SHORT TIME FOURIER TRANSFORMS

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Overview

- Fourier Transforms

Material adapted from lectures by

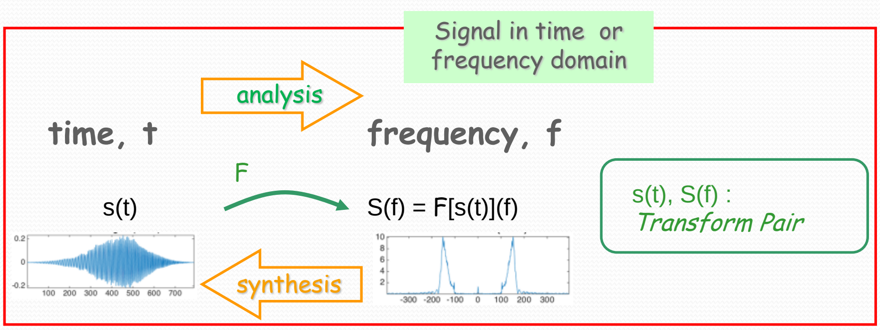
Dr M.E. Angoletta at DISP2003, a DSP course given by CERN and University of Lausanne (UNIL)

Fourier Transforms

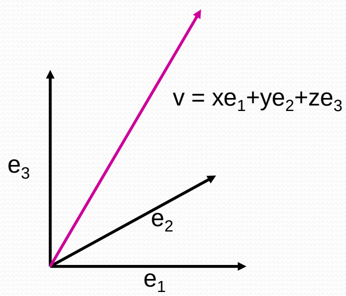
- Frequency analysis
- A tour of Fourier Transforms
- Continuous Fourier Series (FS)
- Discrete Fourier Series (DFS)

Frequency Analysis

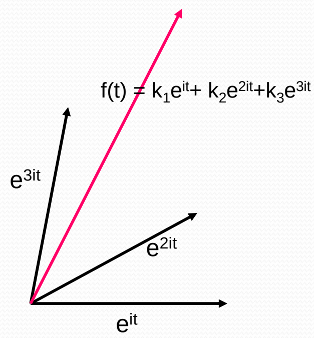
- **Fast & efficient insight on the signal's components**.
- **Powerful & complementary to time domain analysis techniques.**
- But also simplifies original problem - Filtering, solving Part.Diff.Eqns. (PDE),...
- Many transforms: **Fourier, Discrete Cosine, Laplace, z, Wavelet, etc.**



Bases of Vector Spaces

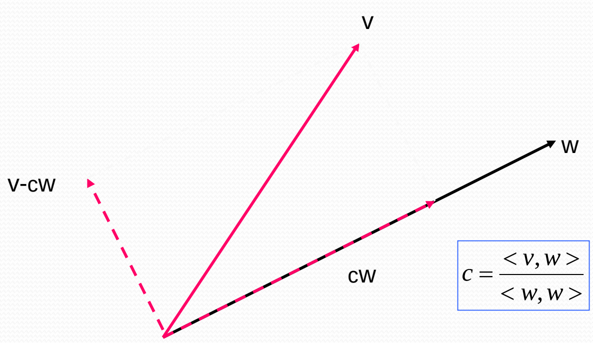


v is a linear combination of the basis vectors e_i (i = 1,2,3)



f is a linear combination of the basis functions e^{it},e^{2it},e^{3it}

Fourier Coefficients



$$c = \frac{<v,w>}{<w,w>}$$ Fourier Coefficient

Let <.,.> an in-product for our vector space V.
Then we calculate the Fourier coefficient **c** of v in V with respect to (basis) vector w by:

$$c = \frac{<v,w>}{<w,w>}$$

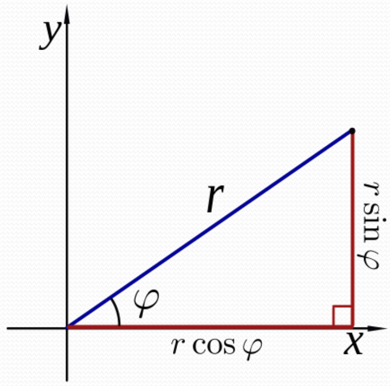
=> cw is the component of v along the direction of w.

