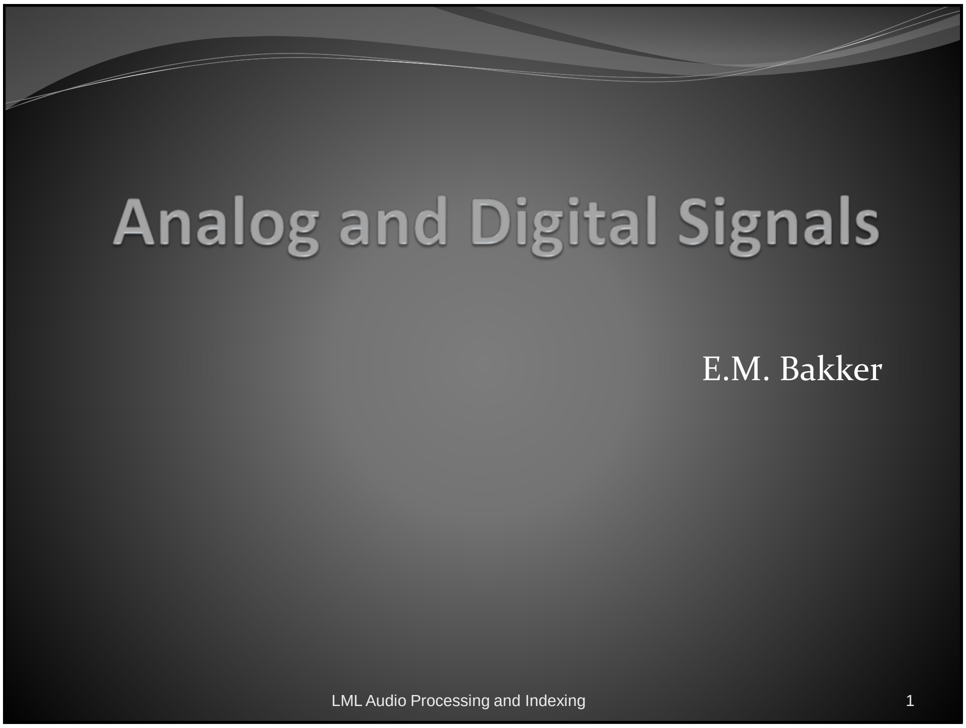
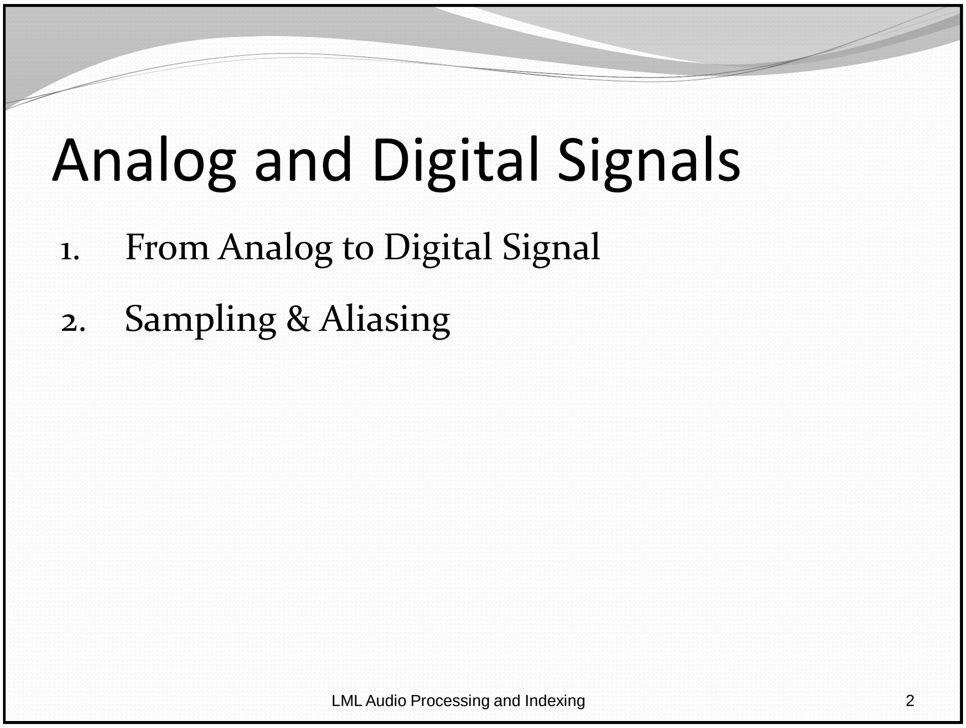


# Analog and Digital Signals

E.M. Bakker



## Analog and Digital Signals

1. From Analog to Digital Signal
2. Sampling & Aliasing

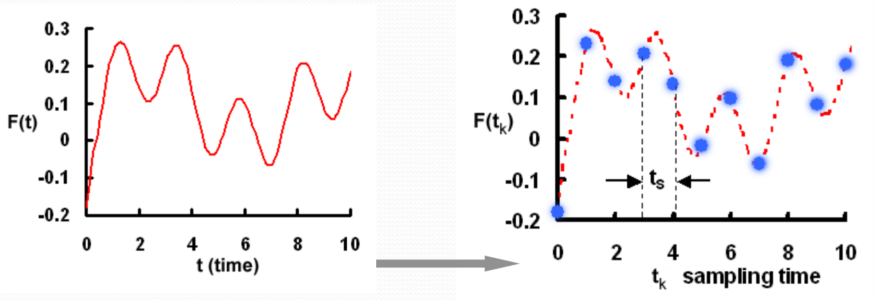
# Analog and Digital Signals

**Analog Signals**

*Continuous function* **F** of a *continuous* variable **t**
(**t** can be time, space etc) : **F(t)**

**Digital Signals**

*Discrete function* $F_k$ of a *discrete* (sampling) variable $t_k$
with **k** an integer: $F_k = F(t_k)$ (F at $t_k$)

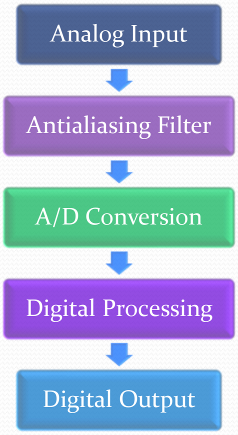


Function F is sampled with sampling frequency $f_S$ *(uniformly and periodic)*

$f_S = 1/ t_S$ Hz, for example,, if $t_s = 0.001$ sec => $f_s = 1000$Hz

---

# Digital System Implementation

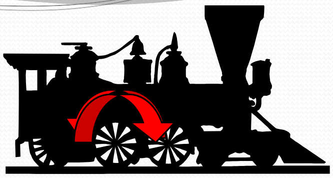


Important issues:

Analysis bandwidth, Dynamic range

- Pass/stop bands
- Sampling rate, Number of bits, and further parameters
- Digital format

# Sampling



How fast must we sample a continuous signal to preserve its information content?

