Biomedical Security

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Schneier on Security 30-3 2021

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System Update: New Android Malware

Researchers have discovered a new Android app called "System Update" that is a sophisticated Remote-Access Trojan (RAT). From a news article:

> The broad range of data that this sneaky little bastard is capable of stealing is pretty horrifying. It includes: instant messenger messages and database files; call logs and phone contacts; Whatsapp messages and databases; pictures and videos; all of your text messages; and information on pretty much everything else that is on your phone (it will inventory the rest of the apps on your phone, for instance).
>
> The app can also monitor your GPS location (so it knows exactly where you are), hijack your phone's camera to take pictures, review your browser's search history and bookmarks, and turn on the phone mic to record audio.
>
> The app's spying capabilities are triggered whenever the device receives new information. Researchers write that the RAT is constantly on the lookout for "any activity of interest, such as a phone call, to immediately record the conversation, collect the updated call log, and then upload the contents to the C&C server as an encrypted ZIP file." After thieving your data, the app will subsequently erase evidence of its own activity, hiding what it has been doing.

This is a sophisticated piece of malware. It feels like the product of a national intelligence agency or — and I think more likely — one of the cyberweapons arms manufacturers that sells this kind of capability to governments around the world.

https://www.schneier.com/

"Brute force and dictionary attacks up 400 percent in 2017"

Feb 28, 2018 by Rene Millman

Your Router's Security Stinks: Here's How to Fix It

by **PAUL WAGENSEIL** May 29, 2018, 5:52 AM

https://www.tomsguide.com

Listen to Your Key: Towards Acoustics-based Physical Key Inference

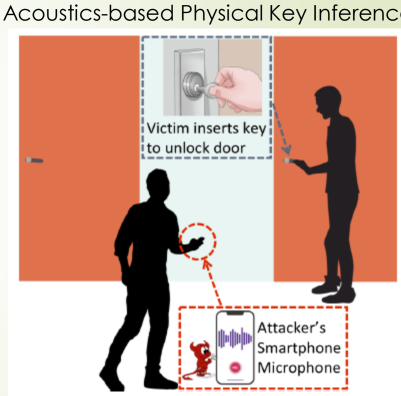


Figure 1: Figure depicts *SpiKey* attack scenario. Attacker records the sound of victim's key insertion to infer the shape, or "secret", of the key.

S. Ramesh et al.

ACM ISBN 978-1-4503-7116-2/20/03...$15.00
https://doi.org/10.1145/3376897.3377853

Nieuws https://www.ncsc.nl/actueel

Kwetsbaarheden in OpenSSL
26-03-2021 | 17:22
Op het internet is een Proof of Concept verschenen waarmee OpenSSL kan worden misbruikt. Het NCSC heeft daarom de inschaling van ...

Call for presentations ONE Conference 2021
25-03-2021 | 15:40
Op 28 en 29 september organiseert het NCSC in samenwerking met het Ministerie van EZK de gemeente Den Haag de 8e editie van de ...

Gevolgen van Microsoft Exchange kwetsbaarheden groot voor Nederlandse organisaties en bedrijven
16-03-2021 | 17:13
De gevolgen van de kwetsbaarheden in Microsoft Exchange Server zijn groot voor Nederlandse organisaties en bedrijven. Het NCSC ...

Misbruik Microsoft Exchange kwetsbaarheid: blijf scannen en bereid je voor
12-03-2021 | 17:07
Het NCSC adviseert om te blijven scannen en monitoren op misbruik van Microsoft Exchange Servers en maatregelen te nemen om ...

40% van Nederlandse Microsoft Exchange Servers nog kwetsbaar
08-03-2021 | 14:14
Het NCSC benadrukt nogmaals dat het van groot belang is om Microsoft Exchange Servers zo snel mogelijk te patchen. Vorige week ...

UPDATE 6 maart: Aanvullend advies misbruikte kwetsbaarheden Microsoft Exchange Server
05-03-2021 | 21:31
Op 6 maart zijn aanvullende tijdelijke mitigerende maatregelen toegevoegd aan dit nieuwsbericht, zie 'Wat kan ik doen', punt 3. ...

Kwetsbaarheden in Microsoft Exchange Server in Nederland actief misbruikt
04-03-2021 | 19:28
Dinsdag 2 maart jl. hebben wij gecommuniceerd over kwetsbaarheden in on-premises installaties van Microsoft Exchange Server. Op ...

Wereldwijd botnet Emotet ontmanteld
12-02-2021 | 17:42
Donderdag 4 februari heeft de Nationale Politie een dataset gedeeld met het NCSC die accounts bevat die door het Emotet botnet ...