Polar Coordinates in $\mathbb{R}^2$

Relation between Polar coordinates (r,φ) and Cartesian coordinates (x,y):

$x = r \cos \varphi$

$y = r \sin \varphi$

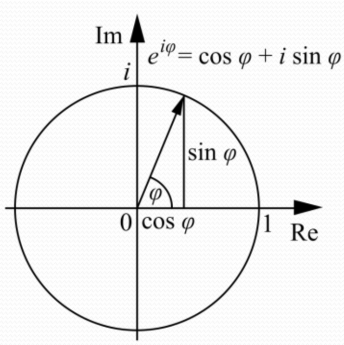


Complex Numbers

Define $e^{i\varphi} = \cos \varphi + i \sin \varphi$

Note: you can write any complex number $z = a + bi$ as:

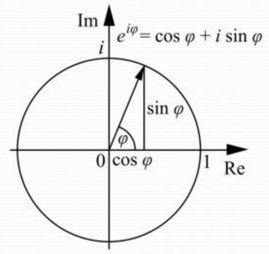
$z = r\, e^{i\varphi}$, with $r = |z|$



Complex Numbers and Functions

Let $z = r\,e^{i\varphi}$, then $\bar{z} = r\,e^{-i\varphi}$ (alternative notation: z^*)

Let $z_1 = r_1\,e^{-i\varphi_1}$, and $z_2 = r_2\,e^{-i\varphi_2}$, then



$$z_1 z_2 = r_1 r_2\,e^{-i(\varphi_1 + \varphi_2)}$$

Let f a given frequency.

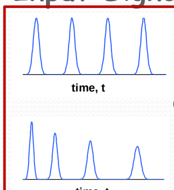
360^0

Let $h(t) = e^{i2\pi ft}$ then $h(t) = \cos 2\pi ft + i \sin 2\pi ft$, thus

$h(t)$ is a function that is 'repeating' over time with frequency f

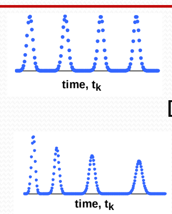
Fourier Analysis – Different 'Flavours'

Input Signal in Time Domain — **Frequency spectrum**



Continuous Time

Periodic *(period T)*	FS	Discrete	$c_k = \frac{1}{T} \cdot \int_0^T s(t) \cdot e^{-ik\omega t} dt$
Aperiodic	FT	Continuous	$S(f) = \int_{-\infty}^{+\infty} s(t) \cdot e^{-i2\pi f t} dt$



Discrete Time

Periodic *(period T)*	DFS**	Discrete	$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-i\frac{2\pi k n}{N}}$
Aperiodic	DTFT	Continuous	$S(f) = \sum_{n=-\infty}^{+\infty} s[n] \cdot e^{-i2\pi f n}$
Aperiodic	DFT**	Discrete	$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-i\frac{2\pi k n}{N}}$

***Note:** $i = \sqrt{-1}$, $\omega = 2\pi/T$, $s[n] = s(t_n)$, N = No. of samples*

***** Calculated using FFT***

History Fourier Transform

- **1669**: Newton: light spectra (*specter* = ghost) but no "frequency" concept (no waves).
- **18th century**: two important problems
  - → celestial bodies orbits: Lagrange, Euler & Clairaut approximate observation data with linear combination of periodic functions; Clairaut,1754(!) first DFT formula.
  - → vibrating strings: Euler describes vibrating string motion by sinusoids (wave equation).
  - → But consensus was: sum of sinusoids *only* represents smooth curves.
- **1807**: Fourier presents his work on heat conduction ⇒ Fourier analysis born.
  - → Diffusion equation ⇔ series (infinite) of sines & cosines.
  - → Strong criticism by peers blocks publication.
  - → **Work published, 1822** *("Theorie Analytique de la chaleur")*.

History Fourier Transform -2

- **19th / 20th century**: two paths for Fourier analysis - Continuous & Discrete.

