Analog and Digital Signals

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Analog and Digital Signals

1. From Analog to Digital Signal
2. Sampling & Aliasing

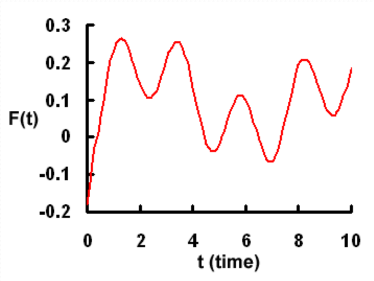
Analog and Digital Signals

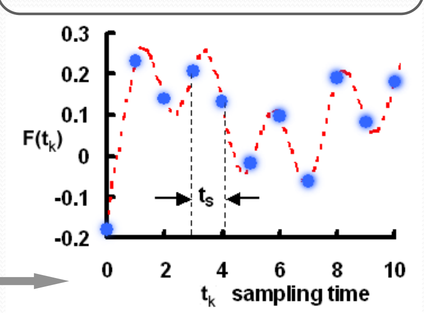
Analog Signals

Continuous function **F** of a *continuous* variable t
(t can be time, space etc) : **F(t)**

Digital Signals

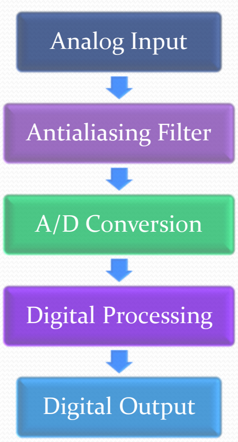
Discrete function F_k of a *discrete* (sampling) variable t_k
with k an integer: $F_k = F(t_k)$ (F at t_k)





Function F is sampled with sampling frequency f_S *(uniformly and periodic)*
$f_S = 1/ t_S$ Hz, for example,, if sampling time is $t_s = 0.001$ sec => $f_s = 1000$Hz

Digital System Implementation

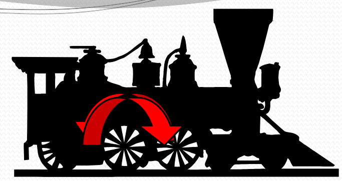


Important issues:

Analysis bandwidth, Dynamic range

- Pass/stop bands
- Sampling rate, Number of bits, and further parameters
- Digital format

Sampling



How fast must we sample a continuous signal to preserve its information content?

Examples:

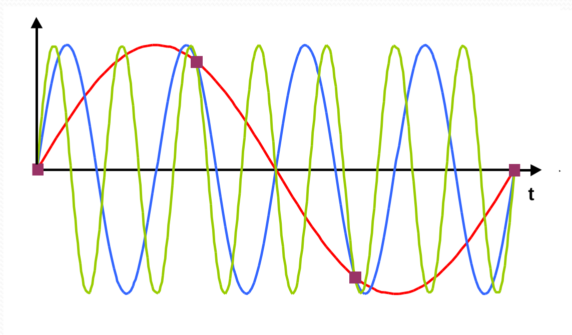
Turning wheels of a car or a train in a movie
- 25 frames per second, i.e., f_s = 25 samples/sec =25 Hz
- Train starts => wheels appear to go clockwise
- Train accelerates => wheels go counter clockwise

Rotating propeller of an airplane captured by a Mobile phone camera.

Both examples: Low sampling frequency leading to Frequency misidentification

Note that, we assume uniform sampling unless stated otherwise.

Sampling a Sine Wave



$s(t) = \sin(2\pi f_0 t)$

$s(t)$ @ f_{Sample}

For example:
$f_0 = 1$ Hz, $f_{Sample} = 3$ Hz

$s_1(t) = \sin(2\pi 4t)$

$s_2(t) = \sin(2\pi 7t)$

$s(t)$ @ f_{Sample} represents exactly all sine-waves $s_k(t)$ defined by:

$s_k(t) = \sin(2\pi (f_0 + k f_{Sample}) t)$, $|k| \in N$, i.e., sin with frequency $f_0 + k f_{Sample}$

The sampling theorem

Theorem

A signal s(t) with maximum frequency f_{MAX} can be recovered if sampled at frequency $f_S > 2\, f_{MAX}$.

* Proposed by: Whittaker(s), Nyquist, Shannon, Kotel'nikov.

Nyquist frequency (rate) $f_N = 2\, f_{MAX}$

Example

$$s(t) = 3 \cdot \underbrace{\cos(25 \cdot 2\pi t)}_{F_1} + 10 \cdot \underbrace{\sin(150 \cdot 2\pi t)}_{F_2} - \underbrace{\cos(50 \cdot 2\pi t)}_{F_3}$$

Condition on f_S?

F_1=25 Hz, F_2 = 150 Hz, F_3 = 50 Hz

f_{MAX}

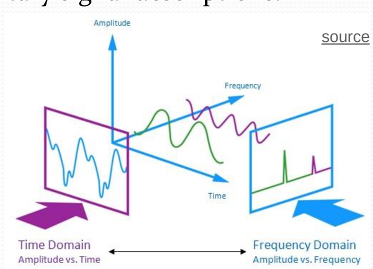
$f_S > 300$ Hz

Frequency Domain

- Time and Frequency are two complementary signal descriptions.