Examples:

Turning wheels of a car or a train in a movie

- 25 frames per second, i.e., $f_s$ = 25 samples/sec =25 Hz
- Train starts => wheels appear to go clockwise
- Train accelerates => wheels go counter clockwise

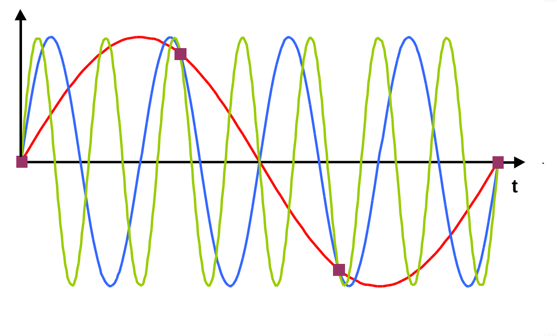
Rotating propeller of an airplane captured by a Mobile phone camera.

**Both examples:** Low sampling frequency leading to Frequency misidentification

Note that, we assume uniform sampling unless stated otherwise.

---

# Sampling a Sine Wave



— $s(t) = \sin(2\pi f_0 t)$

■ $s(t)$ @ $f_{Sample}$

For example:
$f_0 = 1$ Hz, $f_{Sample} = 3$ Hz

— $s_1(t) = \sin(2\pi 4t)$

— $s_2(t) = \sin(2\pi 7t)$

$s(t)$ @ $f_{Sample}$ represents exactly all sine-waves $s_k(t)$ defined by:

$$s_k(t) = \sin(2\pi (f_0 + k f_{Sample}) t) \quad , \quad |k| \in N$$, i.e., sin with frequency $f_0 + k f_{Sample}$

# The sampling theorem

**Theorem**

A signal s(t) with maximum frequency $f_{MAX}$ can be recovered if sampled at frequency $f_S > 2 f_{MAX}$ .

* Proposed by: Whittaker(s), Nyquist, Shannon, Kotel'nikov.

Nyquist frequency (rate) $f_N = 2 f_{MAX}$

**Example**

$$s(t)= \underbrace{3 \cdot \cos(25 \cdot 2\pi t)}_{F_1} + \underbrace{10 \cdot \sin(150 \cdot 2\pi t)}_{F_2} \underbrace{- \cos(50 \cdot 2\pi t)}_{F_3}$$

Condition on $f_S$?

$F_1$=25 Hz, $\underbrace{F_2 = 150 \text{ Hz}}_{f_{MAX}}$, $F_3$ = 50 Hz

$\boxed{f_S > 300 \text{ Hz}}$

---

# Frequency Domain

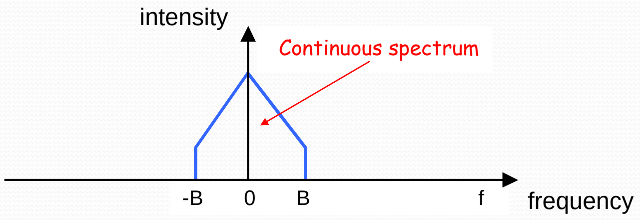
- Time and Frequency are two complementary signal descriptions.

  The signal can be seen as projected onto
  the time domain or the frequency domain.

- Bandwidth indicates the width of a range in the frequency domain.
  - high bandwidth: a range located high up in the frequency domain
  - passband bandwidth: defined by a lower and upper cutoff frequency

Previous lecture:
the inner-ear and early neural circuitry acts as a frequency analyser.
The audio spectrum is split into narrow bands thereby enabling detection of low-power sounds out of louder background sounds.

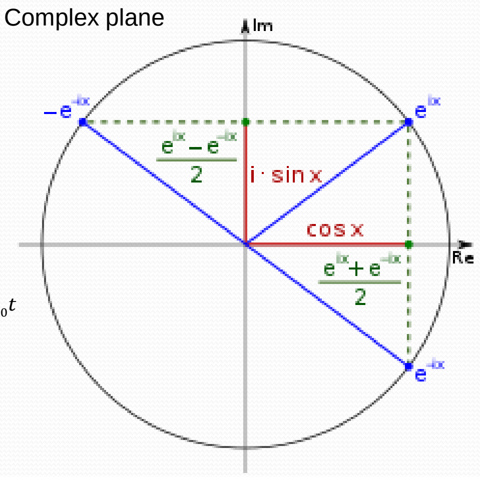
# Spectrum of band-limited signal



Spectrum of a band-limited signal:
frequency components f ∈ [-B,B]

---

# Negative frequencies

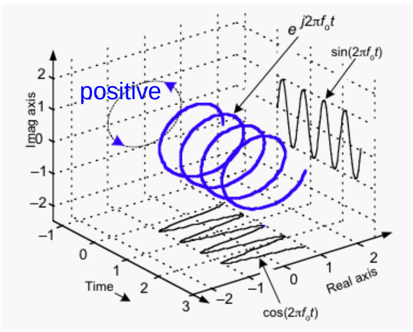


$$\cos(i2\pi f_0 t) + i.sin(i2\pi f_0 t) = e^{i2\pi f_0 t}$$

$$\sin(2\pi f_0 t) = \frac{e^{i2\pi f_0 t} - e^{-i2\pi f_0 t}}{2}$$

$$\cos(2\pi f_0 t) = \frac{e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}}{2}$$

# Negative frequencies

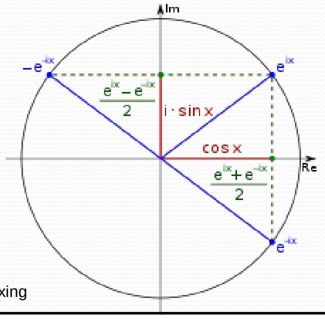


(Source: Richard Lyons)

$$\cos(i2\pi f_0 t) + i.sin(i2\pi f_0 t) = e^{i2\pi f_0 t}$$

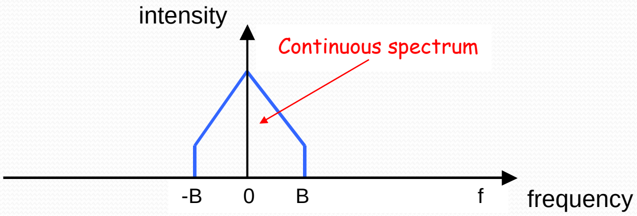
$$\sin(2\pi f_0 t) = \frac{e^{i2\pi f_0 t} \text{-} \ e^{-i2\pi f_0 t}}{2}$$

$$\cos(2\pi f_0 t) = \frac{e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}}{2}$$