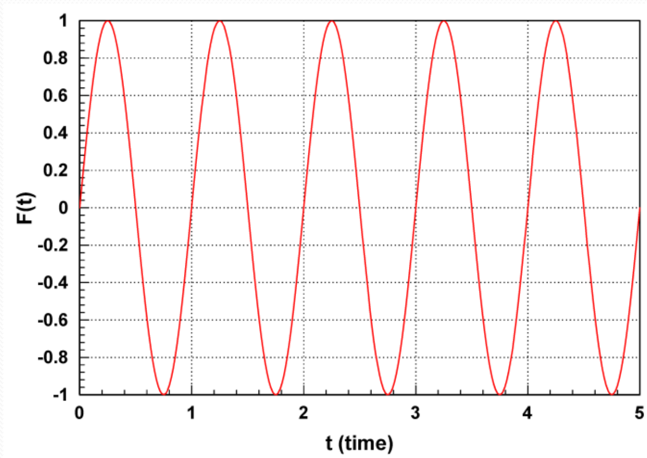
CONTINUOUS

- → Fourier extends the analysis to arbitrary functions (Fourier Transform).
- → Dirichlet, Poisson, Riemann, Lebesgue address Fourier Series convergence.
- → Other FT variants born from varied needs (ex.: Short Time FT - speech analysis).

DISCRETE: Fast calculation methods (FFT)

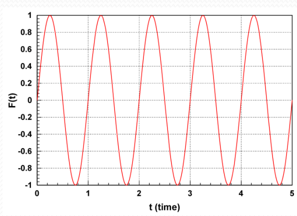
- → **1805** - Gauss, first usage of FFT (manuscript in Latin went unnoticed!!! Published 1866).
- → **1965** - IBM's Cooley & Tukey "rediscover" FFT algorithm (*"An algorithm for the machine calculation of complex Fourier series"*).
- → Other DFT variants for different applications (ex.: Warped DFT - filter design & signal compression).
- → FFT algorithm refined & modified for most computer platforms.
- → *Fastest Fourier Transform in the West (FFTW)*

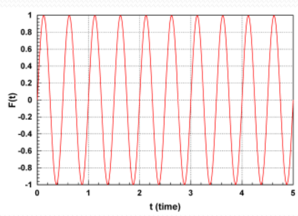
Another Space, Another Base

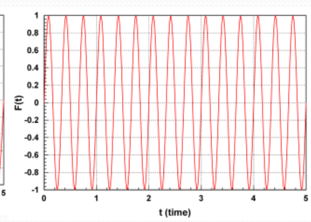


F(t) = sin(2π.t)

Another Space, Another Base





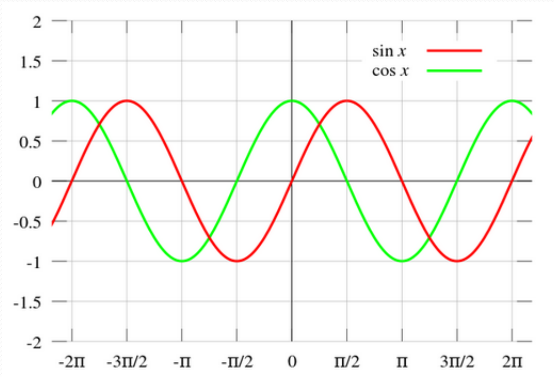


F(t) = sin(2π.t) F(t) = sin(2π.2t) F(t) = sin(2π.3t)

F(t) = cos(2π.t) F(t) = cos(2π.2t) F(t) = cos(2π.3t)

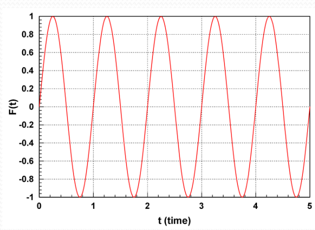
{ cos(2π.kt), sin(2π.kt) }$_k$ forms an orthogonal basis

Sine Cosine Graphs

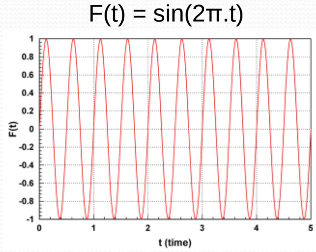


$$\sin(\varphi + \pi/2) = \cos(\varphi)$$

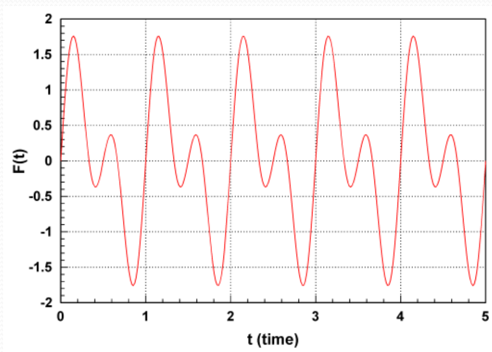
Linear Combination of Functions



F(t) = sin(2π.t)