The signal can be seen as projected onto
the time domain
or
the frequency domain.

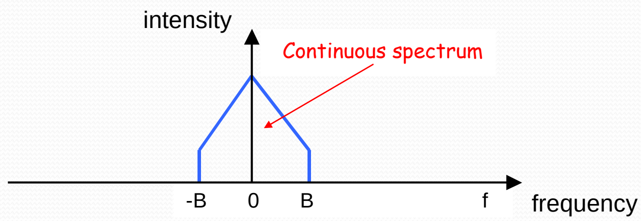
source



- Bandwidth indicates the width of a range in the frequency domain.
  - high bandwidth: a range located high up in the frequency domain
  - passband bandwidth: defined by a lower and upper cutoff frequency

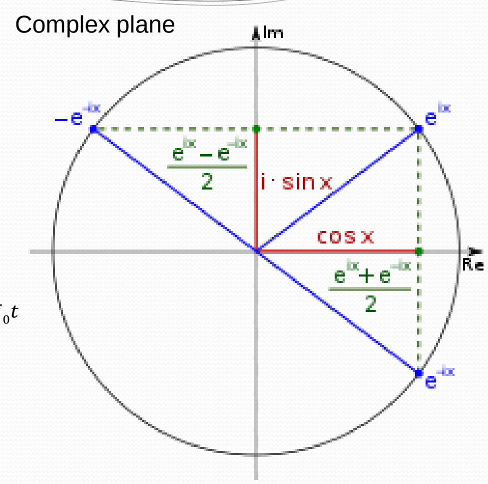
Previous lecture:
the inner-ear and early neural circuitry acts as a frequency analyser.
The audio spectrum is split into narrow bands thereby enabling detection of low-power sounds out of louder background sounds.

Spectrum of band-limited signal



Spectrum of a band-limited signal:
frequency components f ∈ [-B,B]

Negative frequencies

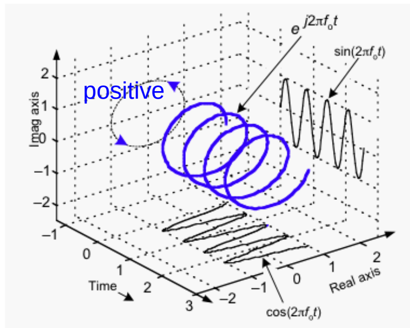


$$\cos(2\pi f_0 t) + i.sin(2\pi f_0 t) = e^{i2\pi f_0 t}$$

$$\sin(2\pi f_0 t) = \frac{e^{i2\pi f_0 t} - e^{-i2\pi f_0 t}}{2}$$

$$\cos(2\pi f_0 t) = \frac{e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}}{2}$$

Negative frequencies



(Source: Richard Lyons)

$$\cos(2\pi f_0 t) + i.sin(2\pi f_0 t) = e^{i2\pi f_0 t}$$

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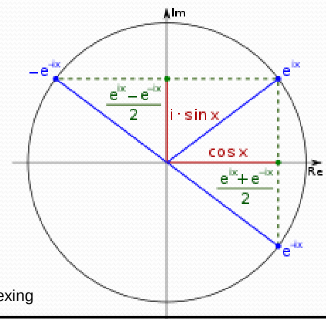
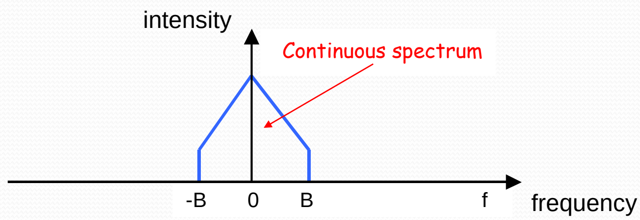
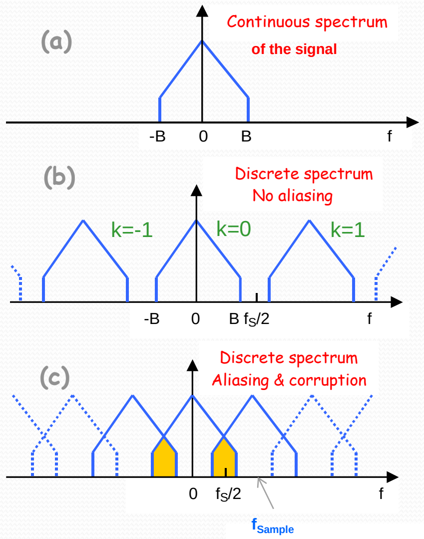


Spectrum of band-limited signal



Spectrum of a band-limited signal:
frequency components f ∈ [-B,B]

Sampling Low-Pass Signals



(a) Given a band-limited signal:
frequencies of the signal in [-B, B] (f_{MAX} = B).

