New Directions for the Tutte Polynomial July 2015
Royal Holloway University of London

Graph Polynomials motivated by Gene Assembly

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background: gene assembly in ciliates

Martin		**Interlace**			
$m(G;y)$	2-in 2-out	$q(G,y)$	simple	two	“directions”
$M(G;y)$	4-regular	$Q(G,y)$		three	

$\hookrightarrow$ via circle graphs

medial graph — fundamental graph — Tutte connection

basic evaluations

explicit vs. recursive formulation

beyond binary matroids?

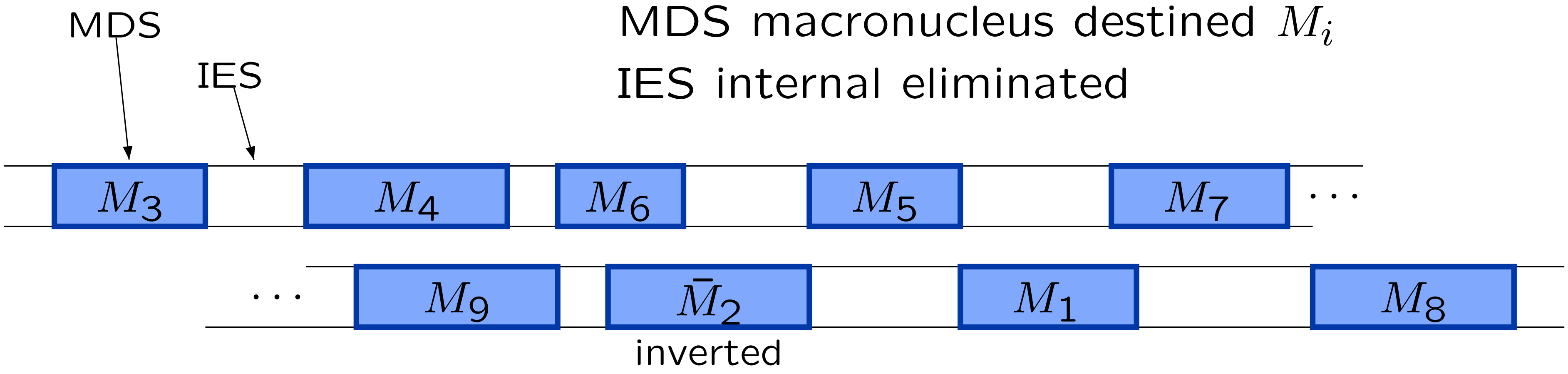
Ciliates: two types of nucleus