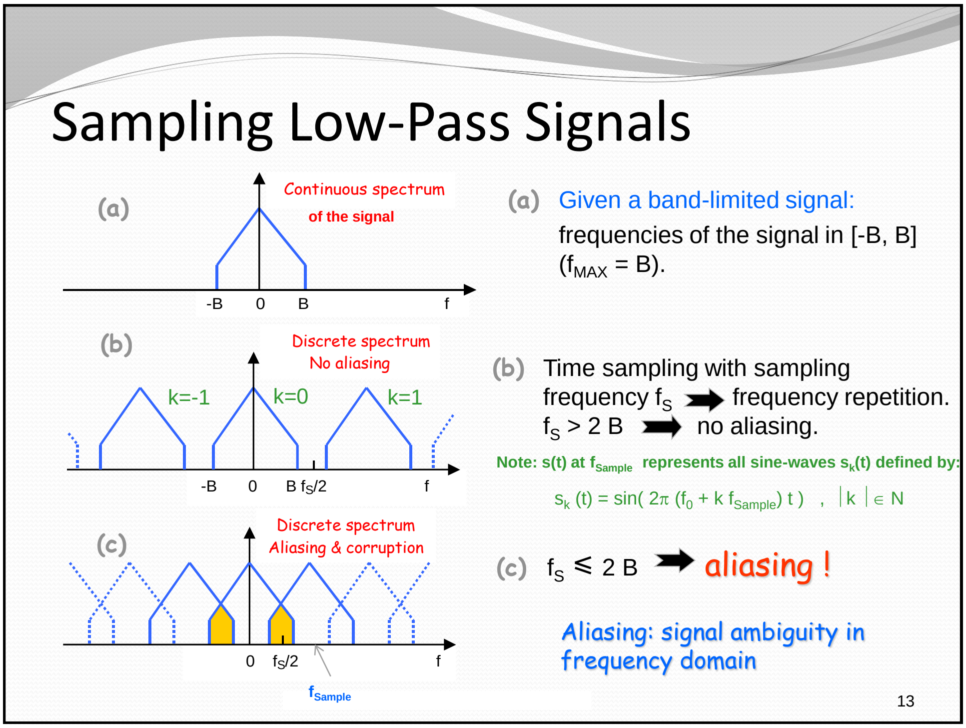


---

# Spectrum of band-limited signal



Spectrum of a band-limited signal:
frequency components f ∈ [-B,B]



# Sampling Low-Pass Signals

(a) Given a band-limited signal: frequencies of the signal in [-B, B] ($f_{MAX}$ = B).

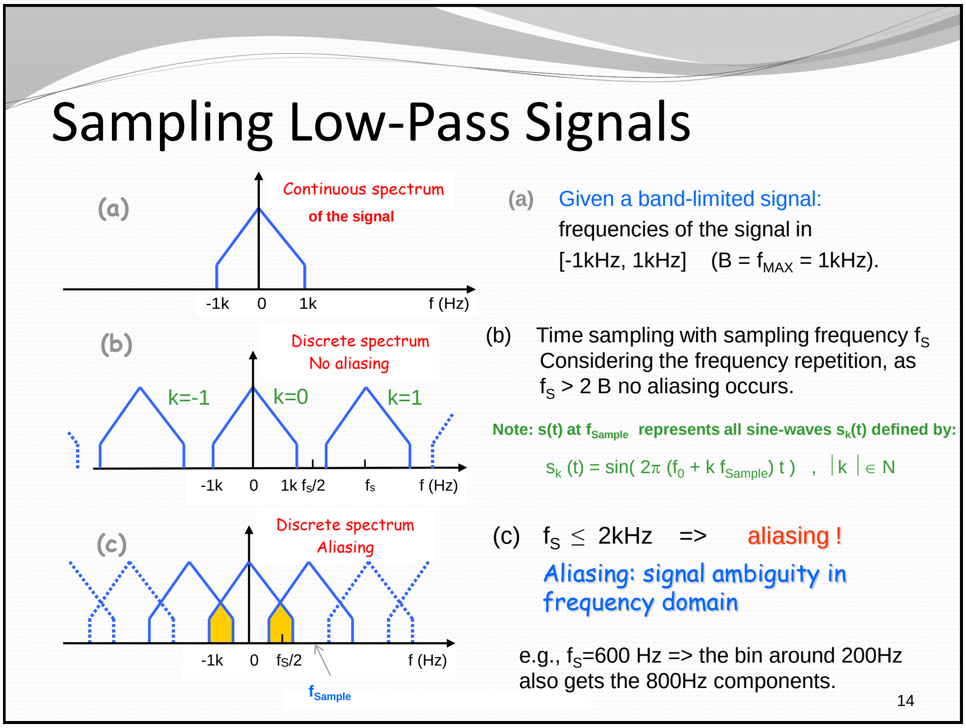
(b) Time sampling with sampling frequency $f_S$ ➡ frequency repetition. $f_S > 2$ B ➡ no aliasing.

Note: s(t) at $f_{Sample}$ represents all sine-waves $s_k(t)$ defined by:

$$s_k(t) = \sin(2\pi(f_0 + k f_{Sample})t) \quad , \quad |k| \in N$$

(c) $f_S \leqslant 2$ B ➡ aliasing !

*Aliasing: signal ambiguity in frequency domain*



# Sampling Low-Pass Signals

(a) Given a band-limited signal: frequencies of the signal in [-1kHz, 1kHz] (B = $f_{MAX}$ = 1kHz).

(b) Time sampling with sampling frequency $f_S$ Considering the frequency repetition, as $f_S > 2$ B no aliasing occurs.

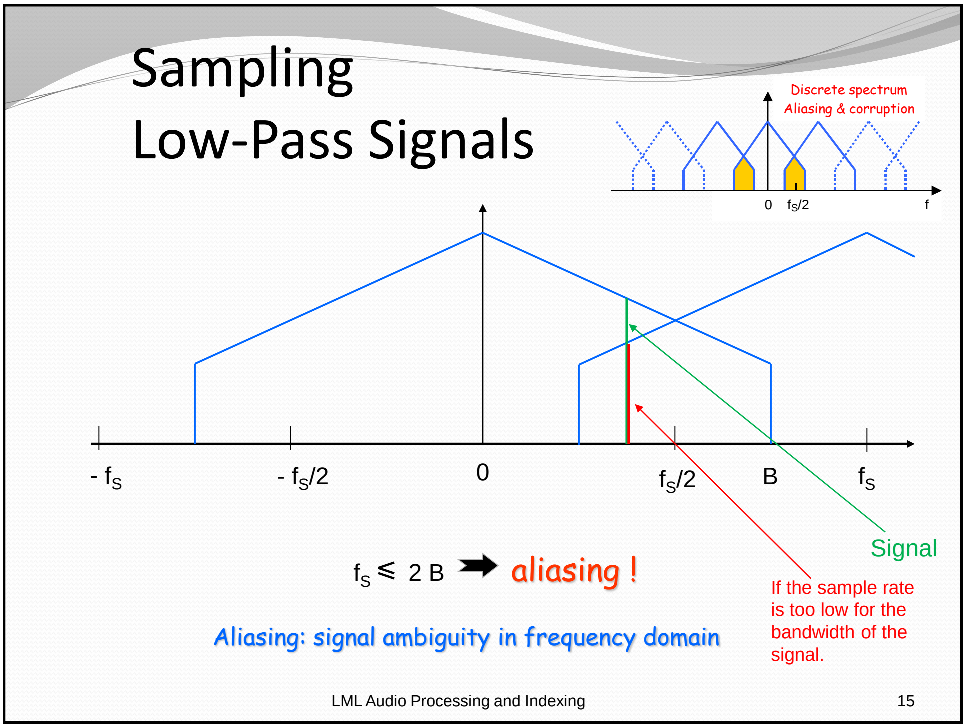
Note: s(t) at $f_{Sample}$ represents all sine-waves $s_k(t)$ defined by:

$$s_k(t) = \sin(2\pi(f_0 + k f_{Sample})t) \quad , \quad |k| \in N$$

(c) $f_S \leq$ 2kHz => aliasing !