(b) Time sampling with sampling frequency f_S ➡ frequency repetition.
$f_S > 2B$ ➡ no aliasing.

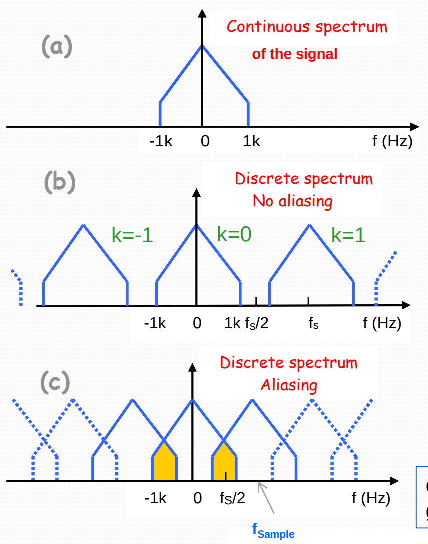
Note: s(t) at f_{Sample} represents all sine-waves $s_k(t)$ defined by:

$s_k(t) = \sin(2\pi (f_0 + k f_{Sample}) t)$, $|k| \in N$

(c) $f_S \leqslant 2B$ ➡ *aliasing !*

Aliasing: signal ambiguity in frequency domain

Sampling Low-Pass Signals



(a) Given a band-limited signal:
frequencies of the signal in [-1kHz, 1kHz] (B = f_{MAX} = 1kHz).

(b) Time sampling with sampling frequency f_S
Considering the frequency repetition, as $f_S > 2B$ no aliasing occurs.

Note: s(t) at f_{Sample} represents all sine-waves $s_k(t)$ defined by:

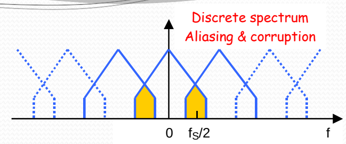
$s_k(t) = \sin(2\pi (f_0 + k f_{Sample}) t)$, $|k| \in N$

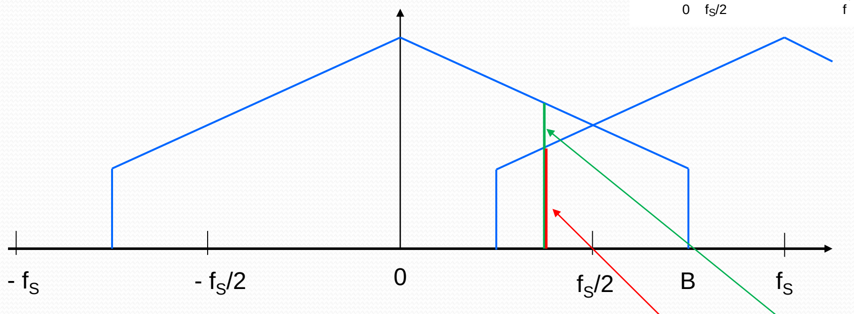
(c) $f_S \leq$ 2kHz => aliasing !

Aliasing: signal ambiguity in frequency domain

e.g., f_S=600 Hz => the bin around 200Hz also gets the contributions of the 800Hz components.

Sampling
Low-Pass Signals





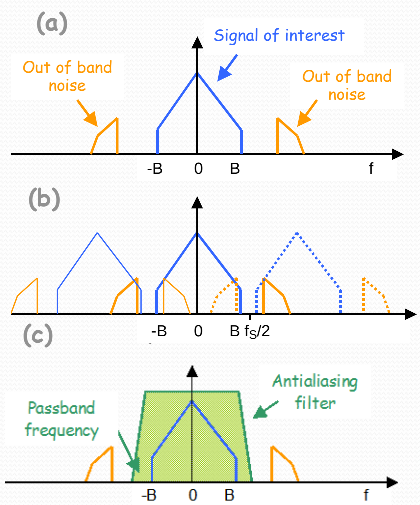
Signal

If the sample rate is too low for the bandwidth of the signal.

$f_S \leqslant 2B$ ➡ *aliasing !*

Aliasing: signal ambiguity in frequency domain

Antialiasing Filter



(a), (b) *Out-of-band* noise can aliase into band of interest. Filter it before!

Out of band noise(t) will be sampled: noise(t) @ f_S thereby mimicking a non-existing contributions of frequencies within the band.

(c) Antialiasing filter

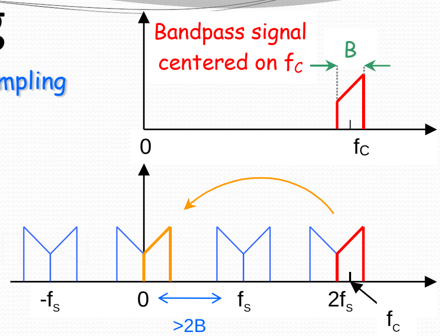
Passband: depends on bandwidth of interest.