gene assembly: splicing and recombination

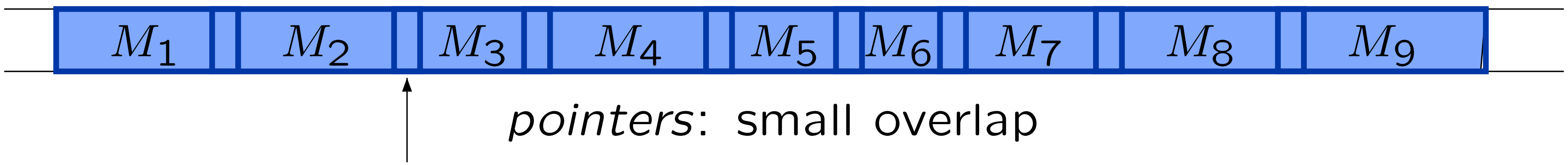
MIC *micronucleus*

MDS macronucleus destined M_i

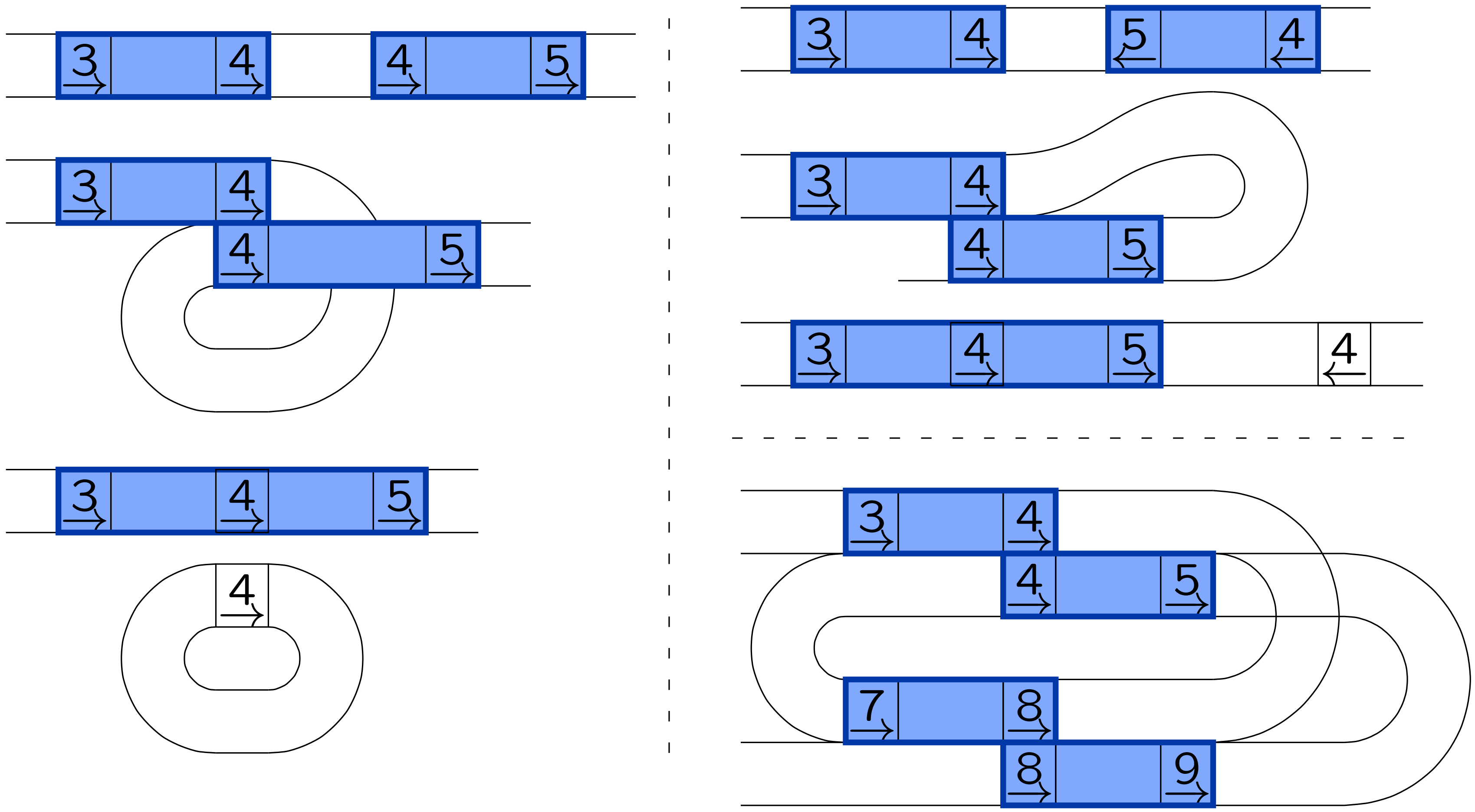
IES internal eliminated



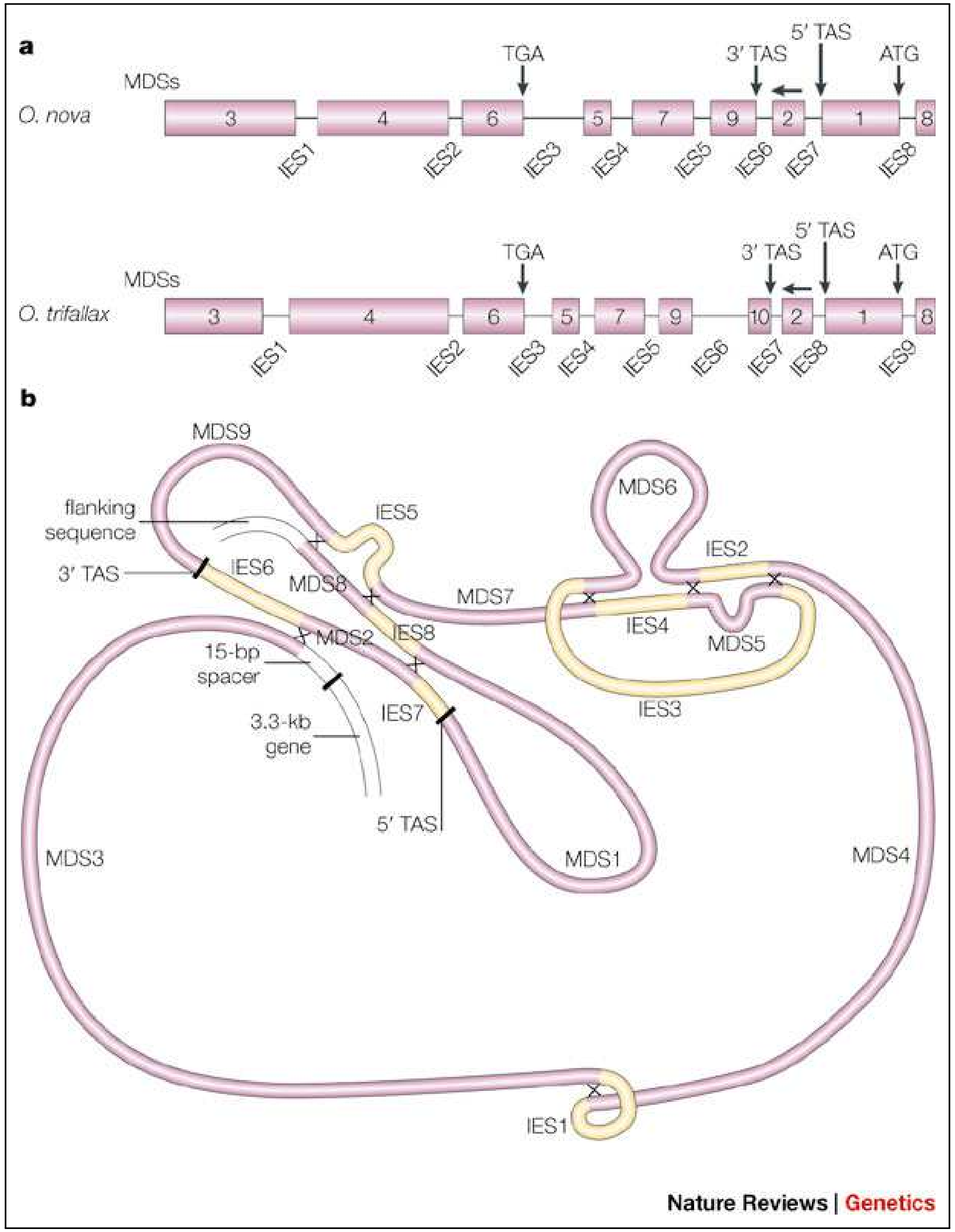
MAC *macronucleus*



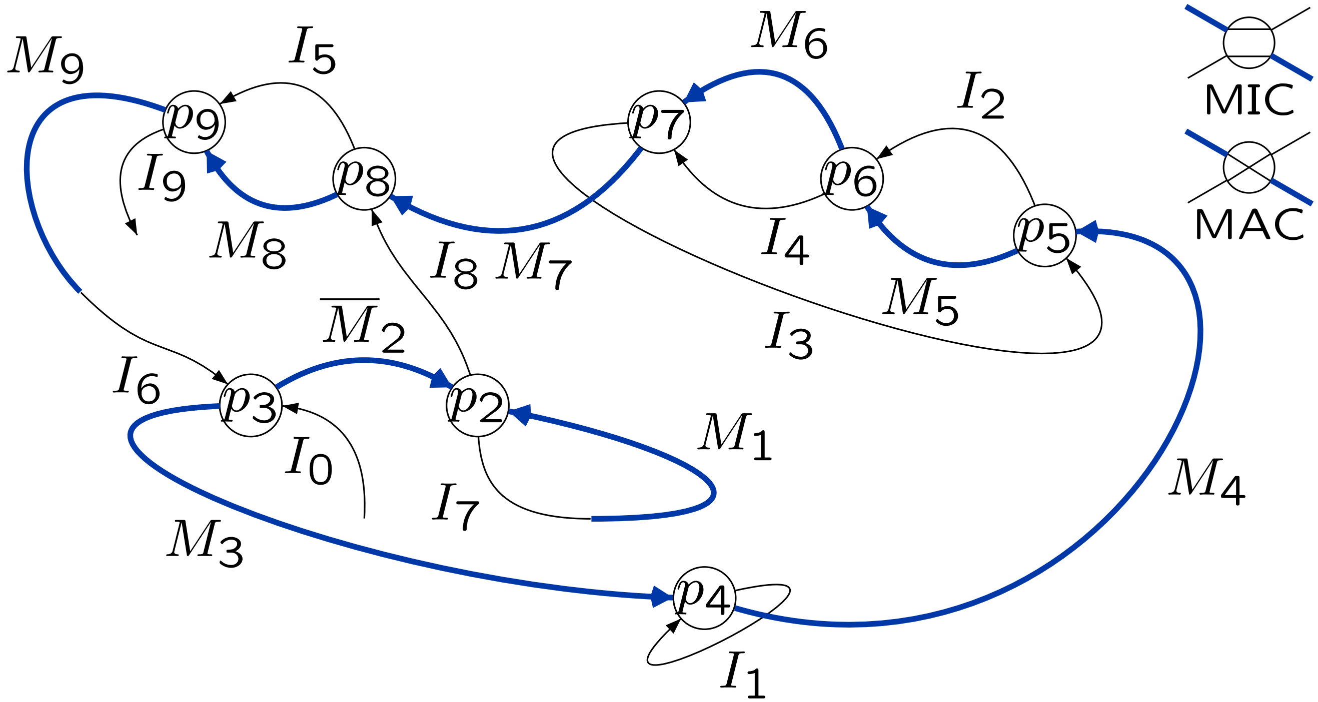
pointers: small overlap



Ehrenfeucht, Harju, Petre, Prescott, Rozenberg:
Computation in Living Cells – Gene Assembly in Ciliates (2004)



David M. Prescott. Genome gymnastics: unique modes of dna evolution and processing in ciliates. Nature Reviews Genetics (December 2000)