*Aliasing: signal ambiguity in frequency domain*

e.g., $f_S$=600 Hz => the bin around 200Hz also gets the 800Hz components.

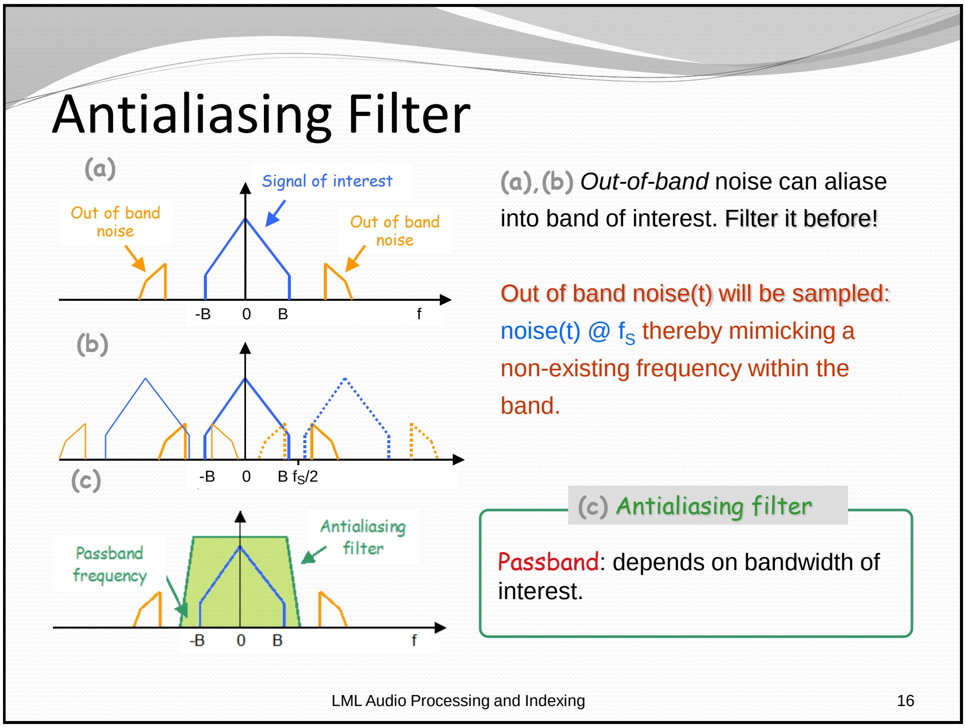


# Sampling Low-Pass Signals

$f_S \leqslant 2\,B$ ➡ aliasing !

Aliasing: signal ambiguity in frequency domain

If the sample rate is too low for the bandwidth of the signal.



# Antialiasing Filter

(a), (b) *Out-of-band* noise can aliase into band of interest. Filter it before!

Out of band noise(t) will be sampled: noise(t) @ $f_S$ thereby mimicking a non-existing frequency within the band.

(c) Antialiasing filter

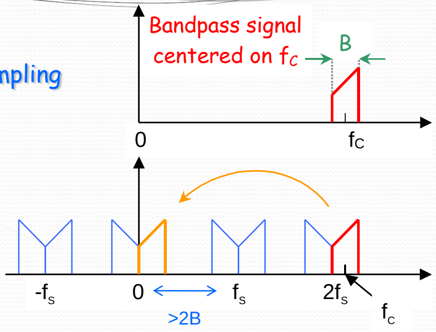
Passband: depends on bandwidth of interest.

# Under-sampling

Using spectral replications to reduce sampling frequency $f_S$ requirements.

$$\boxed{\frac{2\cdot f_C+B}{m+1} \le f_S \le \frac{2\cdot f_C-B}{m}}$$

$m \in \mathbb{N}$, selected so that $f_S > 2B$



**Note: s(t) at $f_{Sample}$ represents all sine-waves $s_k(t)$ defined by:** $s_k(t) = \sin(2\pi(f_0 + k\, f_{Sample})\, t)$ , $|k| \in N$

**Example**

$f_C$ = 20 MHz, B = 5MHz

**Without under-sampling** $f_S$ > 40 MHz.
**With under-sampling:**
$f_S$ = 22.5 MHz (m=1)
$f_S$ = 17.5 MHz (m=2)
$f_S$ = 11.66 MHz (m=3) (last > 10MHz!)

**Advantages**
- Slower ADCs / electronics needed.
- Simpler antialiasing filters.

---

# Over-sampling

Oversampling : sampling at frequencies $f_S >> 2\ f_{MAX}$ .

Over-sampling & averaging may improve ADC resolution

$$\boxed{f_{OS} = 4^w \cdot f_S}$$

$f_{OS}$ = over-sampling frequency,
w = additional bits required.

Each additional bit implies/requires over-sampling by a factor of 4.

# (Some) ADC parameters

1. Number of bits N (~*resolution*)
2. Data throughput (~*speed*)
3. Signal-to-noise ratio (SNR)
4. Signal-to-noise-&-distortion rate

$$\mathrm{SINAD} = \frac{P_{\mathrm{signal}} + P_{\mathrm{noise}} + P_{\mathrm{distortion}}}{P_{\mathrm{noise}} + P_{\mathrm{distortion}}}$$

5. Effective Number of Bits (ENOB)
6. …

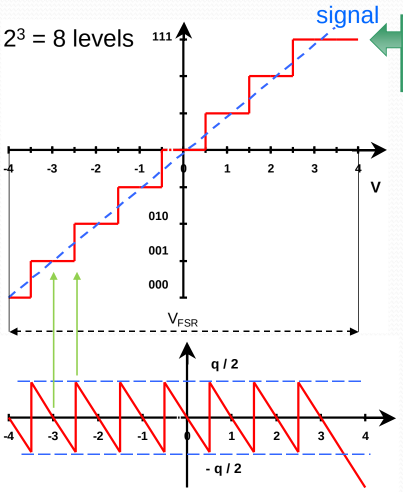
Static distortion

NB: Definitions may be slightly manufacturer-dependent!

---

# ADC - Number of bits N

Continuous input signal digitized into $2^N$ levels.



Uniform, bipolar transfer function
(number of bits N=3 => 8 levels)

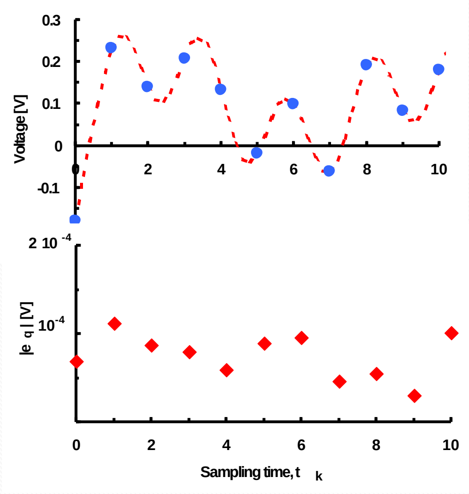
Quantisation step q = $\frac{V_{max}}{2^N}$

Ex: $V_{max}$ = 1V , N = 12 ➡ q = 244.1 µV

Voltage ( = q)
Scale factor (= 1 / $2^N$ )
Percentage (= 100 / $2^N$ )

Quantisation error

# ADC - Quantisation error



Quantisation step q = $\frac{V_{max}}{2^N}$

- Quantisation Error $e_q$ in [-0.5 q, +0.5 q].
- $e_q$ limits ability to resolve small signal.
- Higher resolution (more bits) means lower $e_q$.