Under-sampling

Using spectral replications to reduce sampling frequency f_S requirements.

$$\boxed{\frac{2 \cdot f_C + B}{m+1} \le f_S \le \frac{2 \cdot f_C - B}{m}}$$

$m \in \mathbb{N}$, selected so that $f_s > 2B$



Note: s(t) at f_{Sample} represents all sine-waves $s_k(t)$ defined by: $s_k(t) = \sin(2\pi (f_0 + k f_{Sample}) t)$, $|k| \in N$

Example

f_C = 20 MHz, B = 5MHz

Without under-sampling f_S > 40 MHz.
With under-sampling:
f_S = 22.5 MHz (m=1)
f_S = 17.5 MHz (m=2)
f_S = 11.66 MHz (m=3) last m such that $f_S > 2B$ = 10MHz

Advantages
- Slower ADCs / electronics needed.
- Simpler antialiasing filters.

Over-sampling

Oversampling : sampling at frequencies $f_S >> 2\, f_{MAX}$.

Over-sampling & averaging may improve ADC resolution

$$\boxed{f_{OS} = 4^w \cdot f_S}$$

f_{OS} = over-sampling frequency,
w = additional bits required.

Each additional bit implies/requires over-sampling by a factor of 4.

(Some) ADC parameters

USB Audio Interface
- 24-bit
- 192 kHz

Software Defined Radio (SDR)
- 14-bit ADC
- 2 – 6 Msamples/sec
- Covers 1 kHz – 2 GHz
- Bandwidth 10MHz
- At 8-bit >9.2 MS/sec

1. Number of bits N (~*resolution*)
2. Sample rate (~*speed*)
3. Signal-to-noise ratio (SNR)
4. Signal-to-noise-&-distortion rate

$$\mathrm{SINAD} = \frac{P_{\text{signal}} + P_{\text{noise}} + P_{\text{distortion}}}{P_{\text{noise}} + P_{\text{distortion}}}$$

5. Effective Number of Bits (ENOB)
6. …

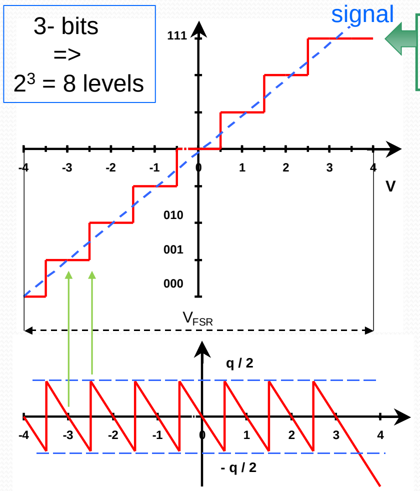
Static distortion

NB: Definitions may be slightly manufacturer-dependent!

ADC - Number of bits N

Continuous input signal digitized into 2^N levels.

3- bits
=>
2^3 = 8 levels



Uniform, bipolar transfer function (number of bits N=3 => 8 levels)

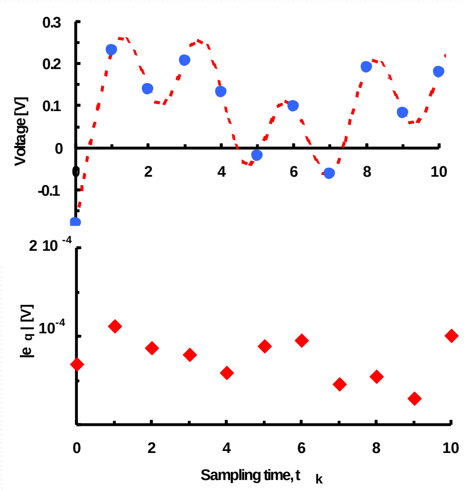
Quantisation step $q = \frac{V_{max}}{2^N}$

Ex: V_{max} = 1V , N = 12 ➡ q = 244.1 µV

Voltage (= q)
Scale factor (= 1 / 2^N)
Percentage (= 100 / 2^N)

Quantisation error digitized signal vs signal

ADC - Quantisation error



Quantisation step $q = \frac{V_{max}}{2^N}$

- Quantisation Error e_q in $[-0.5\ q, +0.5\ q]$.
- e_q limits ability to resolve small signal.
- Higher resolution (more bits) means lower e_q.

QE for
N = 12
$V_{FS} = 1$

SNR of ideal ADC

$$SNR_{ideal} = 20 \cdot \log_{10}\left(\frac{RMS(input)}{RMS(e_q)}\right) \quad (1)$$

Also called SQNR

(signal-to-quantisation-noise ratio)

Assumptions

Ideal ADC:

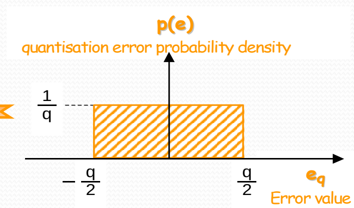
- only quantisation error e_q
  (p(e) = quantisation error probability density is assumed to be constant, uniform, etc.)
- e_q uncorrelated with signal.
- ADC performance constant in time.