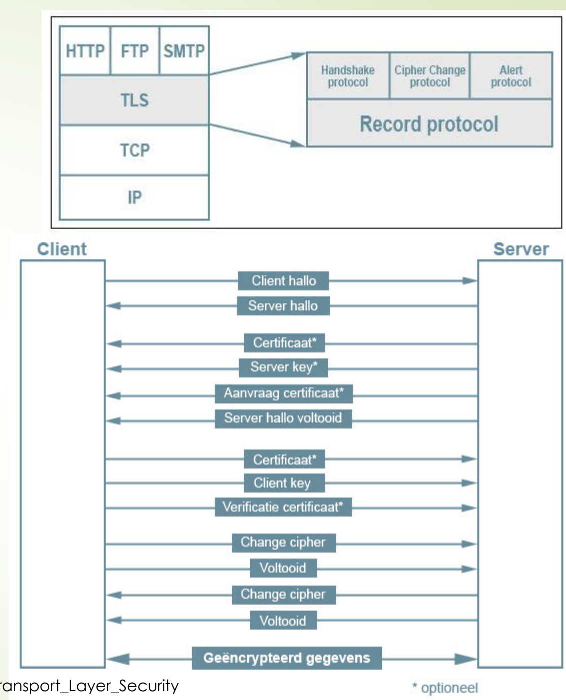
Meer mogelijkheden om informatie te delen voor NCSC
03-02-2021 | 18:11
De afgelopen maanden is de minister van JenV in samenspraak met andere betrokken ministers nagegaan of er aanvullingen nodig zijn ...

SSL -> TLS

Transport Layer Security (TLS)
Secure Sockets Layer (SSL)

Cryptographic protocols for communications security over a network. in applications such as
- Email
- instant messaging
- voice over IP
- Security layer in HTTPS

Note: SSL is depreciated since June 2015
It is insecure as it has several weaknesses, exploits, and can be broken by Poodle (a kind of Man-in-the-Middle) Attack.



https://en.wikipedia.org/wiki/Transport_Layer_Security

Cipher security against publicly known feasible attacks

Cipher			Protocol version						Status
Type	Algorithm	Nominal strength (bits)	SSL 2.0	SSL 3.0 [n 1][n 2][n 3][n 4]	TLS 1.0 [n 1][n 3]	TLS 1.1 [n 1]	TLS 1.2 [n 1]	TLS 1.3	
Block cipher with mode of operation	AES GCM[54][n 5]	256, 128	N/A	N/A	N/A	N/A	Secure	Secure	Defined for TLS 1.2 in RFCs
	AES CCM[55][n 5]		N/A	N/A	N/A	N/A	Secure	Secure	
	AES CBC[n 6]		N/A	Insecure	Depends on mitigations	Depends on mitigations	Depends on mitigations	N/A	
	Camellia GCM[56][n 5]	256, 128	N/A	N/A	N/A	N/A	Secure	N/A	
	Camellia CBC[57][n 6]		N/A	Insecure	Depends on mitigations	Depends on mitigations	Depends on mitigations	N/A	
	ARIA GCM[58][n 5]	256, 128	N/A	N/A	N/A	N/A	Secure	N/A	
	ARIA CBC[58][n 6]		N/A	N/A	Depends on mitigations	Depends on mitigations	Depends on mitigations	N/A	
	SEED CBC[59][n 6]	128	N/A	Insecure	Depends on mitigations	Depends on mitigations	Depends on mitigations	N/A	
	3DES EDE CBC[n 6][n 7]	112[n 8]	Insecure	Insecure	Insecure	Insecure	Insecure	N/A	
	GOST 28147-89 CNT[53][n 7]	256	N/A	N/A	Insecure	Insecure	Insecure	N/A	Defined in RFC 4357
	IDEA CBC[n 6][n 7][n 9]	128	Insecure	Insecure	Insecure	Insecure	N/A	N/A	Removed from TLS 1.2
	DES CBC[n 6][n 7][n 9]	56	Insecure	Insecure	Insecure	Insecure	N/A	N/A	
		40[n 10]	Insecure	Insecure	Insecure	N/A	N/A	N/A	Forbidden in TLS 1.1 and later
	RC2 CBC[n 6][n 7]	40[n 10]	Insecure	Insecure	Insecure	N/A	N/A	N/A	
Stream cipher	ChaCha20-Poly1305[64][n 5]	256	N/A	N/A	N/A	N/A	Secure	Secure	Defined for TLS 1.2 in RFCs
	RC4[n 11]	128	Insecure	Insecure	Insecure	Insecure	Insecure	N/A	Prohibited in all versions of TLS by RFC 7465
		40[n 10]	Insecure	Insecure	Insecure	N/A	N/A	N/A	
None	Null[n 12]	–	Insecure	Insecure	Insecure	Insecure	Insecure	N/A	Defined for TLS 1.2 in RFCs

Overview

- Cryptography: Classical Algorithms,
- Cryptography: Public Key Algorithms
- Cryptography: Protocols
- Cryptography Workshop
- Biomedical Security and Applications
- Student Presentations

Grading:

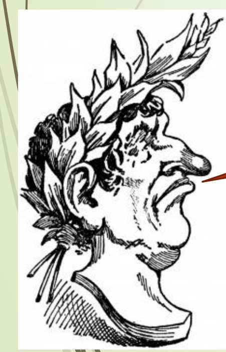
Class participation, assignments (3 out of 4)

(workshop + presentation + technical survey)/3

Cryptography: Sharing Secrets

- CAESAR a substitution cipher

Secret Key:	3
Plain Text:	A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
Cipher Text:	D E F G H I J K L M N O P Q R S T U V W X Y Z A B C



DWWDFFKL

E_3(HELP) = KHOS
D_3(KHOS) = HELP

$D_3 = E_{26\text{-}k}$

Mafia boss Bernardo Provenzano's cipher: 'A' -> 4, 'B' -> 5, etc. In April 2006, Provenzano was captured in Sicily partly because messages encrypted using his cipher, were broken.

https://www.theregister.co.uk/2006/04/19/mafia_don_clueless_crypto/

Cryptography: Sharing Secrets

Alice

C = E_K ('HELLO BOB')

Secret key K

Crypto-text C

Secret Key K?