QE for
N = 12
$V_{FS}$ = 1

---

# SNR of ideal ADC

$$\overline{SNR}_{ideal} = 20 \cdot \log_{10}\left(\frac{RMS(input)}{RMS(e_q)}\right) \quad (1)$$

Also called SQNR

(signal-to-quantisation-noise ratio)

(RMS = root mean square)

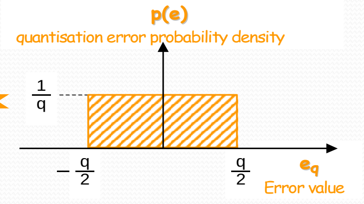
**Assumptions**

Ideal ADC:
- only quantisation error **$e_q$**
( p(e) = quantisation error probability density is assumed to be constant, uniform, etc. )
- **$e_q$** uncorrelated with signal.
- ADC performance constant in time.

$$RMS(input) = \sqrt{\frac{1}{T} \cdot \int_0^T \left(\frac{V_{FSR}}{2} \cdot \sin(\omega t)\right)^2 dt} = \frac{V_{FSR}}{2\sqrt{2}}$$

Input(t) = ½ $V_{FSR}$ sin(ω t).

$$RMS(e_q) = \sqrt{\int_{-q/2}^{q/2} e_q{}^2 \cdot p(e_q) de_q} = \frac{q}{\sqrt{12}} = \frac{V_{FSR}}{2^N \cdot \sqrt{12}}$$



(sampling frequency $f_S = 2 f_{MAX}$)

# SNR of ideal ADC

Substituting in (1) =>

$$\overline{\mathrm{SNR}}_{ideal} = 6.02 \cdot N + 1.76\,[\mathrm{dB}] \quad (2)$$

One additional bit ➡ SNR increased by 6 dB

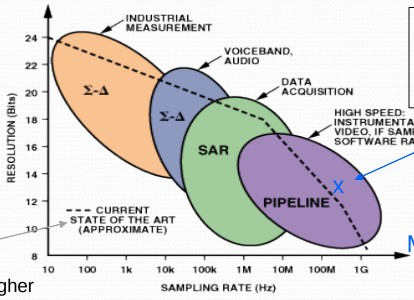
**Real SNR lower because:**

- Real signals have noise.
- Forcing input to full scale unwise.
- Real ADCs have additional noise (aperture jitter, non-linearities etc).

*Actually* (2) needs correction factor depending on ratio between sampling freq & Nyquist freq. Processing gain due to oversampling.

---

# ADC Performance

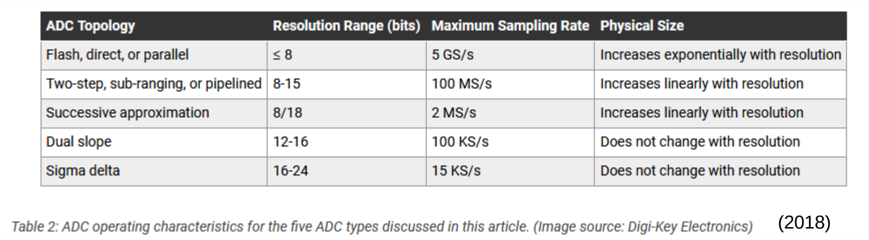


SDR Ettus X310
160MHz Bandwidth
0 - 6GHz, 200MS/s,
14 bit

Mid 2005

Currently:
~3 bits higher

From: http://www.analog.com/library/analogDialogue/archives/39-06/architecture.html



| ADC Topology | Resolution Range (bits) | Maximum Sampling Rate | Physical Size |
|---|---|---|---|
| Flash, direct, or parallel | ≤ 8 | 5 GS/s | Increases exponentially with resolution |
| Two-step, sub-ranging, or pipelined | 8-15 | 100 MS/s | Increases linearly with resolution |
| Successive approximation | 8/18 | 2 MS/s | Increases linearly with resolution |
| Dual slope | 12-16 | 100 KS/s | Does not change with resolution |
| Sigma delta | 16-24 | 15 KS/s | Does not change with resolution |

*Table 2: ADC operating characteristics for the five ADC types discussed in this article. (Image source: Digi-Key Electronics)* (2018)

# Complex Numbers

*The* complex numbers *are given by*:

$$\mathbb{C} = \{c \mid c = a + bi, where, a, b \in \mathbb{R}\}$$

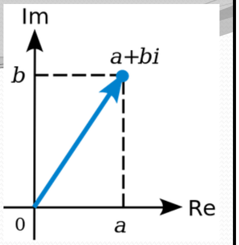
- here $i$ is the imaginary unit that satisfies: $i^2 = -1$
- a is called the real part of c
- b is called the imaginary part of c

If $z=x+yi$, then the complex conjugate $z^*$ is defined as $z^*=x-yi$

---

# Complex Numbers

(see also Wikipedia, and/or your Calculus Book)



*The* complex numbers *are given by*:

$$\mathbb{C} = \{c \mid c = a + bi, where, a, b \in \mathbb{R}\}$$

- here $i$ is the imaginary unit that satisfies: $i^2 = -1$

Addition:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + b - d)i$$

Multiplication:

$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i$$
$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \left(\frac{ab + bd}{c^2 + d^2}\right) + \left(\frac{bc - ad}{c^2 + d^2}\right)i$$

# Complex Numbers

*The complex numbers are given by*:

$$\mathbb{C} = \{c \mid c = a + bi, where, a, b \in \mathbb{R}\}$$

The absolute value (modulus; magnitude) of $z = x + yi$ is:

$$r = |z| = \sqrt{x^2 + y^2}$$

Note that:

$$|z|^2 = zz^* = x^2 + y^2$$

The argument (phase) of $z = x + yi$ is:

$$\varphi = \arg(z) = \{\arctan(y/x), \text{if} \ldots =$$

"the angle of the vector (x,y) with the positive real axis“

Note: $z = r(cos\varphi + isin\varphi) = re^{i\varphi}$

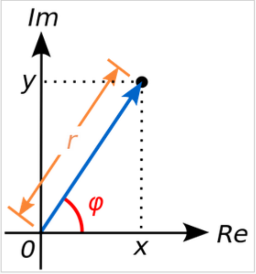


Figure 2: The argument $\varphi$ and modulus $r$ locate a point on an Argand diagram; $r(\cos\varphi + i\sin\varphi)$ or $re^{i\varphi}$ are *polar* expressions of the point.