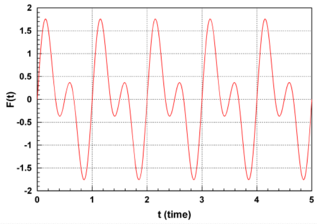


F(t) = sin(2π.2t)

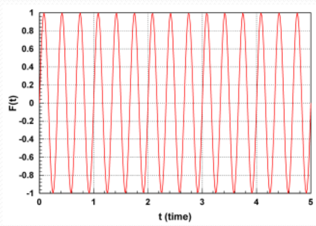


F(t) = sin(2π.t) + sin(2π.2t)

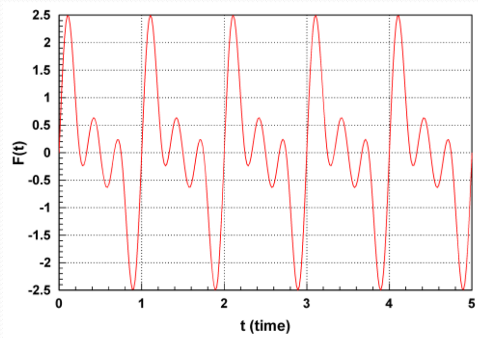
Linear Combination of Functions



F(t) = sin(2π.t) + sin(2π.2t)

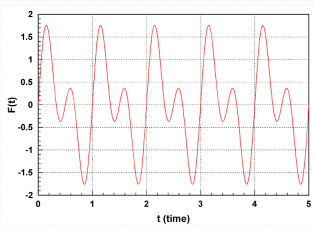


F(t) = sin(2π.3t)

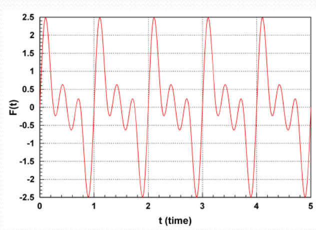


F(t) = sin(2π.t) + sin(2π.2t) + sin(2π.3t)

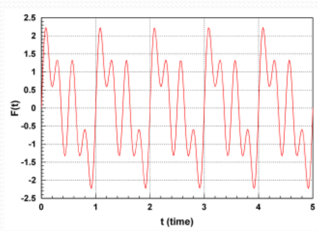
Linear Combination of Functions



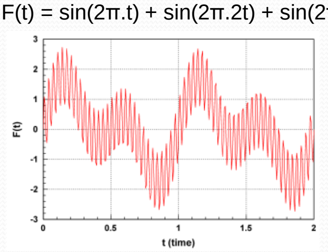
F(t) = sin(2π.t) + sin(2π.2t)



F(t) = sin(2π.t) + sin(2π.2t) + sin(2π.3t)

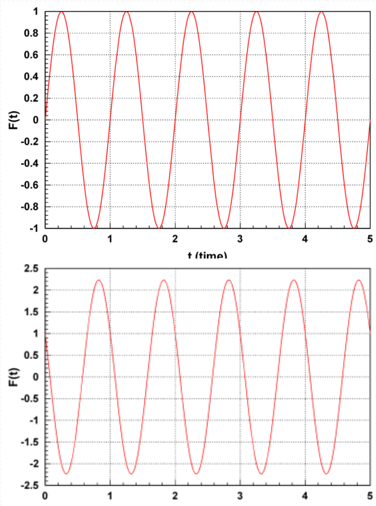


F(t) = sin(2π.t) + sin(2π.2t) + sin(2π.4t)



F(t) = sin(2π.t) + sin(2π.2t) + sin(2π.30t)

Linear Combination of Functions

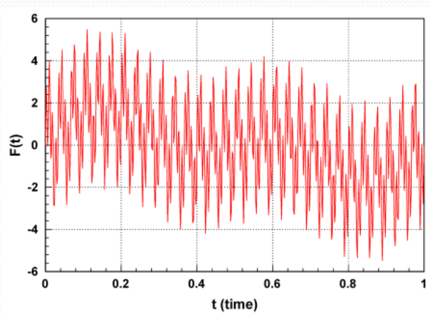


Phase Shift:

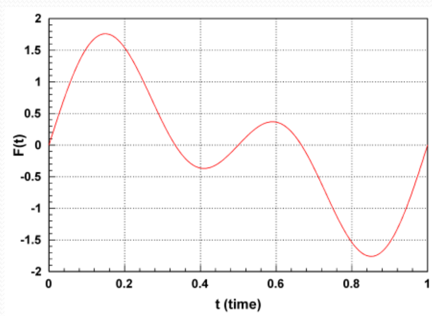
F(t) = sin(2π.t)

F(t) = cos(2π.t) – 2.sin(2π.t)

Low Band Pass Filters



F(t) = sin(2π.t) + sin(2π.2t) + <- low freq.
sin(2π.30t) + sin(2π.120t) <- high freq.