RMS = root mean square

FSR = Full Scale Range

$$RMS(input) = \sqrt{\frac{1}{T} \cdot \int_0^T \left(\frac{V_{FSR}}{2} \cdot \sin(\omega t)\right)^2 dt} = \frac{V_{FSR}}{2\sqrt{2}}$$

Input(t) = ½ V_{FSR} sin(ω t).



$$RMS(e_q) = \sqrt{\int_{-q/2}^{q/2} e_q^2 \cdot p(e_q) de_q} = \frac{q}{\sqrt{12}} = \frac{V_{FSR}}{2^N \cdot \sqrt{12}}$$

(sampling frequency $f_S = 2\ f_{MAX}$)

SNR of ideal ADC

Substituting in (1) => $\overline{SNR}_{ideal} = 6.02 \cdot N + 1.76\,[dB]$ (2)

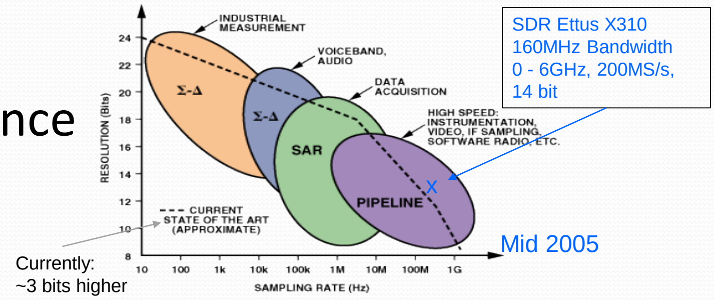
One additional bit ➡ SNR increased by 6 dB

Real SNR lower because:

- Real signals have noise.
- Forcing input to full scale unwise.
- Real ADCs have additional noise (aperture jitter, non-linearities etc).

Actually (2) needs correction factor depending on ratio between sampling freq & Nyquist freq. Processing gain due to oversampling.

ADC Performance



From: http://www.analog.com/library/analogDialogue/archives/39-06/architecture.html

ADC Topology	Resolution Range (bits)	Maximum Sampling Rate	Physical Size
Flash, direct, or parallel	≤ 8	5 GS/s	Increases exponentially with resolution
Two-step, sub-ranging, or pipelined	8-15	100 MS/s	Increases linearly with resolution
Successive approximation	8/18	2 MS/s	Increases linearly with resolution
Dual slope	12-16	100 KS/s	Does not change with resolution
Sigma delta	16-24	15 KS/s	Does not change with resolution

Table 2: ADC operating characteristics for the five ADC types discussed in this article. (Image source: Digi-Key Electronics) (2018)

Complex Numbers

The complex numbers are given by:

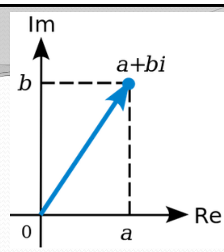
$$\mathbb{C} = \{c \mid c = a + bi, where, a, b \in \mathbb{R}\}$$

- here i is the imaginary unit that satisfies: $i^2 = -1$
- a is called the real part of c
- b is called the imaginary part of c

If z=a+bi, then the complex conjugate z^* is defined as z^*=a-bi

Complex Numbers

(see also your Calculus Book, and/or Wikipedia)



The complex numbers are given by:

$$\mathbb{C} = \{c \mid c = a + bi, where, a, b \in \mathbb{R}\}$$

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Addition:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + b - d)i$$

Multiplication:

$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i$$
$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \left(\frac{ab + bd}{c^2 + d^2}\right) + \left(\frac{bc - ad}{c^2 + d^2}\right)i$$

Complex Numbers

The *complex numbers* *are given by*:

$$\mathbb{C} = \{c\ |c = x + yi, where, x, y \in \mathbb{R}\}$$

The absolute value (modulus; magnitude) of $z = x + yi$ is:

$$r = |z| = \sqrt{x^2 + y^2}$$

Note that:

$$|z|^2 = zz^* = x^2 + y^2$$

The argument (phase) of $z = x + yi$ is:

$$\varphi = \arg(z) = \{\arctan(y/x), \text{if} \ldots =$$

"the angle of the vector (x,y) with the positive real axis"

Note: $z = r(cos\varphi + isin\varphi) = re^{i\varphi}$

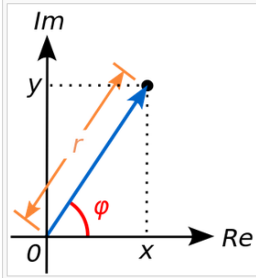


Figure 2: The argument φ and modulus r locate a point on an Argand diagram; $r(\cos\varphi + i\sin\varphi)$ or $re^{i\varphi}$ are *polar* expressions of the point.