MIC $I_0\, M_3\, I_1\, M_4\, I_2\, M_6\, I_3\, M_5\, I_4\, M_7\, I_5\, M_9$
$I_6\, \overline{M}_2\, I_7\, M_1\, I_8\, M_8\, I_9$

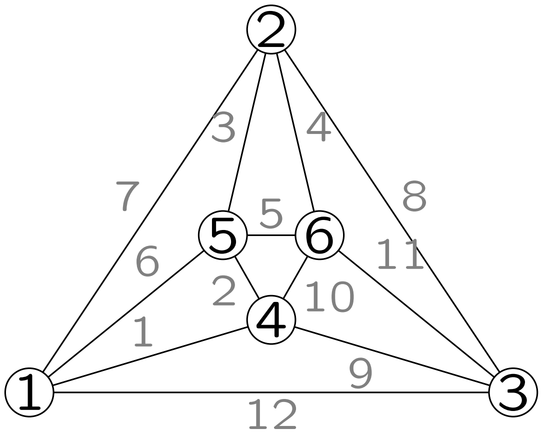
MAC $\overline{I}_9 \overline{I}_5 \overline{I}_8 I_7 \underbrace{M_1 M_2 \cdots M_8 M_9} I_6 \overline{I}_0$, I_1 and $I_2\, I_4\, I_3$

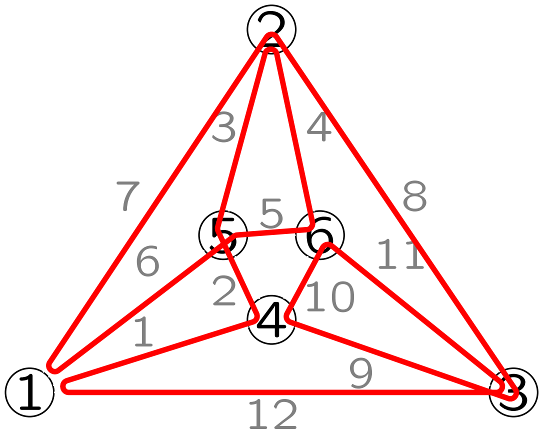
4-regular graph with Euler circuit

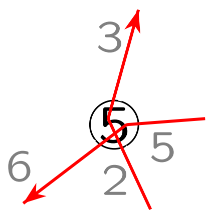
double occurrence string w defines

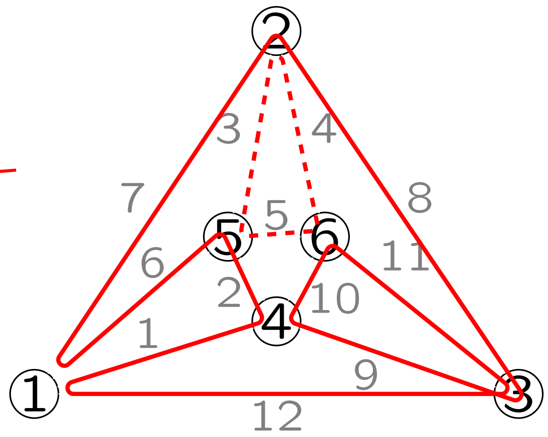
$w =$ 4-regular graph G_w + Euler circuit C_w