F(t) = sin(2π.t) + sin(2π.2t)

Fourier Series (FS)

** see next slide*

A periodic function s(t) satisfying Dirichlet's conditions * can be expressed as a Fourier series, with harmonically related sine/cosine terms.

synthesis

Inverse Fourier Transform

$$s(t) = a_0 + \sum_{k=1}^{+\infty} [a_k \cdot \cos(k\omega t) - b_k \cdot \sin(k\omega t)]$$

For all t but discontinuities

t: ~ time
a_0, a_k, b_k : Fourier coefficients.
k: ~ **frequency**, harmonic number
T: period, $\omega = 2\pi/T$

analysis

Fourier Transform

$$a_0 = \frac{1}{T} \cdot \int_0^T s(t) dt$$

(a_0 is signal average over a period, i.e. Direct Current (DC) term & zero-frequency component.)

$$a_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \cos(k\omega t) dt$$

$$-b_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \sin(k\omega t) dt$$

Note: $\{\cos(k\omega t), \sin(k\omega t)\}_k$ form orthogonal base of function space.

$s(t) \leftrightarrow S(k) = (a_k, b_k)$

Fourier Series Convergence

Dirichlet conditions

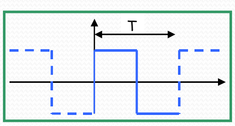
In any period:

(a) s(t) piecewise-continuous;

(b) s(t) piecewise-monotonic;

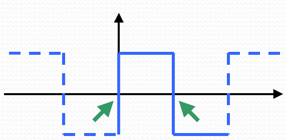
(c) s(t) absolutely integrable , $\int_0^T |s(t)| dt < \infty$

Example: square wave

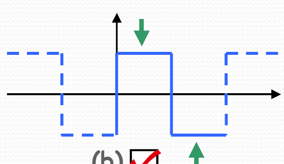


Rate of convergence

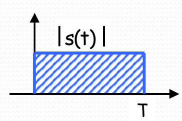
if s(t) discontinuous then $|a_k| < M/k$ for large k (M>0)



(a) ☑



(b) ☑



(c) ☑

Fourier Series Analysis - 1

$$a_0 = \frac{1}{T}\cdot\int_0^T s(t)dt$$

$$a_k = \frac{2}{T}\cdot\int_0^T s(t)\cdot\cos(k\omega t)\,dt$$

$$-b_k = \frac{2}{T}\cdot\int_0^T s(t)\cdot\sin(k\omega t)\,dt$$

Fourier series of square wave sw(t):

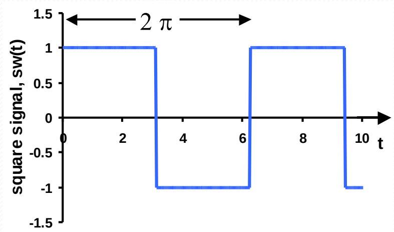
$$a_0 = \frac{1}{2\pi}\cdot\left\{\int_0^{\pi} dt + \int_{\pi}^{2\pi}(-1)dt\right\} = 0$$ *(zero average)*

$$a_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi} \cos kt\,dt - \int_{\pi}^{2\pi}\cos kt\,dt\right\} = 0$$

$$-b_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi} \sin kt\,dt - \int_{\pi}^{2\pi}\sin kt\,dt\right\} = \ldots = \frac{2}{k\cdot\pi}\cdot(1-\cos k\pi) =$$

$$= \begin{cases} \frac{4}{k\cdot\pi}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

sw(t)



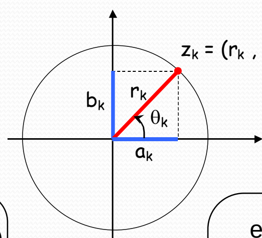
$T = 2\pi \Rightarrow \omega = 1$

$$sw(t) = \frac{4}{\pi}\cdot\sin t + \frac{4}{3\cdot\pi}\cdot\sin 3\cdot t + \frac{4}{5\cdot\pi}\cdot\sin 5\cdot t + \ldots$$