Complex Numbers

Let:

$$z_1 = r_1(cos\varphi_1 + isin\varphi_1) = r_1e^{i\varphi_1}$$
$$z_2 = r_2(cos\varphi_2 + isin\varphi_2) = r_2e^{i\varphi_2}$$

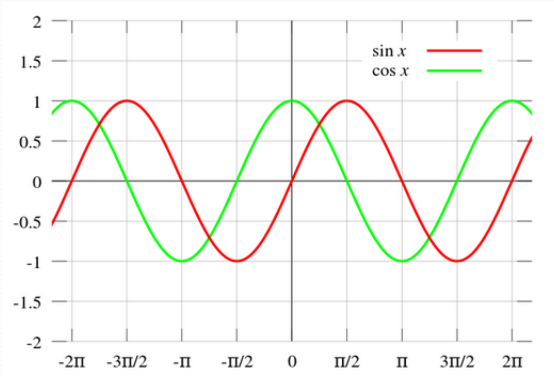
Note:

$$\cos(a)\cos(b) - \sin(a)\sin(b) = \cos(a + b)$$
$$\cos(a)\sin(b) + \sin(a)\cos(b) = \sin(a + b)$$

Hence:

$$z_1z_2 = r_1r_2(cos(\varphi_1{+}\varphi_2) + isin(\varphi_1{+}\varphi_2)) = r_1r_2e^{i(\varphi_1+\varphi_2)}$$

Sine Cosine Graphs



$$\sin(\varphi + \pi/2) = \cos(\varphi)$$

From: S. Lang, Linear Algebra, 2nd Ed.
Addison-Wesley Publ. Comp., Reading, 1970.

Fields

$K \subseteq \mathbb{C}$ is a field if it satisfies:

a) If $x, y \in K$, then $x+y \in K$ and $xy \in K$ — closed under addition and multiplication

b) If $x \in K$, then $-x \in K$ and if $x \neq 0$ also, then $x^{-1} \in K$ — has inverses

c) $0 \in K$ and $1 \in K$ (additive and multiplicative nullelements, resp.) — has null-elements

Examples of Fields:

- $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$
- Note, $\mathbb{Z}$ is not a field

Vector Spaces

a Field

V is a vector space over K if the following is true:

we should define these operations $u+v$ and λu

- if $u, v \in V$, then $u+v \in V$
- if $u \in V$ and $\lambda \in K$, then $\lambda u \in V$

associative 1] $u, v, w \in V$, then $(u+v)+w = u+(v+w)$

2] $\exists\, 0 \in V$ such that $0 + u = u + 0 = u \quad \forall u \in V$

3] given $u \in V$, $\exists -u \in V$ such that $u + (-u) = 0$

commutative 4] $\forall u, v \in V: \quad u+v = v+u$

distributive 5] $\forall c \in K: \quad c(u+v) = cu + cv$ for all $u, v \in V$

6] $\forall a, b \in K: \quad (a+b)v = av + bv$ for all $v \in V$

7] $\forall a, b \in K: \quad (ab)v = a(bv)$ for all $v \in V$

8] $\forall u \in V \quad 1 \cdot u = u$ (where $1 \in K$)

Examples of Vector Spaces

- $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{C}^1, \mathbb{C}^2, \mathbb{C}^3, \ldots$

Examples of Vector Spaces (Cont'd)

Function Spaces

Let S be a set and K a field and $f: S \to K$ a K-valued function, i.e., a rule that associates to each element of S a unique element of K.

Let V be the set of all functions of S into K.

A] If $f, g \in V$ we define $f+g$ as the function whose value at $x \in S$ is the value $f(x)+g(x)$ (again $\in K$ as K is a field).

B] If $c \in K$, we define cf to be the function whose value at $x \in S$ is equal to $c\,f(x)$ (again $\in K$ as K is a field).

- Then it is easy to verify that V is a Vector Space over K.

[$f_0 : S \to K$ where $f(x) = 0$ for all $x \in S$ is the 0 element]

Other Examples of Function Spaces which are Vector Spaces:

- V the set of all functions of $\mathbb{R}$ into $\mathbb{R}$
- V the set of all continuous functions of $\mathbb{R}$ into $\mathbb{R}$
- V the set of all differentiable functions of $\mathbb{R}$ into $\mathbb{R}$
- V the subspace generated by the functions $f(t) = e^t$ and $g(t) = e^{2t}$ (for all $t \in \mathbb{R}$.)

[Just check that A] and B] hold. As $\mathbb{R}$ is a field the claim that V is vector space follows.].

Linearly Dependence

Let V be a vector space over the field K.
Let $v_1, \ldots, v_n \in V$. $v_1, \ldots, v_n$ are linearly dependent over K if $\exists\, a_1, \ldots, a_n \in K$ not all equal **0** such that $a_1 v_1 + \ldots + a_n v_n = 0$.

- If there do not exist such numbers, i.e., if $a_1, \ldots, a_n \in K$ such that $a_1 v_1 + \ldots + a_n v_n = 0$, then $a_i = 0 \;\; \forall i = 1, \ldots, n$. then $v_1, \ldots v_n$ are linearly independent.

