / total number of possibilities (in case of uniformity). Today we learning some rules about how to count possibilities.]
- [Some of the rules and principles are quite simple, but the questions you may ask can very quickly get quite difficult. Here are some:

  In how many ways, can we

  – Pick 6 numbers from 1 to 15 so that no two are consecutive?

  – Can 7 balls be places in 4 boxes if no box is to be left empty?

Fundamental principle of counting

- With m elements a_1, .. ,a_m and n elements b_1, … b_n it is possible to form m x n (ordered) pairs (a_i, b_j) containing one element from each group
- If we have n r successive stages/selections/decisions with exactly n_k choices possible at the kth step, then we can have a total of $N = n_1 \cdot n_2 \cdot \ldots \cdot n_r.$ different results
- Example: 3 starters, 4 mains, 2 dessert gives 24 different menus.

4 methods to generate collections (of size k) from a set of n elements:

- 1. Ordered sampling from the set with replacement
- 2. Ordered sampling from the set without replacement
- 3. Taking a combination from the set without replacement
- 4. Taking a combination from the set with replacement

1. Ordered sampling with replacement

- "order matters, and elements can be drawn more than once", i.e.:
- Draw *k* elements from the set by selecting elements one by one: this gives an ordered sample of length *k*. If each selection is taken from the entire set (i.e. the same element can be drawn more than once), then there exist $N = n^k$ different samples.
- Example: A={a,b,c}. How many words of size 4 can we form? Aabc.

2. Ordered sampling without replacement

- "Order matters, elements can be drawn only once", i.e.
- Draw k elements from the set by selecting elements one by one and remove them from the set once they are chosen.
- There exist a total of $N = (n)_k$ different samples without repetitions, where $(n)_k = n(n-1)\cdots(n-k+1)$

Additional terminology:

- If we select n elements like this we get a permutation of the set. So:
- A permutation of a set A is an ordered listing of the elements of the set. We represent it as a_1 a_2 .. a_n, e.g. A={1, 2, 3} has six permutations 123, 132, 213, 231, 312, 321
- A permutation can be seen as a one-to-one mapping of A onto itself

(continued)

- The total number of different orderings=permutations of n elements is: n! (n factorial) 0! =1
- A k-permutation of a set A is an ordered listing of a k-subset of A; we represent it as a_1 a_2 … a_k, e.g. there are 6 2-permutations of A={1,2,3}: 12, 13, 21, 23, 31, 32
- As we have seen, a set of n elements has $(n)_k = n!/(n-k)!$ k-permutations
- Give the probability of sampling without any repetitions: $(n)_k/n^k$

Assignments

- 1,2,3

3. Combinations without replacement

- "order does not matter, elements can be used only once", i.e.
- Take k elements without minding their order. Without replacement this corresponds to taking a subset of the set
- Example: A={1,2,3}. There are 3 such combinations: {1,2}, {2,3} and {1,3}
- Define the binomial coefficient (pronounced as "n choose k") as $$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}$$
- A set of n elements has $\binom{n}{k}$ subsets (combinations without repetition) of size $k \leq n$
- Note $\binom{n}{0} = 1$

Assignment

- 3 (on page 113)

Combinations with replacement

- "Order does not matter, and elements can be chosen more than once".
- Example: A={a,b,c,d}. Pick 4 elements, e.g. a,a,b,d.

  Encode as 1 (pick one),1,/ (next element), 1, /,/,1; b,c,c,d: /,1,/,1,1,/,1. Every selection corresponds to such strings. Strings always have length n+k-1; there are always (n-1) next-symbols, and k 1's.
- The total number of combinations with repetition (also knows as bags or multisets) of size k from a set of n elements is:

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

- Example: 3 types of pies. In how many ways can we pick 10 pies. (10 + 3 – 1)! / 3! (10 – 1)! = 220

Bernouilli trials

- Bernouilli trials process consisting of n experiments:
  - Each experiment has two possible outcomes, e.g. 0 or 1, success or failure: $\Omega_i = \{0, 1\}, i = 1, \ldots n$
  - The probability p of success (1) is the same for each experiment; the probability q of failure(0) is 1 – p:

$$m_i(1) = m_i(S) = p; m_i(0) = m_i(F) = q = 1 - p$$

- Tree: probabilities of outcomes of entire experiment (see figure page 96)

Binomial distribution

- Given *n* Bernouilli trials with probability *p* of success on each experiment, the probability of exactly *k* successes is:

$$b(n,p,k) = \binom{n}{k} p^k q^{n-k}$$

- Explanation:

$$P(E) \quad \text{with} \quad E = \{\omega | \omega \,\text{has}\, k \,\text{successes}\}\,.$$

$$P(E) = \sum_{\omega \in E} m(\omega)$$

Using the tree: every path with *k* successes and *n-k* failures:

$$m(\{k\,\text{successes}, n-k\,\text{failures}\}) = p^k q^{n-k}$$

How many such paths are there? n possible trials, k should be successes: $\binom{n}{k}$

- If B is a random variable counting the number of successes in a Bernouilli trials process with parameters n and p. Then the distribution m(k) = b(n,p,k) is called the Binomial distribution.

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Examples

- A die is rolled four times. What is the probability that exactly one 6 turns up: b(4, 1/6, 1)= (4 1) (1/6) (5/6)^3

Assignment

- When time, make assignment 6 (p114).