Fourier Series Analysis - 2

Fourier spectrum representations (in **k**)

$$s(t) = \sum_{k=0}^{\infty} v_k(t)$$



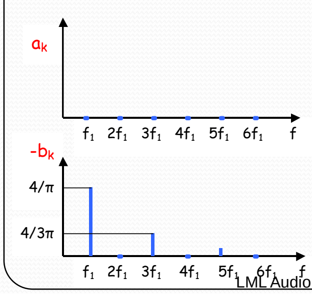
$z_k = (r_k, \theta_k)$

$r_k = \sqrt{a_k^2 + b_k^2}$

$\theta_k = \arctan(b_k/a_k)$

Rectangular

$e_k(t) = a_k\cos(\omega k\,t) - b_k\sin(\omega k\,t)$

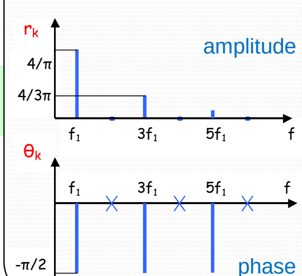


r_K = amplitude,
θ_K = phase

$f_k = k\,\omega/2\pi$

Polar

$e_k(t) = r_k\cos(\omega k\,t + \theta_k)$

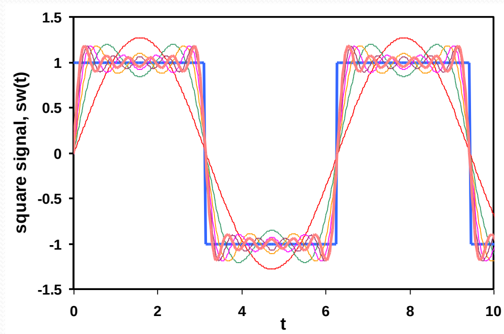


Fourier spectrum of square-wave.

Fourier Series Synthesis

Square wave reconstruction
from spectral terms

$$sw_9(t) = \sum_{k=1}^{9} [-b_k \cdot \sin(kt)]$$

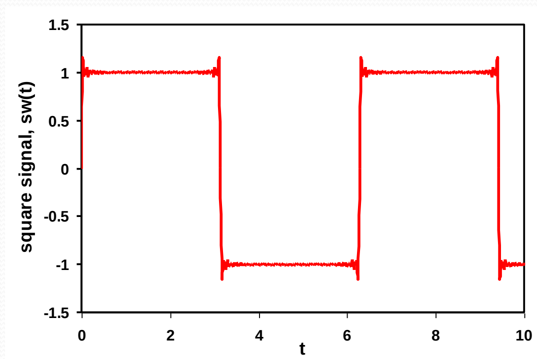


Convergence may be slow (~1/k) - ideally need infinite terms.
Practically, series truncated when remainder below computer tolerance
(⇒ *error*). ***BUT*** … Gibbs' Phenomenon.

Gibbs Phenomenon

Overshoot exist at each
discontinuity

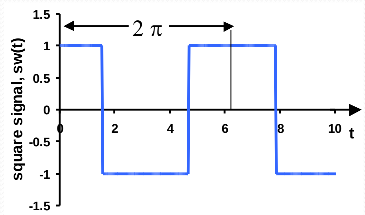
$$sw_{79}(t) = \sum_{k=1}^{79} [-b_k \cdot \sin(kt)]$$



- First observed by Michelson, 1898. Explained by Gibbs.
- Max overshoot pk-to-pk = 8.95% of discontinuity magnitude.
- FS converges to (-1+1)/2 = 0 at discontinuities, *in this case*.

Fourier Series Time Shifting

FS of even function:
$\pi/2$-advanced square-wave

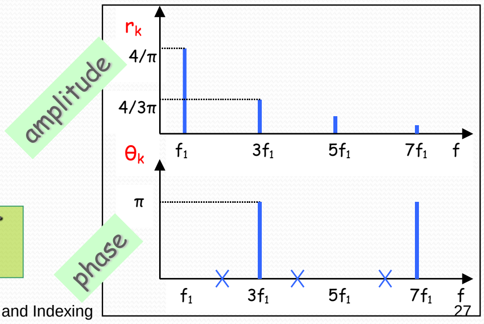


$a_0 = 0$ *(zero average)*

$$a_k = \begin{cases} \frac{4}{k\cdot\pi}, & k \text{ odd}, \; k = 1,5,9\ldots \\ -\frac{4}{k\cdot\pi}, & k \text{ odd}, \; k = 3,7,11\ldots \\ 0, & k \text{ even}. \end{cases}$$

$-b_k = 0$ *(even function: s(-x) = s(x))*

Note: amplitudes unchanged ***BUT*** phases advance by $k\cdot\pi/2$.