Eve

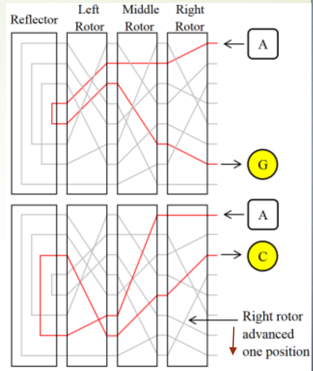
Bob

D_K (C) = 'HELLO BOB'

Secret key K

Crypto-Analyst Eve
- Crypto-text only
- Known Plaintext
- Chosen Plaintext

Enigma



Encryption as a product of permutations:

- P the plug-board transformation
- U the reflector
- L, M, and R the three rotors
- Then encryption is $E = PRMLUL^{-1}M^{-1}R^{-1}P^{-1}$
- After each key press the rotors turn i positions changing the transformation: R becomes C^iRC^{-i}, where C is the cyclic permutation (A->B, B-> C, etc. …)
- the military Enigma has 158,962,555,217,826,360,000 settings **(?)**

https://en.wikipedia.org/wiki/Enigma_machine

http://enigmamuseum.com/replica/

How would you proof that an encryption system has perfect secrecy?

ONE-TIME PAD

- A crypto system with perfect secrecy

Plaintext:	01000110101110100110
Key:	11010100001100010010
Crypto-text:	10010010100010110100

Uses XOR for both encryption and decryption.

Classical Symmetric or Two-way Crypto Systems

- A shared secret key K used for both encryption as well as decryption.

Secret Key	K
Plaintext	P
Crypto-text	C

$C = E_K(P)$

$P = E^{-1}_K(C)=D_K(C)$

Classical Symmetric Crypto System: Data Encryption Standard (DES)

- March 17, 1975 published by the National Bureau of Standards (NBS)
- NSA reduced key-size from the original 128-bit to 56-bit
- At the time NSA studied it and said it was secure to use as a standard SKCS.
- Next government standard was classified: Skipjack

- Block cipher encrypting data in 64-bit blocks
- Key length 56-bits
- 16 rounds: in each round a substitution is followed by a permutation

Permutations & Substitutions

Permutation: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{bmatrix}$

In general: $\begin{bmatrix} 1 & 2 & \dots & N \\ p_1 & p_2 & \dots & p_N \end{bmatrix}$

$$with\ \{p_1, \dots, p_N\} = \{1, \dots, N\}$$

Permutations & Substitutions

Substitution $\mathbf{S_5(0}1101\mathbf{1) = 1001}$

S_5		Middle 4 bits of input															
		0000	**0001**	**0010**	**0011**	**0100**	**0101**	**0110**	**0111**	**1000**	**1001**	**1010**	**1011**	**1100**	**1101**	**1110**	**1111**
Outer bits	**00**	0010	1100	0100	0001	0111	1010	1011	0110	1000	0101	0011	1111	1101	0000	1110	1001
	01	1110	1011	0010	1100	0100	0111	1101	0001	0101	0000	1111	1010	0011	1001	1000	0110
	10	0100	0010	0001	1011	1010	1101	0111	1000	1111	1001	1100	0101	0110	0011	0000	1110
	11	1011	1000	1100	0111	0001	1110	0010	1101	0110	1111	0000	1001	1010	0100	0101	0011

Feistel Networks

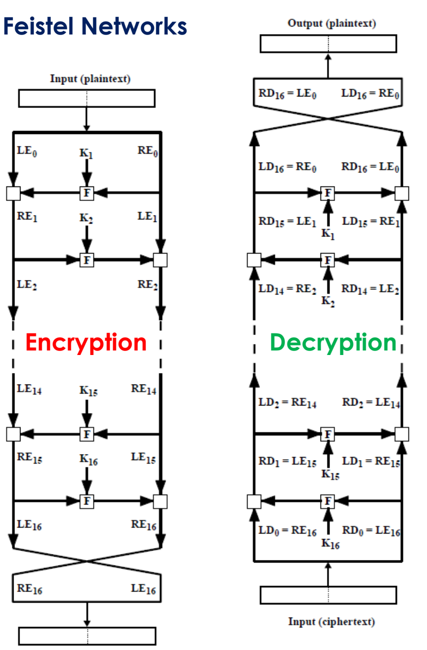
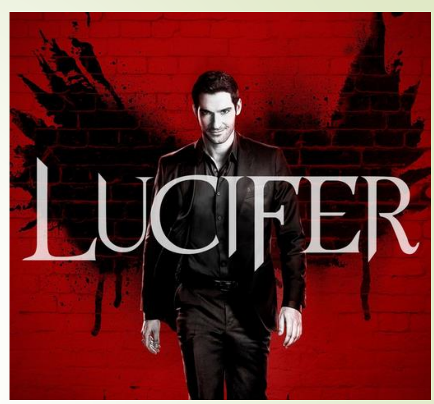


Figure 3.6 Feistel Encryption and Decryption



IBM's Cipher LUCIFER designed by H. Feistel and D. Coppersmith in 1973 used Feistel Networks for encryption and decryption.

