## GC-Gadgets in the Rush Hour

## Game Complexity

Gadgets in the Rush Hour


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IPA, Eindhoven; Friday, January 25, 2019



Deep Blue (with minimax/ $\alpha-\beta$ ) vs. Garry Kasparov, 1997


$10^{3}$ Simulations


December 2018
AlphaZero

Silver et al.
Science 362, 1140-1144


In 2016 John Tromp showed at CG2016 that there are 2081681993819799846 9947863334486277028 6522453884530548425 6394568209274196127 3801537852564845169 8519643907259916015 6281285460898883144 2712971531931755773 6620397247064840935

$\approx 2 \cdot 10^{170}$ legal positions in $19 \times 19$ Go, using dynamic programming and HARDWARE.

> https://tromp.github.io/go/legal.html

In 2011 IBM used a computer to play "Jeopardy!":

| \% | wome | = | " | ? | mam |
| :---: | :---: | :---: | :---: | :---: | :---: |
| \$100 | \$100 | \$100 | \$100 | \$100 | \$100 |
| \$200 | \$200 | \$200 | \$200 | \$200 | \$200 |
| \$300 | \$300 | \$300 | \$300 | \$300 | \$300 |
| \$400 | \$400 | \$400 | \$400 | \$400 | \$400 |
| \$500 | \$500 | \$500 | \$500 | \$500 | \$500 |

> IN 2013 ROB FORD, MAYOR OF THIS 4thLARGEST CITY IN N. AMERICA, FIRST SAID HE SMOKED WEED, NOT CRACK...THEN YES, OK, CRACK, TOO


We study the complexity of games (puzzles, ...). We want to make statements like

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Tetris is NP-complete.
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In order to do so, we examine reductions between appropriate games, with the help of gadgets.

Games studied include TipOver, Plank puzzles, Sokoban $\rightarrow$, Rush Hour, Mahjongg, ...


We want to reduce a known problem to a new one, for example, 3SAT to VC (so VertexCover is NP-hard).

For every Boolean variable $x_{i}$ we make a variable gadget (left) and for every clause $C_{j}$ a clause gadget (right):


We connect these gadgets in the intuitive way; satisfying assignments (left) correspond to vertex covers (right):

$\wedge \underbrace{\left(\overline{x_{1}} \vee \overline{x_{2}} \vee x_{3}\right)}_{C_{2}}$


Satisfying assignment $x_{1}=$ true, $x_{2}=x_{3}=$ false gives a VC X of size $3+2 \cdot 2=7$, for 3 literals and 2 clauses.

## GC——adgets

Basic idea: gadgets


Suppose we want to show a game to be NP/PSPACE-hard (formally: some related ( $\mathrm{y} / \mathrm{n}$ )-decision problem $\Pi$ ).

For this purpose we produce a reduction from a known well-chosen graph game (formally: some related (y/n)decision problem $\Pi^{\prime}$, hopefully with planar graphs) to $\Pi$.


The less complicated $\Pi^{\prime}$ is, the better. If we are lucky, we only have to show how certain basic constructs are "emulated" by means of gadgets. Plus many details...

We also have gadgets to emulate certain (sub)graph behaviour in the graphs themselves.


Constraint graphs consist of AND- and OR-nodes:


Edges are always directed such that every node $=$ vertex receives a total input $\geq 2$, where incoming blue edges contribute 2 and incoming red edges 1.

Examples:


An edge can be reversed if all total inputs remain $\geq 2(\mathbf{X})$.

## AND- and OR-node



External behavior of these gadgets can be described by the statespaces below (where 1: points in; 0: points out):


We have several simple gadgets available:

- free blue-edge terminator (FBET)
- constrained blue-edge terminator (CBET):
- free red-edge terminator (do we need this?)

> Exercise: Explain the CBET (arrows? statespace?).

Exercise: Develop a FBET.