Example: - Let $V = \mathbb{R}^n$, then $E_1 = (1, 0, \ldots, 0)$ $\vdots$ $E_n = (0, 0, \ldots, 1)$ are linearly independent.
- also e^t, e^{2t} are linearly independent.

Basis

If elements $v_1, \dots, v_n \in V$ generate V, and $v_1, \dots, v_n$ are linearly independent. $\{v_1, \dots, v_n\}$ is called a basis of V.

- $v_1, \dots, v_n \in V$ generate V, that is every element of V can be expressed as a linear combination of $v_1, \dots, v_n$.
- and indeed if $x_1 v_1 + \dots + x_n v_n = x = y_1 v_1 + \dots + y_n v_n$ with $x_1, \dots, x_n, y_1, \dots, y_n \in K$ and for a $x \in V$,

then $(x_1 - y_1) v_1 + \dots + (x_n - y_n) v_n = 0$

thus $x_1 = y_1, \dots, x_n = y_n$. => in a unique way

Scalar Products

Let V a vector space over a field K. (real)

A scalar product on V is an association which to any pair $v, w \in V$ associates a scalar $\langle v, w \rangle$ (also $v \cdot w$) satisfying:

1) $\forall v, w \in V \quad \langle v, w \rangle = \langle w, v \rangle$

2) Let $u, v, w \in V$, then $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

3) Let $c \in K$, then $\langle cu, v \rangle = c \langle u, v \rangle$ and $\langle u, cv \rangle = c \langle u, v \rangle$

A scalar product is non-degenerate, if also:

4) If $v \in V$ and $\langle v, w \rangle = 0$ for all $w \in V$, then $v = 0$.

Examples of Scalar Products:

- $V = K^n$ $\langle x, y \rangle$: $x, y \to x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$ is a scalar product. [this is the 'standard' dot-product.]
- Let V be the space of continuous real-valued functions on the interval $[0, 1]$. If $f, g \in V$,

$\langle f, g \rangle = \int_0^1 f(t) g(t) \, dt$.

Then $\langle f, g \rangle$ is a scalar product. <= Homework I: Proof

Orthogonality

$v, w \in V$ are orthogonal: $v \perp w$ if $\langle v, w \rangle = 0$.

Norm

- The norm of $v \in W$ can be defined by $\|v\| = \sqrt{\langle v, v \rangle}$

  It is clear that: $\|c\,v\| = |c|\,\|v\|$
- $v \in V$ is a unit vector if $\|v\| = 1$

  ($v/\|v\|$ is always a unit vector, if $v \neq 0$) .

Some Theorems (easy)

We have the following theorems:

Th. If $v, w \in V$ and $v \perp w$ (i.e. $\langle v, w \rangle = 0$), then (Pythagoras)

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

Proof: $\|v + w\|^2 = \langle v+w, v+w \rangle = \langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle \underset{v \perp w}{=} \|v\|^2 + \|w^2\|$ (as $\langle v, w \rangle = 0$). ☒

Parallelogram law:

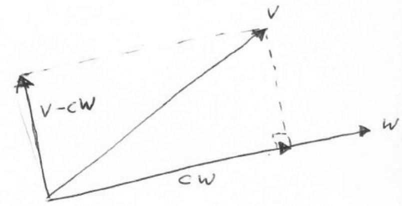
$\forall\, v, w \in V$ we have $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$.

Homework II: Proof Parallelogram Law.

Fourier Coefficients

Observation:

Let $w \in V$ such that $\|w\| \neq 0$.
For any $v \in V$ there exists a unique $c \in K$ such that $v - cw$ is perpendicular to w.



Now $v - cw$ perpendicular to w means that $\langle v - cw, w \rangle = 0$.

$$\Rightarrow \langle v, w \rangle - c \langle w, w \rangle = 0 \quad \Rightarrow \quad \boxed{c = \frac{\langle v, w \rangle}{\langle w, w \rangle}}$$

Conversely, if $c = \dfrac{\langle v, w \rangle}{\langle w, w \rangle}$ then $c\langle w, w \rangle = \langle v, w \rangle \Rightarrow$
$\langle v, w \rangle - c\langle w, w \rangle = 0 \Rightarrow$
$\langle v - cw, w \rangle = 0$

hence $v - cw$ perpendicular to w.

We call c the component of v along w, or
the Fourier coefficient of v with respect to w.

Example Fourier Coefficients.

Let V be the space of continuous functions on $[-\pi, \pi]$.
Let $f : x \to \sin kx$, where $k \in \mathbb{Z}_{>0}$.
Then $\|f\| = \sqrt{\langle f, f \rangle} = \left(\int_{-\pi}^{\pi} \sin^2 kx \, dx \right)^{1/2} = \sqrt{\pi}$

In this case, if g is any continuous function on $[-\pi, \pi]$, then the Fourier coefficient of g with respect to f is

$$\frac{\langle g, f \rangle}{\langle f, f \rangle} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin kx \, dx$$

The Complex ($\mathbb{C}$) Case

Let V be a vector space over the complex numbers.