LUCIFER is one of the first commercial block ciphers on which DES is based.

Encryption of the Text

Processing of the Key

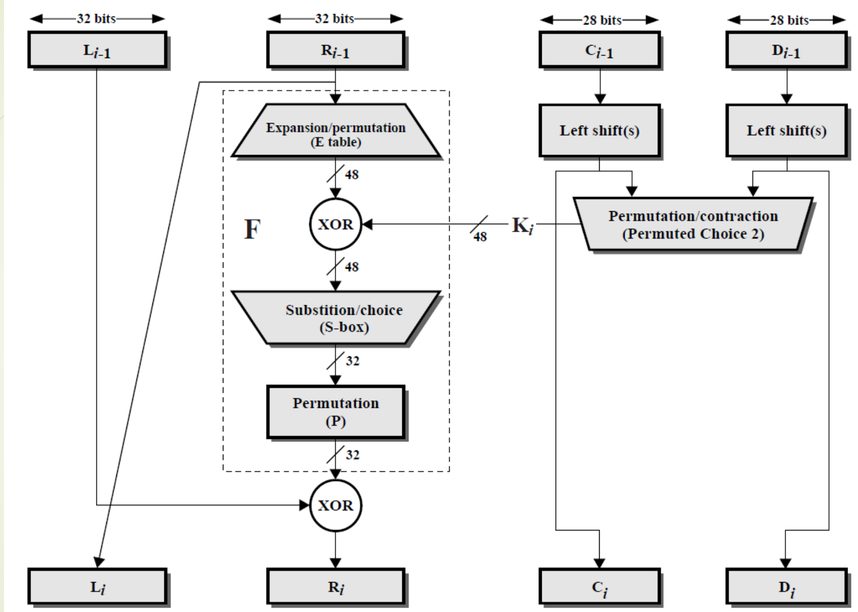
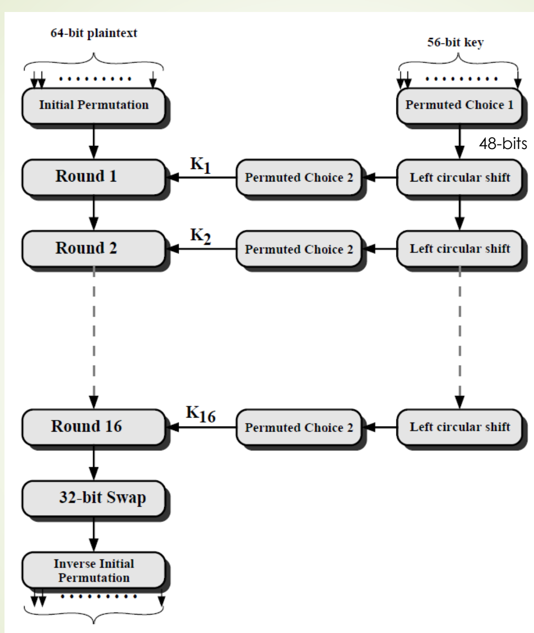


Figure 3.8 Single Round of DES Algorithm



Encryption in DES

Classical Symmetric Crypto System: International Data Encryption Algorithm (IDEA)

IDEA is a Block Cipher designed by X. Lai and J. Massey in 1990. Revised in 1991 to withstand differential cryptanalysis.

- **Block Length**

  64-bit Data Blocks Is considered safe against statistical attacks. Cipher Feedback Mode enhances cryptographic strength.

- **128-bit Key**

  Safe against brute-force attacks.

- **Good Confusion**

  By using three operations: XOR, Addition mod 2^{16}, Multiplication mod $2^{16}+1$ (compare with DES: XOR, small S-Boxes)

- **Good Diffusion**

  Every plaintext bit and every key-bit influences every ciphertext bit.

Symmetric Cryptosystem: BLOWFISH

Blowfish is a symmetric block cipher

designed by Bruce Schneier in 1993.

- **Block Length**

  64-bit data blocks encrypted in 64-bit ciphertext Blocks.

- **Key Length**

  32- 448 bits (1 to 14 32-bit key-blocks).

- **Variable Security**

  Key generates 18 (32-bit) subkeys, and 4 (8x32 bit) S-boxes. The algorithm itself is used for this.

- **Fast, simple, and compact**

  On a 32-bit processor: 18 clock cycles per encrypted byte. Uses less than 5K of memory (was at the time too big for smart-cards).