The CHOICE-vertex (left) can be emulated by the gadget on the right:


Exercise: Show that the emulation works.

Don't worry about the fact that $A, B$ and $C$ are all red or blue. What matters now is whether they point in or out.

And in reality edges are always directed (have arrows)!

In many graphs we have (unavoidable) edge crossings.

We now want a gadget that can replace such a crossing. So assume that we have two crossing blue edges. (There is no node where the edges cross.)


If we have such a gadget, we need only emulate planar graphs in our reductions to specific games - and these are often planar (flat)!


Exercise: Show that A and B may not both point out.
Exercise: Show there are $\leq 2^{4}-7=9$ states for $A B C D$.
Exercise: Show that this emulates two crossing edges.
Exercise: And if each edge may be reversed at most once?

Wait a minute: did we just use "4-red-nodes"!?
This gadget requires any 2 from $A / B / C / D$ to go in:


Exercise: Show that this can replace a "4-reds-node".
Exercise: Still OK if each edge may be reversed at most once?
In that case we (unfortunately) need a "race condition".

## Protected-OR

For a protected-OR vertex two of the three incident edges are special: they are not both allowed to be directed inward (by some outside force).


Exercise: Show that this emulates an OR-node.

Remember again that A and B can "change to blue".
Exercise: Where are the protected-OR-nodes in the gadgets?
Exercise: Describe the statespace of a protected-OR-node.

Having seen the general picture and some gadgetry, we now examine particular games and puzzles, like Rush Hour $®$ :

www.puzzles.com/products/rushhour.htm

The rules of Rush Hour are easy: cars may move either horizontally or vertically (left/right and up/down), in their natural direction, as long as they do not bump/crash through other cars or the walls.

Target: get the red car out of the garage through the exit.


Theorem Rush Hour is PSPACE-complete.
(Remember Savitch: PSPACE $=$ NPSPACE.)
$\uparrow$
non-deterministic Turing machine with polynomial space

The proof proceeds by reduction from Nondeterministic Constraint Logic (NCL): NCL is PSPACE-complete for planar graphs using only ANDs and protected-ORs.


The decision problem is: Given a constraint graph $G$ (including arrows) and a distinguished edge $e$ in $G$; is there a sequence of edge reversals that eventually reverses $e$ ? Moves may be repeated: it is an unbounded game.


## Exercise: Fill in the proof details.

This includes

- proper inner working of the gadgets,
- proper communication between gadgets,
- proper glueing together (in polynomial space),
- check that walls do not move (or hardly),

The statespace for the Rush-Hour protected-OR gadget is somewhat strange (where again 1: car out; 0: car in):



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And how about Plank puzzle \(=\) River Crossing \({ }^{\text {TM }}\) (link) ?
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You must travel from Start to End; you can carry and move one plank at a time (if you "have" it), and traverse them in the obvious way.

The Plank puzzle is also PSPACE-complete:

(a) AND

(b) OR

In these gadgets, for the correct behavior it is important that plank $A$ and/or $B$ are inside. You can freely walk around the squares with a length 3 plank.



Game rules: two visible stones may be removed if they are the same and they are "free" to one or two sides.

Exercise: Provide AND- and OR-gadgets for Mahjongg.


Hint: keep it simple; find a small set of stones, such that a special one can be "opened" exactly if one (for OR, or both for AND) of two others can be removed.

Exercise: And a CHOICE-gadget?

(a) AND gadget

(c) FANOUT gadget

(b) OR gadget

(d) CHOICE gadget

The statespaces for AND, OR and protected-OR:




Reductions between problems concerning games are based on simple gadgets, technique and peculiarities. Many games can be proven to be NP-hard, PSPACEhard, etc., using the Constraint Logic machinery.

Thanks: Erik Demaine \& Bob Hearn (book: Games, Puzzles \& Computation, AK Peters, 2009) and Jan van Rijn.
www.liacs.leidenuniv.nl/~kosterswa/19gadgets.pdf