---

# Complex Numbers

*Let:*

$$z_1 = r_1(cos\varphi_1 + isin\varphi_1) = r_1e^{i\varphi_1}$$
$$z_2 = r_2(cos\varphi_2 + isin\varphi_2) = r_2e^{i\varphi_2}$$

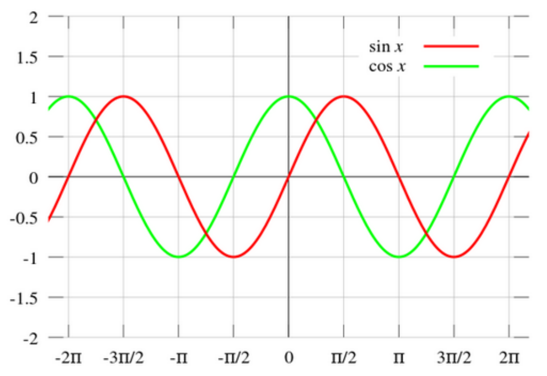
Note:

$$\cos(a)\cos(b) - \sin(a)\sin(b) = \cos(a + b)$$
$$\cos(a)\sin(b) + \sin(a)\cos(b) = \sin(a + b)$$

Hence:

$$z_1z_2 = r_1r_2(cos(\varphi_1{+}\varphi_2) + isin(\varphi_1{+}\varphi_2)) = r_1r_2e^{i(\varphi_1+\varphi_2)}$$

# Sine Cosine Graphs



$$\sin(\varphi + \pi/2) = \cos(\varphi)$$

---

From: S. Lang, Linear Algebra, 2nd Ed.
Addison-Wesley Publ. Comp., Reading, 1970.

## Fields

$K \subseteq \mathbb{C}$ is a field if it satisfies:

a) If $x, y \in K$, then $x+y \in K$ and $xy \in K$

closed under addition and multiplication

b) If $x \in K$, then $-x \in K$ and if $x \neq 0$ also, then $x^{-1} \in K$

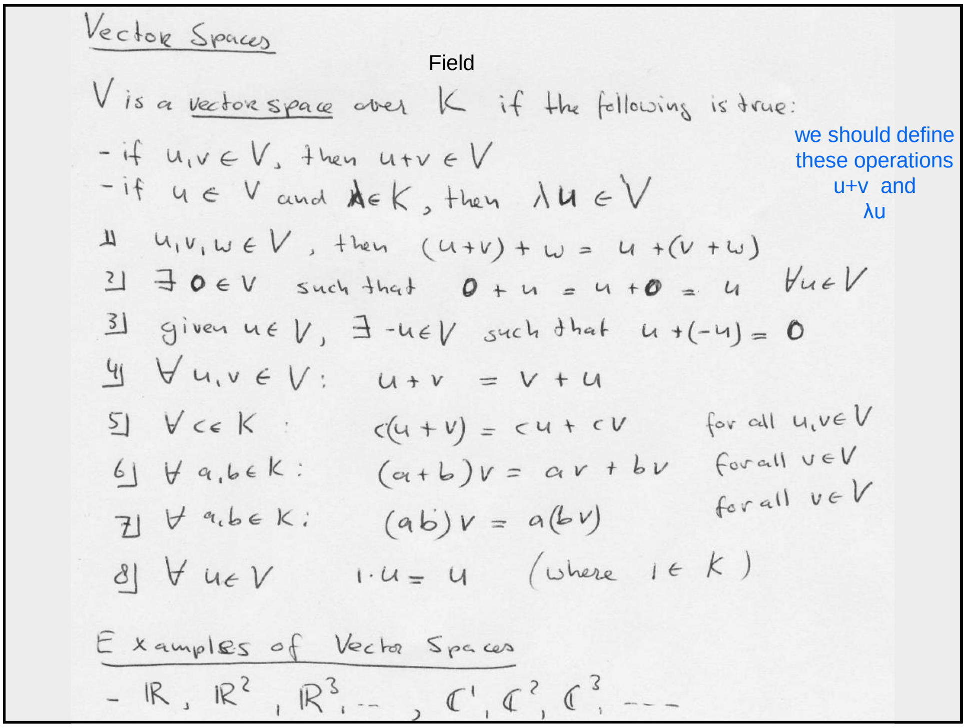
has inverses

c) $0 \in K$ and $1 \in K$ (additive and multiplicative nullelements, resp.)

has null-elements

Examples of Fields:

- $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$
- Note, $\mathbb{Z}$ is not a field



## Vector Spaces

Field

$V$ is a vector space over $K$ if the following is true:

we should define these operations $u+v$ and $\lambda u$

- if $u, v \in V$, then $u+v \in V$
- if $u \in V$ and $\lambda \in K$, then $\lambda u \in V$

1) $u, v, w \in V$, then $(u+v)+w = u+(v+w)$

2) $\exists\, 0 \in V$ such that $0 + u = u + 0 = u \quad \forall u \in V$

3) given $u \in V$, $\exists\, -u \in V$ such that $u+(-u) = 0$

4) $\forall u, v \in V: \quad u+v = v+u$

5) $\forall c \in K: \quad c(u+v) = cu + cv$ for all $u, v \in V$

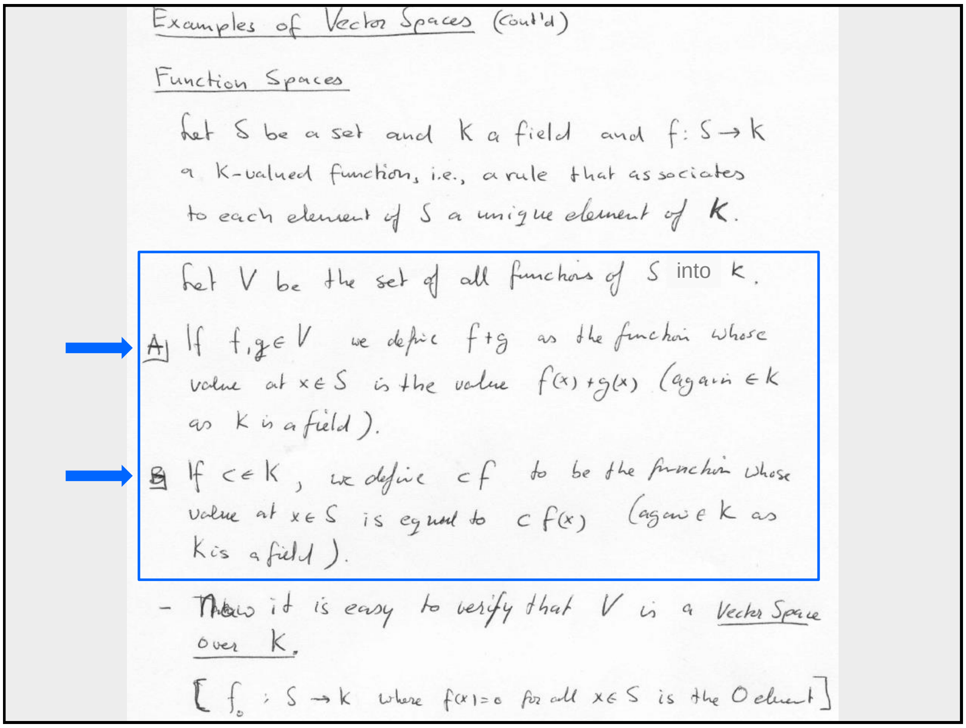
6) $\forall a, b \in K: \quad (a+b)v = av + bv$ for all $v \in V$

7) $\forall a, b \in K: \quad (ab)v = a(bv)$ for all $v \in V$

8) $\forall u \in V \quad 1 \cdot u = u$ (where $1 \in K$)

### Examples of Vector Spaces

- $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, ---, $\mathbb{C}^1$, $\mathbb{C}^2$, $\mathbb{C}^3$, ---



## Examples of Vector Spaces (Cont'd)

### Function Spaces

Let $S$ be a set and $K$ a field and $f: S \to K$ a $K$-valued function, i.e., a rule that associates to each element of $S$ a unique element of $K$.