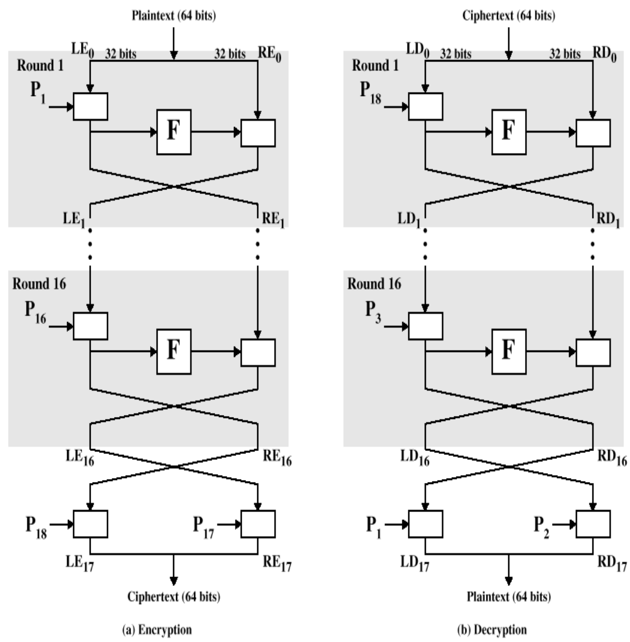


Figure 4.9 Blowfish Encryption and Decryption

Rivest Cipher 5 (RC5)

RC5 is a block-cipher by R. Rivest in 1994.

- **Efficient Hard and Software Implementations**

  Simple structure, simple operations, low memory requirements, fast and simple implementations.
- **Variable Word Length:**

  w = 16, 32,or 64 Length of the plaintext blocks is 2w
- **Variable Key-Length**

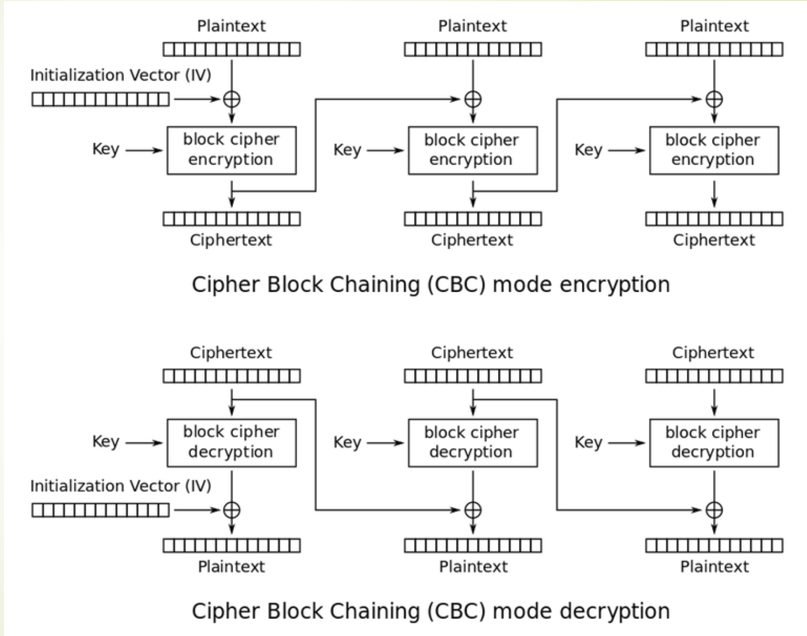
  b = 0,...,255 bytes
- **Variable Security**

  Depending on the parameters, number of rounds: r = 0,…,255
- **Data-Dependent Rotations**

  Circular Bit Shifts. RC5-w/r/b = RC5-32/12/16 considered to have “Nominal” Security. Incorporated in the products BSAFE, JSAFE, and S/MAIL of RSA Data Security, Inc.

Rivest Cipher 5 (RC5) Modes

- Block Cipher Mode
- Cipher Block Chaining Mode
- RC5-CBC-Pad
- RC5-CTS Ciphertext Stealing Mode: CBC style.



CAST-128

A symmetric encryption cipher by

C. Adams and S. Tavares in 1997.

- Uses four primitive operations

  addition and substraction mod 2^{32}, XOR, left circular rotations.
- Uses fixed non-linear S-boxes, also for sub-key generation.
- A function F is used with good confusion, diffusion, and avalanche properties.
  - its strength is based on the S-boxes. F differs per round.
  - increase of strength of CAST-128 using more rounds is not (yet) demonstrated.
- 64-bits data blocks
- 40- 128-bits key

CAST-128 is used in PGP.

Rivest Cipher 2 (RC2)

A symmetric encryption cipher by
R. Rivest in 1997.

- Designed for 16-bit microprocessors
- Uses 6 primitive operations

  addition and subtraction mod 2^{32}, XOR, COMPL, AND, and Left Circular Rotation.
- No Feistel Structure.
- 18 rounds: 16 mixing rounds, and 2 mashing rounds.
- 64-bits data blocks
- 8 - 1024-bits key

RC2 is used in S/MIME with 40-, 64-, and 128-bits keys.

RC2 is vulnerable to a related-key attack using 2^{34} chosen plaintexts (Kelsey et al., 1997).

Characteristics of Advanced Symmetric Block Ciphers

- **Variable Key Length** Blowfish, RC5, CAST-128, and RC2
- **Mixed Operators**
- **Data-Dependent Rotation** An alternative to S-boxes. No dependence on sub-keys. RC5.
- **Key-Dependent Rotation** CAST-128
- **Key-Dependent S-Boxes** Blowfish
- **Lengthy Key Schedule Algorithm** Against brute-force attacks. Blowfish
- **Variable F** to complicate cryptanalysis. CAST-128

Advanced Symmetric Block Ciphers

- **Variable Plaintext/Ciphertext Block Length**

  For convenience and cryptographic strength (longer blocks is better) RC5
- **Variable Number of Rounds**

  More rounds increase cryptographic strength. Trade-off between execution time and security. RC5
- **Operation on Both Data Halves in Each Round**

  AES, IDEA, Blowfish, and RC5

Advanced Encryption Standard (AES)

- Block-size: 128
- Key-sizes: 128, 192, 256
- NIST Specification 2001
- Origin: a subset of 3 out of the Rijndael Cipher by V. Rijmen and J. Daemen (NIST paper 2003)
- Substitution-permutation network
- From the cipher key the keys per round are derived.
- Each round
  - has a non-linear substitution step implemented using a lookup table
  - Followed by transposition using cyclic shifts
  - And a mixing step on the columns of the internal state matrix.
  - The round ends with an add key operation.