1 4 <u>**5** 2 6 **5**</u> 1 2 3 4 6 3 or 2-in 2-out graph + directed circuit

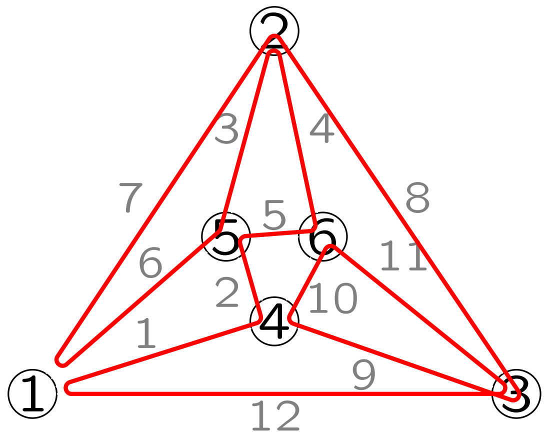








segment split

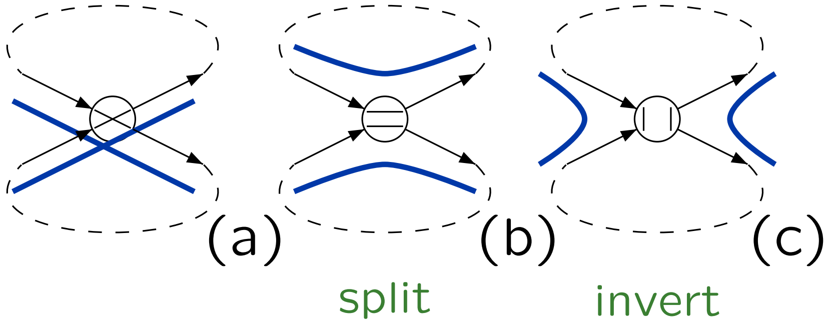


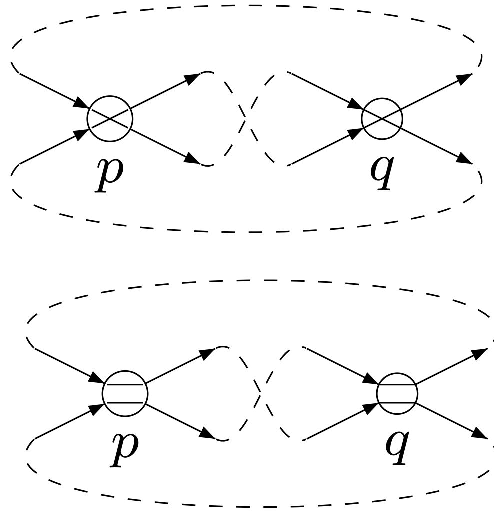
segment inverted

(a) follows C

(b) orientation consistent

(c) orientation inconsistent

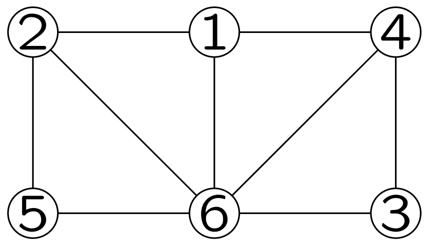




interlaced $\dots \underbrace{p \dots q} \dots \underbrace{p \dots q} \dots$

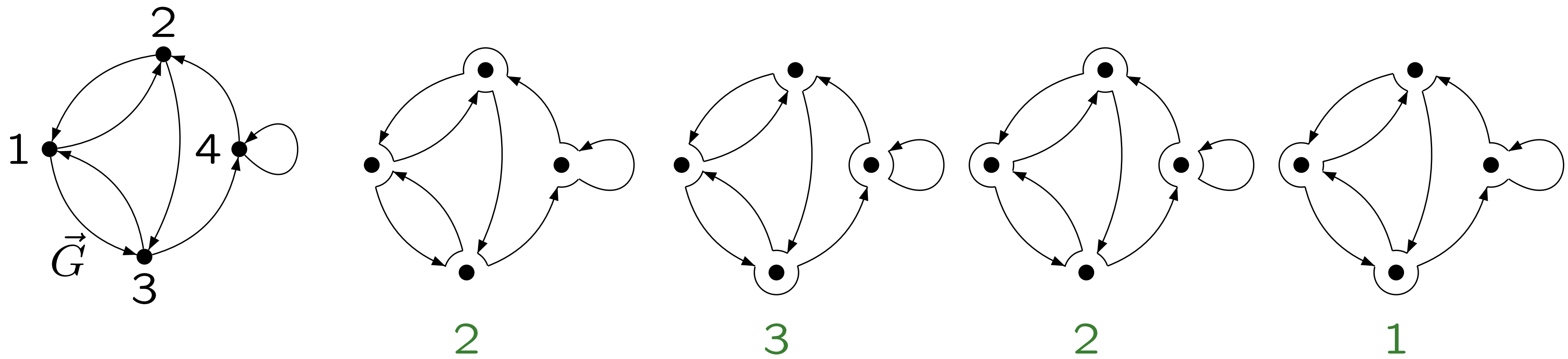
segments are swapped

interlace graph $I(C)$



$w = 1\,4\,\underline{\mathbf{5}\,2\,6\,\mathbf{5}}\,1\,2\,3\,4\,6\,3$

Kotzig. Eulerian lines in finite 4-valent graphs (1966)



transition system (graph state)

Martin polynomial of 2-in 2-out digraph $\vec{G}$

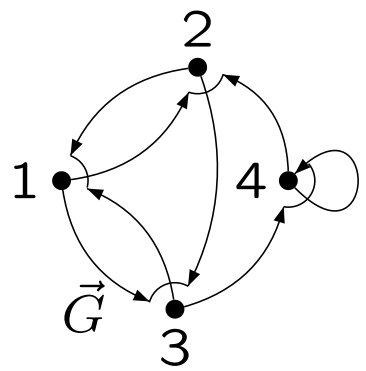
$$m(\vec{G}; y) = \sum_{T \in \mathcal{T}(\vec{G})} (y-1)^{k(T)-c(\vec{G})}$$

$c(\vec{G})$ components
$k(T)$ circuits for transition system T

Pierre Martin, Enumérations eulériennes dans les multigraphes et invariants de Tutte-Grothendieck, PhD thesis, 1977