Let $V$ be the set of all functions of $S$ into $K$.

A) If $f, g \in V$ we define $f+g$ as the function whose value at $x \in S$ is the value $f(x)+g(x)$ (again $\in K$ as $K$ is a field).

B) If $c \in K$, we define $cf$ to be the function whose value at $x \in S$ is equal to $c\,f(x)$ (again $\in K$ as $K$ is a field).

- Then it is easy to verify that $V$ is a Vector Space over $K$.

[ $f_0 : S \to K$ where $f(x)=0$ for all $x \in S$ is the $0$ element ]

## Other Examples of Function Spaces which are Vector Spaces:

- V the set of all functions of $\mathbb{R}$ into $\mathbb{R}$
- V the set of all continuous functions of $\mathbb{R}$ into $\mathbb{R}$
- V the set of all differentiable functions of $\mathbb{R}$ into $\mathbb{R}$
- V the subspace generated by the functions $f(t)=e^t$ and $g(t)=e^{2t}$ (for all $t\in\mathbb{R}$.)

[Just check that A] and B] hold. As $\mathbb{R}$ is a field the claim that V is vector space follows.].

## Linearly Dependence

Let V be a vector space over the field K.

Let $v_1,\dots,v_n\in V$. $v_1,\dots,v_n$ are linearly dependent over K if $\exists\, a_1,\dots,a_n\in K$ not all equal $0$ such that $a_1v_1+\dots+a_nv_n=0$.

- If there do not exist such numbers, i.e., if $a_1,\dots,a_n\in K$ such that $a_1v_1+\dots+a_nv_n=0$, then $a_i=0$ $\forall i=1,\dots,n$. then $v_1,\dots,v_n$ are linearly independent.

Example:- Let $V=\mathbb{R}^n$, then $E_1=(1,0,\dots,0)$ $\vdots$ $E_n=(0,0,\dots,1)$ are linearly independent.

- also $e^t, e^{2t}$ are linearly independent.

## Basis

If elements $v_1, \dots, v_n \in V$ generate $V$, and $v_1, \dots, v_n$ are linearly independent. $\{v_1, \dots, v_n\}$ is called a basis of $V$.

- $v_1, \dots, v_n \in V$ generate $V$, that is every element of $V$ can be expressed as a linear combination of $v_1, \dots, v_n$.
- and indeed if $x_1 v_1 + \dots + x_n v_n = x = y_1 v_1 + \dots + y_n v_n$ with $x_1, \dots, x_n, y_1, \dots, y_n \in K$ and for a $x \in V$,

then $(x_1 - y_1) v_1 + \dots + (x_n - y_n) v_n = 0$

thus $x_1 = y_1, \dots, x_n = y_n$.

=> in a unique way

## Scalar Products

Let $V$ a vector space over a field $K$. (real)

A scalar product on $V$ is an association which to any pair $v, w \in V$ associates a scalar $<v, w>$ (also $v \cdot w$) satisfying:

1. $\forall v, w \in V \quad <v, w> = <w, v>$
2. Let $u, v, w \in V$, then $<u, v+w> = <u, v> + <u, w>$
3. Let $c \in K$, then $<cu, v> = c <u, v>$ and $<u, cv> = c <u, v>$

A scalar product is non-degenerate, if also:

4. If $v \in V$ and $<v, w> = 0$ for all $w \in V$, then $v = 0$.

### Examples of Scalar Products:

- $V = K^n$ $\quad <x, y>: \; x, y \to x \cdot y = x_1 y_1 + x_2 + y_2 + \dots + x_n y_n$ is a scalar product. [this is the 'standard' dot-product.].
- Let $V$ be the space of continuous real-valued functions on the interval $[0, 1]$. If $f, g \in V$,

$$<f, g> = \int_0^1 f(t) g(t) \, dt.$$

Then $<f, g>$ is a scalar product. Homework

## Orthogonality

$v, w \in V$ are orthogonal: $v \perp w$ if $\langle v, w \rangle = 0$.

## Norm

- The norm of $v \in W$ can be defined by $\|v\| = \sqrt{\langle v, v \rangle}$

  It is clear that: $\|c\,v\| = |c|\,\|v\|$

- $v \in V$ is a unit vector if $\|v\| = 1$

  ( $v/\|v\|$ is always a unit vector, if $v \neq 0$ ) .

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## Some Theorems (easy)

We have the following theorems:

Th. If $v, w \in V$ and $v \perp w$ (i.e. $\langle v, w \rangle = 0$), then (Pythagoras)

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

Proof: $\|v + w\|^2 = \langle v+w, v+w \rangle = \langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle$

$\underset{v \perp w}{=} \|v\|^2 + \|w^2\|$ (as $\langle v, w \rangle = 0$). ∎

Parallelogram law:

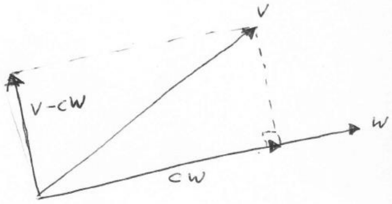
$\forall\, v, w \in V$ we have $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$.

Homework: Proof Parallelogram Law.

# Fourier Coefficients

Observation:

Let $w \in V$ such that $\|w\| \neq 0$.
For any $v \in V$ there exists a unique $c \in K$ such that $v - cw$ is perpendicular to $w$.



Now $v - cw$ perpendicular to $w$ means that $\langle v - cw, w \rangle = 0$.

$$\Rightarrow \langle v, w \rangle - c\langle w, w \rangle = 0 \quad \Rightarrow \quad \boxed{c = \frac{\langle v, w \rangle}{\langle w, w \rangle}}$$

Conversely, if $c = \frac{\langle v, w \rangle}{\langle w, w \rangle}$ then $c\langle w, w \rangle = \langle v, w \rangle \Rightarrow$

$\langle v, w \rangle - c\langle w, w \rangle = 0 \Rightarrow$

$\langle v - cw, w \rangle = 0$

hence $v - cw$ perpendicular to $w$.

We call $c$ the component of $v$ along $w$, or
the Fourier coefficient of $v$ with respect to $w$.

# Example Fourier Coefficients.

Let $V$ be the space of continuous functions on $[-\pi, \pi]$.

Let $f : x \to \sin kx$, where $k \in \mathbb{Z}_{>0}$.