Complex Fourier Series

Euler's notation:
$e^{-jt} = (e^{jt})^* = \cos(t) - i\cdot\sin(t)$ → "phasor" $\cos(t) = \frac{e^{it} + e^{-it}}{2}$ $\sin(t) = \frac{e^{it} - e^{-it}}{2\cdot i}$

analysis
$$c_k = \frac{1}{T}\cdot\int_0^T s(t)\cdot e^{-ik\omega t}dt$$

synthesis
$$s(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{ik\omega t}$$

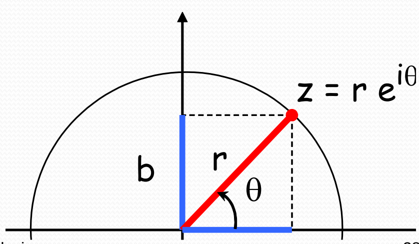
Complex form of FS (Laplace 1782). Harmonics c_k separated by $\Delta f = 1/T$ on frequency plot.

Note: $c_{-k} = (c_k)^*$

Link to FS real coeffs.

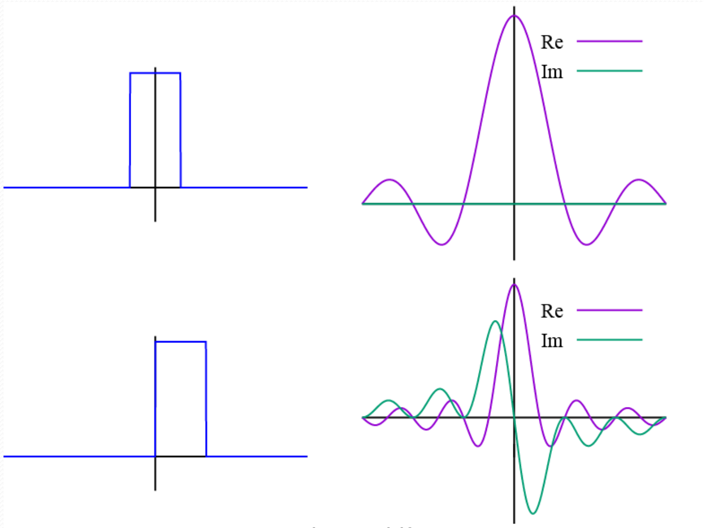
$c_0 = a_0$

$$c_k = \frac{1}{2}\cdot(a_k + i\cdot b_k) = \frac{1}{2}\cdot(a_{-k} - i\cdot b_{-k})$$



Signal

Fourier Transformed Signal



Phase Shift

Fourier Series Properties

	Time (t)	Frequency (f)
Homogeneity	$a\cdot s(t)$	$a\cdot S(f)$
Additivity	$s(t) + u(t)$	$S(f)+U(f)$
Linearity	$a\cdot s(t) + b\cdot u(t)$	$a\cdot S(f) + b\cdot U(f)$
Time reversal	$s(-t)$	$S(-f)$
Multiplication	$s(t)\cdot u(t)$	$\frac{1}{T}\cdot\int_0^T S(f-t)\cdot U(t)\,dt$
Convolution	$\sum_{m=-\infty}^{\infty} s(m)u(t-m)$	$S(f)\cdot U(f)$
Time shifting	$s(t-\bar{t})$	$e^{-i\frac{2\pi f\cdot t}{T}}\cdot S(f)$
Frequency shifting	$e^{+i\frac{2\pi m t}{T}}\cdot s(t)$	$S(f - m)$

Discrete Fourier Series (DFS)

Band-limited signal s[n], period = N.

DFS generate periodic c_k with same signal period

DFS defined as:

FT: analysis

$$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-i\frac{2\pi k n}{N}}$$

Note: $\tilde{c}_{k+N} = \tilde{c}_k \Leftrightarrow$ **same period N**

i.e. time periodicity propagates to frequencies!