$$m(\vec{G};y) = \sum_{T\in\mathcal{T}(\vec{G})}(y-1)^{k(T)-c(\vec{G})}$$

$\vec{G}$ 2-in 2-out digraph and $n = |V(\vec{G})|$



$a(\vec{G})$ anti circuits

Thm. $m(\vec{G};-1) = (-1)^n(-2)^{a(\vec{G})-1}$

$m(\vec{G};0) = 0$, when $n > 0$

$m(\vec{G};1)$ number of Eulerian systems

$m(\vec{G};2) = 2^n$

$m(\vec{G};3) = k\,|m(\vec{G};-1)|$ for odd k

graph reductions: glueing edges

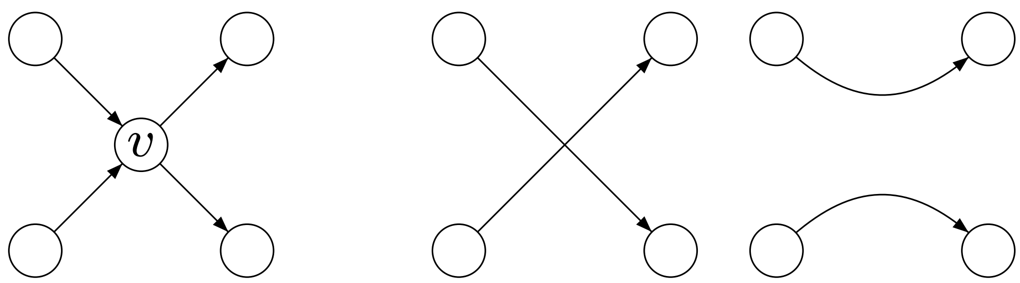
$\vec{G}$ 2-in 2-out digraph

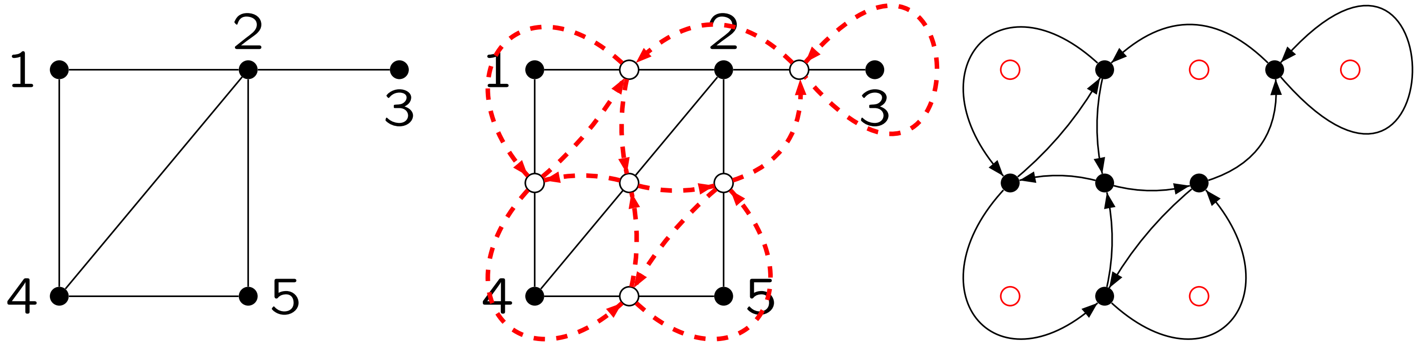
Thm. $m(\vec{G};y) = 1$ for $n = 0$

$m(\vec{G};y) = y\, m(\vec{G}';y)$ cut vertex v

$m(G;y) = m(\vec{G}'_v;y) + m(\vec{G}''_v;y)$

vertex v without loops

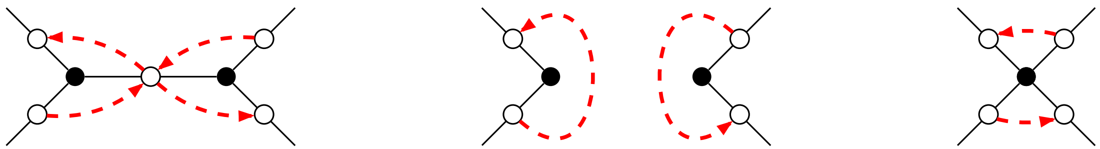


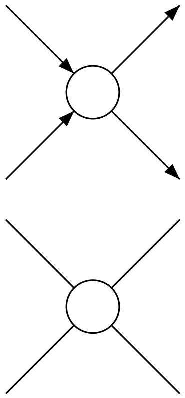


plane graph G, with medial graph $\vec{G}_m$

Thm. $m(\vec{G}_m; y) = T(G; y, y)$

proof: deletion-contraction





three directions

Martin polynomial of 4-regular graph G

$$M(G;y) = \sum_{T \in \mathcal{T}(G)} (y-2)^{k(T)-c(G)}$$

$c(G)$ components

$k(T)$ circuits for transition system T

three graph reductions

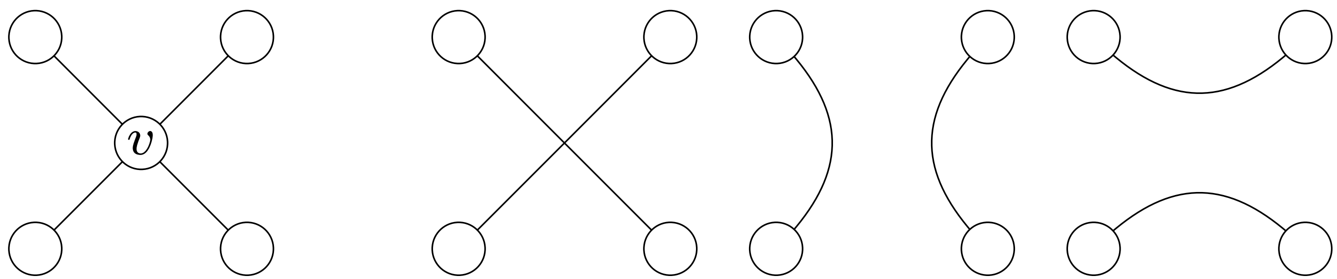
G 4-regular graph

Thm. $M(G; y) = 1$ for $n = 0$

$M(G; y) = y\, M(G'; y)$ cut vertex v

$M(G; y) = M(G'_v; y) + M(G''_v; y) + M(G'''_v; y)$

vertex v without loops

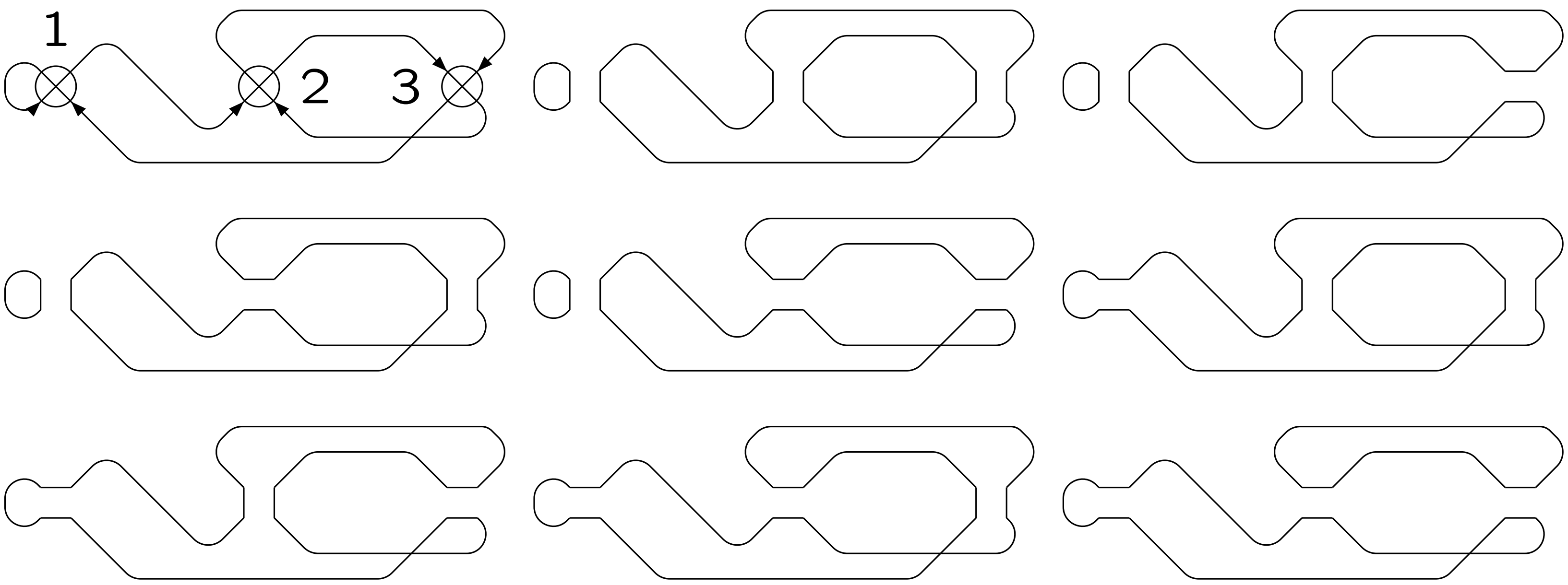