Then $\|f\| = \sqrt{\langle f, f \rangle} = \left( \int_{-\pi}^{\pi} \sin^2 kx \, dx \right)^{1/2} = \sqrt{\pi}$

In this case, if $g$ is any continuous function on $[-\pi, \pi]$, then the Fourier coefficient of $g$ with respect to $f$ is

$$\frac{\langle g, f \rangle}{\langle f, f \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin kx \, dx$$

## The Complex ($\mathbb{C}$) Case

Let V be a vector space over the complex numbers.

A hermitian product on V is a rule $\langle v,w\rangle$ satisfying:

1] $\langle v,w\rangle = \overline{\langle w,v\rangle}$ for all $v,w \in V$

2] $u,v,w \in V$, then $\langle u, v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$

3] if $\alpha \in \mathbb{C}$, then $\langle \alpha u, v\rangle = \alpha \langle u,v\rangle$

$\langle u, \alpha v\rangle = \overline{\alpha}\langle u,v\rangle$.

$\langle \cdot,\cdot\rangle$ is positive definite if $\langle v,v\rangle \geq 0$ for all $v \in V$ and $\langle v,v\rangle > 0$ if $v \neq 0$.

Note: Orthogonal, perpendicular, orthogonal basis, orthogonal complement, as before!

Also the Fourier coefficient and the projection of v along w are as before.

## Example

Let V be the space of continuous complex-valued functions on the interval $[-\pi, \pi]$.

- If $f,g \in V$, we define $\langle \cdot,\cdot\rangle$ as follows:

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt$$

This can be shown, using standard properties of the integral, to be a positive definite hermitean product.

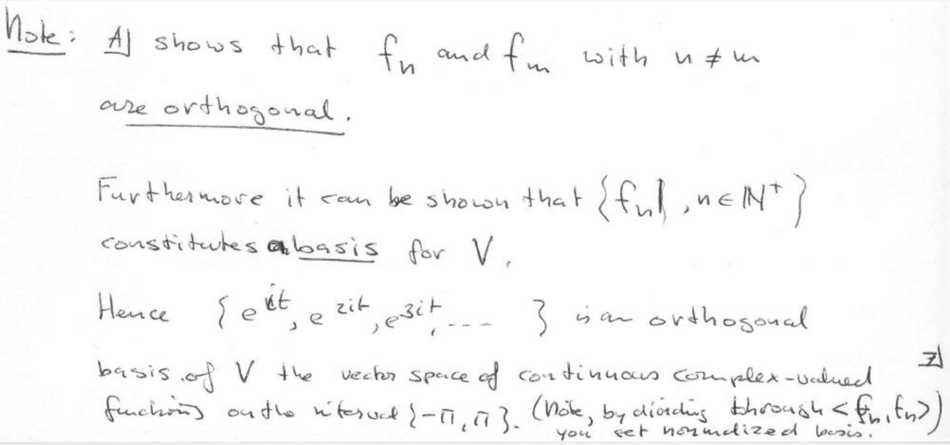
- Let $\boxed{f_n(t) = e^{int}}$

A] if $n \neq m$, then $\langle f_n, f_m\rangle = \int_{-\pi}^{\pi} e^{int}\overline{e^{imt}}\,dt = \int_{-\pi}^{\pi} e^{ikt}\,dt = 0$

if $n = m$, then we have $\langle f_n, f_n\rangle = \int_{-\pi}^{\pi} e^{int}e^{-int}\,dt = \int_{-\pi}^{\pi} 1\,dt = 2\pi$

- If $f \in V$, then its Fourier coefficient with respect to $f_n$ is equal to:

$$\frac{\langle f, f_n\rangle}{\langle f_n, f_n\rangle} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt.$$



Note: A] shows that $f_n$ and $f_m$ with $n \neq m$ are orthogonal.

Furthermore it can be shown that $\{f_n\}, n \in \mathbb{N}^+\}$ constitutes ~~a~~ basis for V.

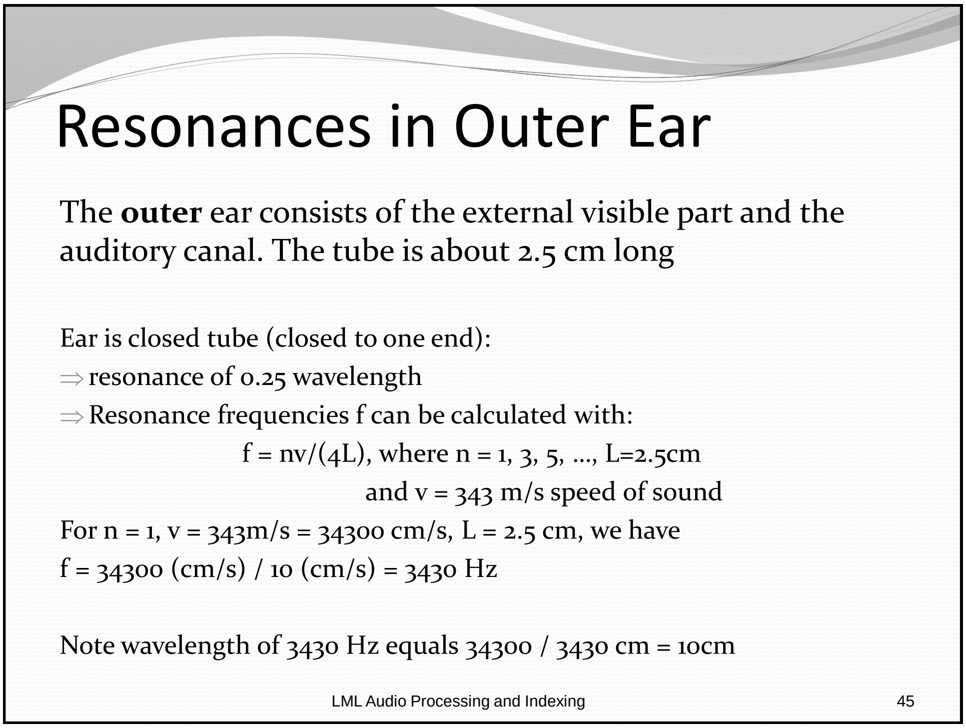
Hence $\{e^{it}, e^{2it}, e^{3it}, ---\}$ is an orthogonal basis of V the vector space of continuous complex-valued functions on the interval $\{-\pi, \pi\}$. (Note, by dividing through $<f_n, f_n>$ you get normalized basis.)

Z]

# References

This presentation uses a selection of slides that are adapted from original slides by

Dr M.E. Angoletta at DISP2003, a DSP course given by CERN and University of Lausanne (UNIL)



# Resonances in Outer Ear

The **outer** ear consists of the external visible part and the auditory canal. The tube is about 2.5 cm long

Ear is closed tube (closed to one end):
$\Rightarrow$ resonance of 0.25 wavelength
$\Rightarrow$ Resonance frequencies f can be calculated with:

$f = nv/(4L)$, where $n = 1, 3, 5, \ldots$, $L$=2.5cm
and $v$ = 343 m/s speed of sound

For $n = 1$, $v$ = 343m/s = 34300 cm/s, $L$ = 2.5 cm, we have
$f$ = 34300 (cm/s) / 10 (cm/s) = 3430 Hz

Note wavelength of 3430 Hz equals 34300 / 3430 cm = 10cm