Orthogonality in DFS:

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{i\frac{2\pi\, n(k-m)}{N}} = \delta_{k,m}$$

↑ Kronecker's delta

IFT: synthesis

$$s[n] = \sum_{k=0}^{N-1} \tilde{c}_k \cdot e^{i\frac{2\pi k n}{N}}$$

N consecutive samples of s[n] completely describe s in time or frequency domains.

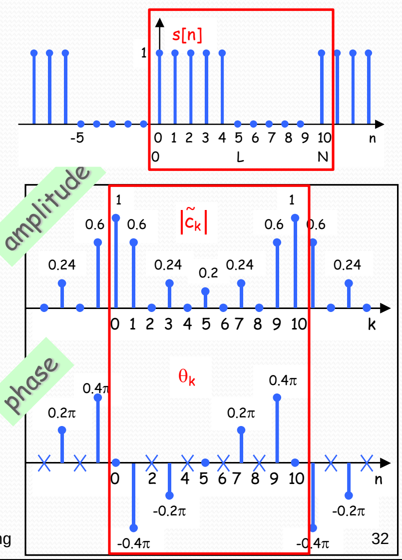
Synthesis: finite sum ⇐ band-limited s[n]

Discrete Fourier Series Analysis

DFS of periodic discrete 1-Volt square-wave

s[n]: period N, duty factor L/N

$$\tilde{c}_k = \begin{cases} \frac{L}{N}, & k = 0, +N, \pm 2N, \ldots \\ \frac{e^{-i\frac{\pi k(L-1)}{N}}}{N}\cdot\frac{\sin\left(\frac{\pi kL}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)}, & \text{otherwise} \end{cases}$$



Discrete signals ⇒ periodic frequency spectra.
Compare to continuous rectangular function (slide # 20)

Fourier Transforms

Let s(.) a signal in the time domain: s(t) values as a function of time t ($-\infty < t < \infty$)

The same signal can be described as amplitudes and phases (complex values)
S(.) in the frequency domain: S(f) values as a function of frequency f ($-\infty < f < \infty$)

One can transform the representation s(t) in the time domain to the representation S(f) in the frequency domain by using the Fourier Transform equation:

$$S(f) = \int_{-\infty}^{\infty} s(t).e^{2\pi i f t}\,dt$$

And back, using the inverse FT-equation:

$$s(t) = \int_{-\infty}^{\infty} S(f).e^{-2\pi i f t}\,df$$

Some Discrete Fourier Series Properties

	Time (n)	Frequency (k)
Homogeneity	$a \cdot s[n]$	$a \cdot S(k)$
Additivity	$s[n] + u[n]$	$S(k)+U(k)$
Linearity	$a \cdot s[n] + b \cdot u[n]$	$a \cdot S(k)+b \cdot U(k)$
Multiplication	$s[n] \cdot u[n]$	$\frac{1}{N} \cdot \sum_{h=0}^{N-1} S(h)U(k-h)$
Convolution	$\sum_{m=0}^{N-1} s[m] \cdot u[n-m]$	$S(k) \cdot U(k)$
Time shifting	$s[n - m]$	$e^{-i\frac{2\pi k \cdot m}{T}} \cdot S(k)$
Frequency shifting	$e^{+i\frac{2\pi h n}{T}} \cdot s[n]$	$S(k - h)$

References

1. Serge Lang, *Linear Algebra*, Springer Verlag New York Inc, 3rd Edition 1987.

Schedule (tentative, visit regularly):

8-2	Organization and Introduction
15-2	Audio Production and Processing
22-2	ADC and an Algebraic Introduction to FT
1-3	No Class (See Brightspace)
4-3	Question hour: Project Proposals (See Brightspace: 13.00 Kaltura)
8-3	Project Proposals (presentations by students)
15-3	FFT & FFT Workshop
22-3	Audio Features & workshop and data
29-3	Machine Learning + Workshop
5-4	Student Paper Presentations I.
12-4	Student Paper Presentations II.
19-4	Student Paper Presentations III.
26-4	Project Progress Reports
3-5	Team Meetings
10-5	Final Project Presentations Demo I
17-5	Final Project Presentations Demo II
23-5	Final Technical Project Paper (4-8 pages), code, and Web Site

Assignments (workshops):

1. Vocal Tract Workshop. Due: 22-2 2022.
2. FFT Workshop and audio_data. Due 22-3 2022 before 23.59.
3. Audio Features Workshop and data. Due 29-3 2022, 23.59.
4. Machine Learning Workshop. Due 8-4 2022, 23.59