A hermitian product on V is a rule $\langle v, w \rangle$ satisfying:

1) $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in V$

2) $u, v, w \in V$, then $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

3) if $\alpha \in \mathbb{C}$, then $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$

$\langle u, \alpha v \rangle = \overline{\alpha} \langle u, v \rangle$.

$\langle \cdot, \cdot \rangle$ is positive definite if $\langle v, v \rangle \geq 0$ for all $v \in V$ and $\langle v, v \rangle > 0$ if $v \neq 0$

Note: Orthogonal, perpendicular, orthogonal basis, orthogonal complement, as before!

Also the Fourier coefficient and the projection of v along w are as before.

Example

Let V be the space of continuous complex-valued functions on the interval $[-\pi, \pi]$.

- If $f, g \in V$, we define $\langle \cdot, \cdot \rangle$ as follows:

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt$$

This can be shown, using standard properties of the integral, to be a positive definite hermitian product.

- Let $\boxed{f_n(t) = e^{int}}$

A) if $n \neq m$, then $\langle f_n, f_m \rangle = \int_{-\pi}^{\pi} e^{int}\overline{e^{imt}}\, dt = \int_{-\pi}^{\pi} e^{ikt}\, dt = 0$

if $n = m$, then we have $\langle f_n, f_n \rangle = \int_{-\pi}^{\pi} e^{int} e^{-int}\, dt = \int_{-\pi}^{\pi} 1\, dt = 2\pi$

- If $f \in V$, then its Fourier coefficient with respect to f_n is equal to:

$$\frac{\langle f, f_n \rangle}{\langle f_n, f_n \rangle} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt.$$

Note: A] shows that f_n and f_m with $n \neq m$ are orthogonal.

Furthermore it can be shown that $\{f_n, n \in \mathbb{N}^+\}$ constitutes ~~a~~ a basis for V.

Hence $\{e^{it}, e^{2it}, e^{3it}, \ldots\}$ is an orthogonal basis of V the vector space of continuous complex-valued functions on the interval $\{-\pi, \pi\}$. (Note, by dividing through $<f_n, f_n>$ you get normalized basis.) 7]

References

This presentation uses a selection of slides that are adapted from original slides by

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Resonances in Outer Ear

The **outer** ear consists of the external visible part and the auditory canal. The tube is about 2.5 cm long

Ear is closed tube (closed to one end):
$\Rightarrow$ resonance of 0.25 wavelength
$\Rightarrow$ Resonance frequencies f can be calculated with:
f = nv/(4L), where n = 1, 3, 5, …, L=2.5cm
and v = 343 m/s speed of sound
For n = 1, v = 343m/s = 34300 cm/s, L = 2.5 cm, we have
f = 34300 (cm/s) / 10 (cm/s) = 3430 Hz

Note wavelength of 3430 Hz equals 34300 / 3430 cm = 10cm

*)

$$\int_{-\pi}^{\pi} \sin^2 kx \, dx = \left[\frac{x}{2} - \frac{\sin(2kx)}{4k}\right]_{-\pi}^{\pi}$$

$$= \frac{\pi}{2} - \frac{\sin(2k\pi)}{4k} - \left(\frac{-\pi}{2} - \frac{\sin(-2k\pi)}{4k}\right)$$

$$= \pi$$

Recap:

Complex numbers $\mathbb{C}$:

1) $\mathbb{R} \subset \mathbb{C}$; sum and products for these numbers $\in \mathbb{R} \subset \mathbb{C}$ as before.

2) $\exists$ complex number i such that $i^2 = -1$

3) Every complex number can be uniquely expressed as $a + bi$, with $a, b \in \mathbb{R}$

4) $\alpha, \beta, \gamma \in \mathbb{C}$, then

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$
$$(\alpha+\beta)+\gamma = \alpha + (\beta+\gamma)$$
$$\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$$
$$(\beta+\gamma)\alpha = \beta\alpha + \gamma\alpha$$
$$\alpha\beta = \beta\alpha$$
$$\alpha+\beta = \beta+\alpha$$

If $1 \in \mathbb{R}$ then $1\alpha = \alpha$

If $0 \in \mathbb{R}$ then $0\alpha = 0$

Furthermore, $\alpha + (-1)\alpha = 0$.

$\alpha = a+bi$, $\beta = c+di$, then

$$\alpha+\beta = (a+bi)+(c+di) = (a+c) + (b+d)i$$

and

$$\alpha\cdot\beta = (a+bi)(c+di) = ac + adi + bci + bdi^2 = (ac-bd) + (ad+bc)i$$

, and if $\lambda \in \mathbb{R}$ $\lambda\alpha = \lambda(a+bi) = \lambda a + \lambda bi$

$\bar{\alpha} = \overline{a+bi} = a - bi$ $\Rightarrow \alpha\bar{\alpha} = a^2+b^2 \in \mathbb{R}$

$\bar{\alpha}$ is conjugate of α $\Rightarrow \alpha^{-1} = \dfrac{\bar{\alpha}}{\alpha\bar{\alpha}}$

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