A Short Introduction to Number Theory

- Primes
- Factorization
- Euclid's Algorithm
- Modular Arithmetic and Groups
- Fast Exponentiation
- Discrete Logarithms
- Euler Phi

Number Theory

Definition (Divisors):

$b \neq 0$ divides a, if a = mb for some m (where a, b, and m are integers)

Notation: $b \mid a$

Example: divisors of 24 are

1, 2, 3, 4, 6, 8, 12, and 24

The following relations hold:

- if $a \mid 1$, then $a = \pm 1$
- if $a \mid b$ and $b \mid a$, then $a = \pm b$
- any $b \neq 0$ divides 0
- if $b \mid g$ and $b \mid h$, then $b \mid (mg+nh)$ for arbitrary integers m and n

Number Theory

Definition (Prime Numbers):

An integer p>1 is a prime number if its only divisors are ±1 and ±p.

Theorem: Any positive integer a>1 can be factored in a unique way as:

$$a = p_1^{a1}.p_2^{a2}\ldots p_t^{at},$$

where $p_1 > p_2 > \ldots > p_t$ are prime,

and $a_i > 0$

$$\text{or } a = \prod_{i=1\ldots t} p_i^{a_i}, \text{ where } p_1 > p_2 > \ldots > p_t \text{ are prime and each } a_i \geq 0$$

Example: 91 = 7 x 13,

11011 = 7 x 11^2 x 13

Number Theory

Definition1 (GCD):

The positive integer c is said to be the greatest common divisor of a and b if:

1) c | a and c | b
2) if d | a and d | b, then d | c

Notation: c = gcd(a,b)

Definition2 (GCD):

gcd(a,b) = max[k, such that k | a and k | b]

Example: 192 = 2^2 x 3^1 x 4^2

18 = 2^1 x 3^2

gcd(18,192) = 2^1 x 3^1 x 4^0 = 6

Number Theory

Definition1 (Relative Prime):
The integers a and b are said to be relatively prime if gcd(a,b) = 1.

Example:
192 and 18 are not relatively prime:

$192 = 2^2 \times 3^1 \times 4^2$

$18 = 2^1 \times 3^2$

$\gcd(18,192) = 2^1 \times 3^1 \times 4^0 = 6$

74 and 75 are relatively prime:

$74 = 2 \times 37$

$75 = 3 \times 5^2$

$\gcd(74,75) = 1$

Number Theory: Modular Arithmetic

Given any positive integer n and any integer a we can write:

$a = qn + r$, where $0 \leq r < n$, $q = \lfloor a/n \rfloor$

r is called the **residue** (mod n)

Definition: If a is an integer and n is a positive integer we define *a mod n* to be the remainder when a is divided by n.

Thus, $a = \lfloor a/n \rfloor \times n + (a \bmod n)$

Definition: Two integers are said to be congruent modulo n if

$(a \bmod n) = (b \bmod n)$

Notation: $a \equiv b \bmod n$

Number Theory: Modular Arithmetic

Examples: $73 \equiv 4 \bmod 23$ as
$73 = 3 \times 23 + 4$, hence
$4 = 73 \bmod 23$, and clearly
$4 = 4 \bmod 23$

$21 \equiv -9 \bmod 10$ as
$1 = 21 \bmod 10$ and
$1 = -9 \bmod 10$

Properties (Check):
- $a \equiv b \bmod n$ if $n \mid (a-b)$
- $(a \bmod n) = (b \bmod n)$ implies $a \equiv b \bmod n$
- $a \equiv b \bmod n$ implies $b \equiv a \bmod n$
- $a \equiv b \bmod n$ and $b \equiv c \bmod n$ implies $a \equiv c \bmod n$

Number Theory: Modular Arithmetic

The mod n operator maps all integers into the set of integers $Z_n = \{0,1,\ldots,(n-1)\}$, the set of all residues modulo n.