assembly polynomial of G_w for doc-word w

$$S(G_w)(p,t) = \sum_s p^{\pi(s)} t^{c(s)-1},$$

follow/consistent/inconsistent

$w = 1\,1\,2\,3\,2\,3$



Burns, Dolzhenko, Jonoska, Muche, Saito:
Four-regular graphs with rigid vertices associated to DNA recombination (2013)

transition polynomials $W = (a, b, c)$

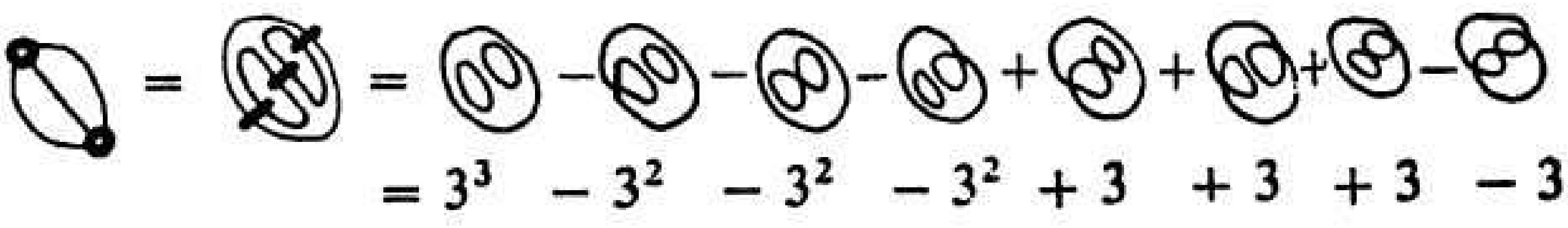
transition T defines partition V_1, V_2, V_3

eg wrt fixed cycle

weight $W(T) = a^{|V_1|} b^{|V_2|} c^{|V_3|}$

$$M(G, W; y) = \sum_{T \in \mathcal{T}(\vec{G})} W(T) y^{k(T) - c(\vec{G})}$$

polynomial	a	b	c
Martin	1	1	0
(3-way)	1	1	1
assembly	0	p	1
Penrose	0	1	-1



F. Jaeger: On transition polynomials of 4-regular graphs (1990) where are the Δ-matroids?

2-in 2-out graph
fix euler cycle C
represent all cycles by the vertices that differ

de Bruijn Graphs for DNA Sequencing
originally recursive definition

simple graph G (with loops)

interlace polynomial

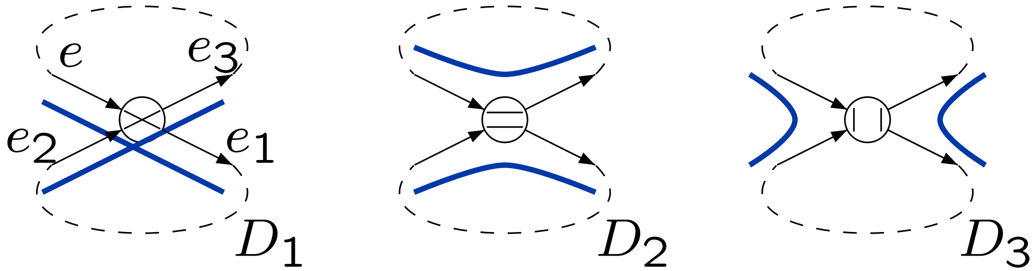
(single-variable, vertex-nullity)

$$q(G;y) = \sum_{X \subseteq V(G)} (y-1)^{n(\,A(G)[X]\,)}$$

Arratia, Bollobás, Sorkin: The interlace polynomial: a new graph polynomial (2000)

Aigner, van der Holst: Interlace polynomials (2004)

Bouchet: TutteMartin polynomials and orienting vectors of isotropic systems (1991)