The following properties hold for modular arithmetic within Z_n:
- $(w + x) \bmod n = (x + w) \bmod n$
- $((w+x)+y) \bmod n = (w+(x+y)) \bmod n$
- $(0+w) \bmod n = w \bmod n$
- $\forall w \in Z_n \, \exists z \in Z_n$ such that $w + z \equiv 0 \bmod n$

- $(w \times x) \bmod n = (x \times w) \bmod n$
- $((w \times x) \times y) \bmod n = (w \times (x \times y)) \bmod n$
- $(1 \times w) \bmod n = w \bmod n$
- $(w \times (x+y)) \bmod n = ((w \times x)+(w \times y)) \bmod n$

Number Theory: Modular Arithmetic

Z_8:	0	1	2	3	4	5	6	7
×6:	0	6	12	18	24	30	36	42
mod 8:	0	6	4	2	0	6	4	2

Z_8:	0	1	2	3	4	5	6	7
×5:	0	5	10	15	20	25	30	35
mod 8:	0	5	2	7	4	1	6	3

Note: gcd(6,8) = 2, and gcd(5,8) = 1

Notation: $Z_p^* = \{1,2,\ldots,(p-1)\}$

Theorem: Let p prime, then for each $w \in Z_p^*$ there exists a z such that $w \times z \equiv 1 \bmod p$, z is equal to the multiplicative inverse w^{-1} of w

Public-Key Cryptography Fast Exponentiation

Calculate $a^b \bmod n = 7^{560} \bmod 561$

a = 7, b = 560 = 1000110000, n = 561

I	Bit b_i	Exponent c	result d	$\rightarrow 7^{560}$
9	1	1	7	7^1
8	0	2	49	7^2
7	0	4	157	7^4
6	0	8	526	7^8
5	1	17	160	7^{16+1}
4	1	35	241	7^{32+2+1}
3	0	70	298	7^{64+4+2}
2	0	140	166	$7^{128+8+4}$
1	0	280	67	$7^{256+16+8}$
0	0	560	1	$7^{512+32+16}$

```
c ← 0; d ← 1
for i ← k downto 0
    do c ← 2 × c
       d ← (d × d) mod n
       if  b_i = 1
           then  c ← c + 1
                 d ← (d × a) mod n
return d
```

Number Theory: Euler Totient Function

Definition: The Euler's totient function $\Phi(n)$ of n is equal to the number of positive integers <n that are relative prime to n.

Examples:

8: {1,3,5,7} are relative prime to 8 and <8, thus $\Phi(8) = 4$

11: {1,2,3,4,5,6,7,8,9,10} are relative prime to 11 and <11, thus $\Phi(11) = 10$

Lemma: If p is prime, then $\Phi(p) = p - 1$.

Lemma: If n = pq, with p and q prime, then $\Phi(n) = (p-1)(q-1)$.

Proof: {p,2p,…,(q-1)p}, {q,2q,…,(p-1)q}, and 0 are not relatively prime. Thus $\Phi(n) = pq - (q-1) - (p-1) - 1 = (p-1)(q-1)$.

Number Theory: Euler's Totient Function

Fermat's Theorem (1640): For every prime p and any integer a, the following holds:

$$a^{p-1} \equiv 1 \bmod p.$$

Euler's Theorem (~1740): For any positive integer n, and any integer a relative prime to n, the following holds:

$$a^{\Phi(n)} \equiv 1 \bmod n$$

Corollary: Let p,q be prime, and n = pq, m an integer such that gcd(m,n)=1, then

$$m^{(p-1)(q-1)} \equiv 1 \bmod n$$

Examples:

$\mathbf{2^6 = 64 = 63 + 1} \equiv 1 \bmod 7$

$4^{(5-1)(7-1)} = 4^{24} = (4^8)^3 \bmod 35 \equiv 16^3 \bmod 35 \equiv 4096 \bmod 35 \equiv 1 \bmod 35$

Number Theory: Testing for Primality

[Miller'75, Rabin'80]

Procedure Witness(a,n) n is to be tested for primality, a is some integer less than n.

if (not $a^{n-1} \equiv 1 \bmod n$) or
($\exists x$: $x^2 \equiv 1 \bmod n$ and $x \neq \pm 1$)
then return TRUE {n is no prime}
else return FALSE {n may be prime}

If n is no prime the probability that Witness returns FALSE is <0.5.
Thus, if Witness returns FALSE s times the probability that n is prime is at least $1 - 2^{-s}$.

Number Theory: Number of Primes

Definition: $\pi(n)$ is equal to the number of primes p that satisfy $2 \le p \le n$.

Theorem (The Prime Number Theorem, conjectured by Legendre, Gauss, Dirichlet, Chebyshev, and Riemann; proven by Hadamard and de la Vallee Poussin in 1896).

$$\pi(n) \sim n/\ln(n)$$

Thus there are about

$$10^{100}/\ln(10^{100}) - 10^{99}/\ln(10^{99}) =$$
$$0.039 \times 10^{99} \text{ 100-digit primes}$$

There are 4.5×10^{99} 100-digit odd numbers.

That is, about 1 of every 115 100-digit odd numbers is prime.

Number Theory: Euclid's Algorithm
Finding the Greatest Common Divisor

Theorem: For any integer a≥0, and any integer b>0: gcd(a,b) = gcd(b,a mod b)

Proof: Let d = gcd(a,b) => d | a and d | b => a = kb + a mod b for some integer k => (a mod b) = a -kb => d | (a mod b) (as d | a and d | kb). Thus d is a common divisor of b and (a mod b).

Conversely, if d = gcd(b, a mod b), then d | kb and thus also d | (kb + a mod b) => d | a. Thus d is also a common divisor of a and b.

qed

Example (Calculation of GCD):

- gcd(12,18) = gcd(18,6) = gcd(6,0) = 6
- gcd(10,11) = gcd(11,1) = gcd(1,0) = 1

Number Theory: Euclid's Extended Algorithm
Finding the Multiplicative Inverse

If gcd(d,n) = 1, then (d^{-1} mod n) exists.

I.e., dd^{-1} = 1 mod n.

Complexity: The multiplicative inverse can be found in $O(\log^2 n)$ time.

Number Theory: Discrete Logarithm

Definition: Let $Z_n^*=\{1,2,\ldots,(n-1)\}$, and g in Z_n^*. Then any integer x such that:

$$g^x = y \bmod n$$

is called a *discrete logarithm of y to base g.*

Example:

Z_7^*	1	2	3	4	5	6
	3^1	3^2	3^3	3^4	3^5	3^6
g=3	3	2	6	4	5	1

Z_7^*	1	2	3	4	5	6
$\log_3$	6	2	1	4	5	3

N.B. g=3 is a generator of Z_7^*

Definition: If for g in Z_p^* $\{g^1,\ldots,g^{(p-1)}\} = Z_p^*$ holds, then g is a *generator* of Z_p^*.

Number Theory: Complexity of PRIMES, Discrete Log, FACTORIZE, etc.

- Finding Primes (PRIMES is in P, AKS-Algorithm, August 2002)

After 1/115 tries success. Each try fastexp and some tests are executed => O(log n) time.

- Finding Safe Primes

It is unknown whether there exist infinitely many safe primes.

- Calculating the Discrete Logarithm

If the prime factors of (p-1) are small there exist efficient algorithms, otherwise roughly the same complexity as factorising.

- Factorising n (b-bits)

  Peter Shor(1994): $O(b^3)$ and O(b) space on a quantum computer.

  Kleinjung et al. (2010) used general number field sieve GNFS- approach, $O(e^{\sqrt[3]{\frac{64}{9}b(\log b)^2}}$ time, for the factorization of a 768-bit RSA modulus n.

- Calculating Euler's Phi Function of n

It is unknown if this can be done without factorising n.

- Finding the multiplicative inverse mod n

$O(\log^2 n)$

Public Key Crypto Systems

Idea by Diffie and Hellman [1976]

- Encryption method made public.
- Decryption method kept secret.

Closely related to the idea of cryptographic one-way functions:

$$x \underset{\text{intractable}}{\overset{\text{easy}}{\rightleftarrows}} f(x)$$

But there exists a trapdoor that makes it easy to calculate x given f(x).

Note: No known cryptographic one-way functions. Only likely to be intractable!

Public Key Crypto System: RSA

- Key Generation

    Select p, q, both primes.

    Calculate $n = p * q$

    Calculate $\varphi(n) = (p-1)(q-1)$

    Select integer e, such that $\gcd(\varphi(n), e) = 1$, and $1 < e < \varphi(n)$

    Calculate d, such that $d = e^{-1} \bmod \varphi(n)$

  Public key is (e, n)

  Secret key is (d, n)

- Encryption of plaintext $M < n$

    $C = M^e \bmod n$

- Decryption of ciphertext/crypto-text C

    $M = C^d \bmod n$

Diffie-Hellman Key Exchange Algorithm

- Global and Public
  Prime q, and $\alpha < q$ a primitive root of q
- Alice Key Generation
  Select private $X_A < q$
  Calculate public $Y_A = \alpha^{X_A} mod\ q$
- Bob Key Generation
  Select private $X_B < q$
  Calculate public $Y_B = \alpha^{X_B} mod\ q$
- Generation of the Exchanged Secret Key by Alice
  $K = (Y_B)^{X_A} mod\ q$
- Generation of the Exchanged Secret Key by Bob
  $K = (Y_A)^{X_B} mod\ q$

Digital Signatures

Using a public key crypto system.

(M,S)

Alice

Bob

Message M and signature S = D_{Alice} (M)

Checks with public E_{Alice}
if $E_{Alice}(S) = E_{Alice}(D_{Alice}(M)) = M$

Public Key Crypto System ElGamal

- Public Key
  - p a prime
  - g < p
  - $y = g^x mod\ p$
- Private Key
  - x < p
- Encryption of message M
  - random k, with gcd(k,p-1)=1
  - $C = (a, b),\ where\ \ a = g^k mod\ p\ \ \text{and}\ \ b = y^k M\ mod\ p$
- Decryption
  - $M = b/a^x \text{mod p}$

ElGamal Signatures

- Public Key
  - p a prime
  - g < p
  - $y = g^x mod\ p$
- Private Key
  - x < p
- Signing of message M
  - random k, with gcd(k,p-1)=1
  - Signature S = $(a, b),\ where$
  - $a = g^k mod\ p\ \ \text{and}\ \ b\ such\ that\ M = (xa + kb)\ mod\ p\text{-}1$
- Verification
  - Accept as valid if $y^a a^b\ mod\ p =\ g^M mod\ p$

Cryptographic Hash Functions

An **hash function** H has the following properties:

- H can be applied to data of any size.
- H produces fixed length output.
- H(x) is easy to compute for any given x.

- **One-way**

  for any given hash-code h, it is computationally infeasible to find x such that H(x) = h.

- **Weak collision resistance**

  for any given x, it is computationally infeasible to find y (not equal to x) such that H(y) = H(x).

- **Strong collision resistance**

  it is computationally infeasible to find any pair (x,y) such that H(x) = H(y).