4-regular graph G with Eulerian system C

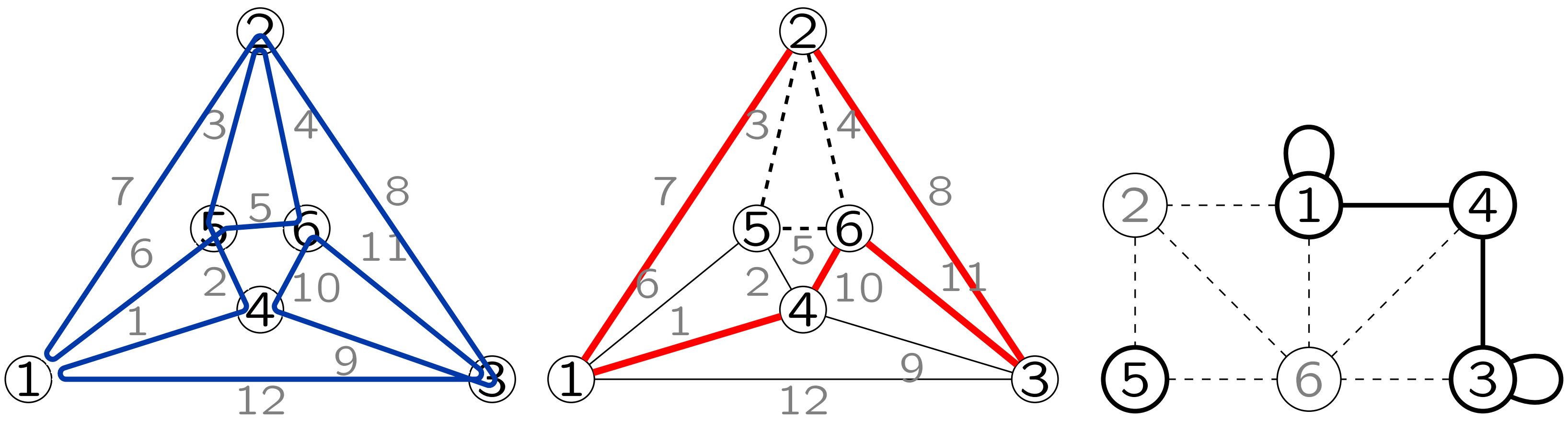
P circuit partition of $E(G)$, partition vertices:

D_1 follows C

D_2 orientation consistent

D_3 orientation inconsistent

Thm. Then $|P| - c(G) = n(\,(I(C) + D_3) \setminus D_1\,)$



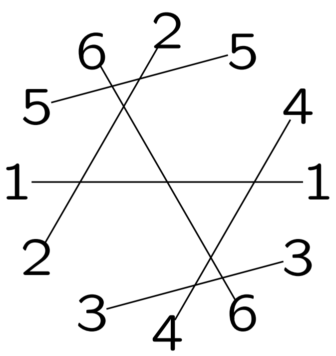
$$\begin{array}{c} \\ 1 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

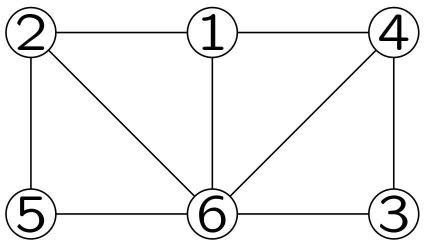
$$q(I(C);y) = \sum_{X \subseteq V(G)} (y-1)^{n(\,A(I(C))[X]\,)}$$

$$|P| - c(G) = n(\,(I(C) + \underset{=\varnothing}{D_3}) \setminus D_1\,)$$

$$m(\vec{G};y) = \sum_{T \in \mathcal{T}(\vec{G})} (y-1)^{k(T)-c(\vec{G})}$$

Thm. $m(\vec{G};y) = q(I(C);y)$





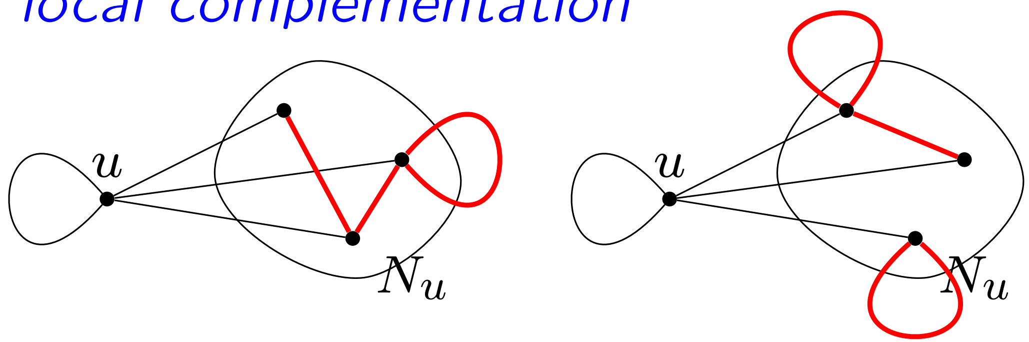
$w = 1\,4\,\underline{\mathbf{5}\,2\,6\,\mathbf{5}}\,1\,2\,3\,4\,6\,3$

$G \mapsto G * u$

looped vertex u

$N_u = N_G(u) \setminus \{u\}$

local complementation



$G \mapsto G * \{u, v\}$

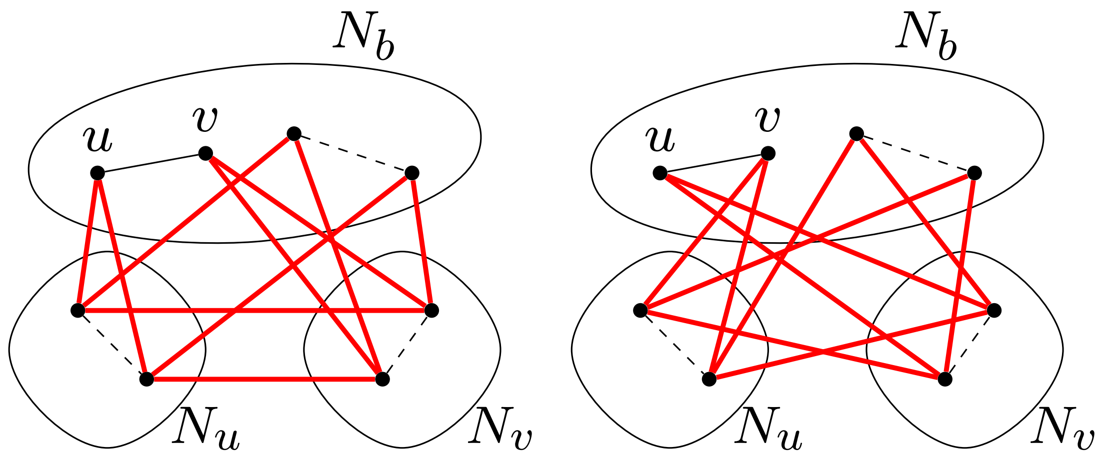
unlooped edge $\{u, v\}$

$N_u = N_G(u) \setminus N_G(v)$

$N_v = N_G(v) \setminus N_G(u)$

$N_b = N_G(u) \cap N_G(v)$

edge complementation



special cases of *principal pivot transform*

(partial inverse)

invert $I(C * u) = I(C) * v$

swap $I(C * \{u, v\}) = I(C) * \{u, v\}$ when defined

Thm. $q(G; y) = 1$ if $n = 0$

$q(G; y) = y\, q(G \setminus v; y)$ if v isolated (unlooped)

$q(G; y) = q(G \setminus v; y) + q((G * v) \setminus v; y)$

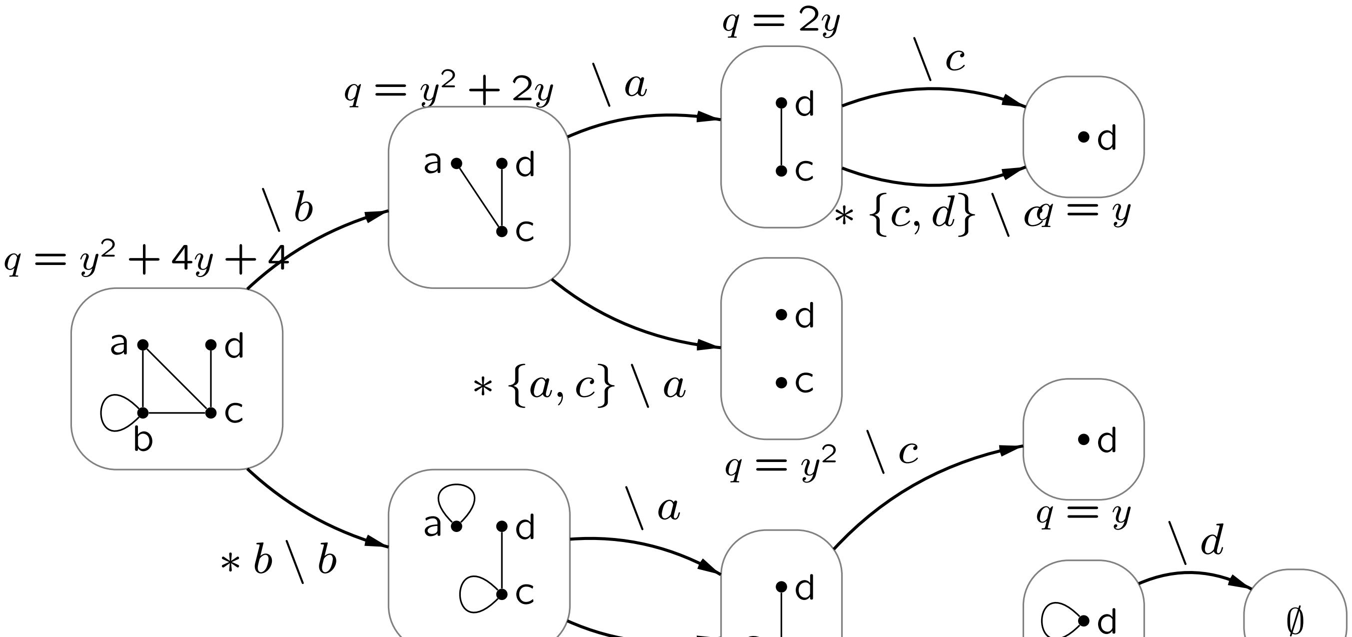
if v looped

$q(G; y) = q(G \setminus v; y) + q((G * e) \setminus v; y)$

if $e = \{v, w\}$ unlooped edge

Thm. $q(G; y) = q(G \setminus v; y) + q((G * X) \setminus v; y)$

$A(G[X])$ nonsingular, $v \in X$



Thm. $q(G;y) = q(G*v;y)$ if v looped

$q(G;y) = q(G*e;y)$

if $e = \{v,w\}$ unlooped edge

Thm. $q(G;y) = q(G*X;y)$

$A(G[X])$ nonsingular

Thm.

$m(\vec{G};-1) = (-1)^n(-2)^{a(\vec{G})-1}$	$q(G;-1) = (-1)^n(-2)^{n(A(G)+I)}$
$m(\vec{G};0) = 0$, when $n > 0$	$q(G;0) = 0$ if $n > 0$, no loops
$m(\vec{G};1)$ #Eulerian systems	$q(G;1)$ #induced subgraphs with odd number of perfect matchings
$m(\vec{G};2) = 2^n$	$q(G;2) = 2^n$
$m(\vec{G};3) = k\,\lvert m(\vec{G};-1)\rvert$ odd k	$q(G;3) = k\,\lvert q(G;-1)\rvert$ odd k

$$q(G;y) = \sum_{X \subseteq V(G)} (y-1)^{n(A(G)[X])}$$

$$Q(G;y) = \sum_{X \subseteq V(G)} \sum_{Y \subseteq X} (y-2)^{n((A(G+Y))[X])}$$

Cohn-Lempel-Traldi

$$|P| - c(G) = n(\,(I(C) + D_3) \setminus D_1\,)$$

third direction

$e = \{v, w\}$ unlooped edge

$$Q(G;y) = Q(G \setminus v; y) + Q((G * e) \setminus v; y) + Q(((G+v)*v) \setminus v; y)$$

operations $*$ and $+$

Aigner, vander Holst; Bouchet

M binary matroid over E

G *fundamental graph* wrt basis B of M

$(B, E \setminus B)$-bipartite graph

edge iff $B \setminus \{v\} \cup \{w\}$ basis of M

Thm. $T(M; y, y) = q(G; y)$.

Question: generalization for $T(M; y, y)$ and $q(G; y)$?

binary *bipartite*

THANKS