SHORT TIME FOURIER TRANSFORMS

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Overview

- Fourier Transforms

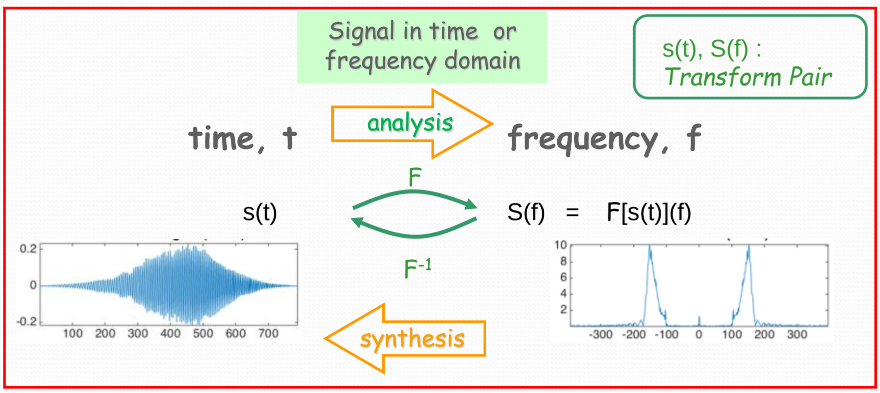
Some slides adapted from lectures by
Dr M.E. Angoletta at DISP2003,
a DSP course given by CERN and University of Lausanne (UNIL)

Fourier Transforms

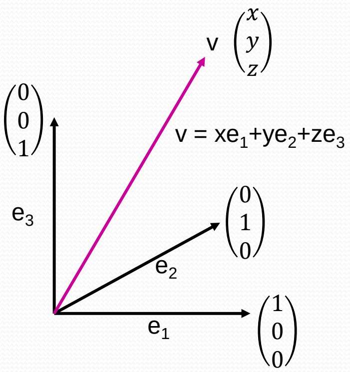
- Frequency analysis
- A tour of Fourier Transforms
- Continuous Fourier Series (FS)
- Discrete Fourier Series (DFS)

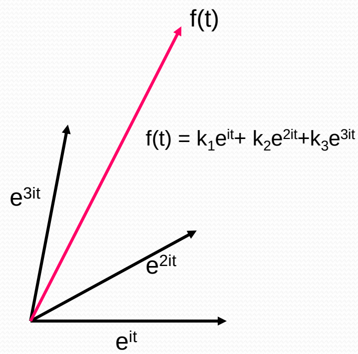
Frequency Analysis

- **Fast & efficient insight on the signal's components**.
- **Powerful & complementary to time domain analysis techniques.**
- Simplifies the original problem - Filtering, solving Part.Diff.Eqns. (PDE),...
- Many transforms: **Fourier, Discrete Cosine, Laplace, z, Wavelet, etc.**



Bases of Vector Spaces

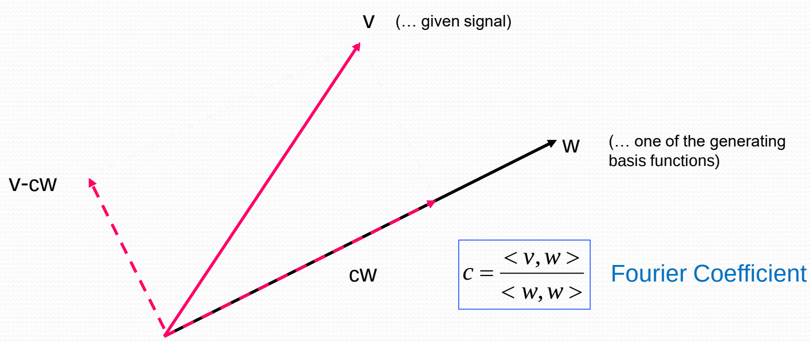




v is a linear combination of the basis vectors e_i (i = 1,2,3)

f is a linear combination of the basis functions e^{it},e^{2it},e^{3it}

Fourier Coefficients



Let <.,.> an in-product for our vector space V.
Then we calculate the Fourier coefficient **c** of v in V with respect to (basis) vector w by: $c = \frac{<v,w>}{<w,w>}$

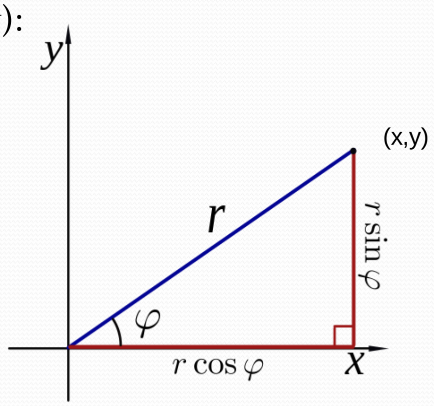
=> cw is the component of v along the direction of w.

Polar Coordinates in $\mathbb{R}^2$

Relation between Polar coordinates (r,φ) and Cartesian coordinates (x,y):

$x = r \cos \varphi$

$y = r \sin \varphi$

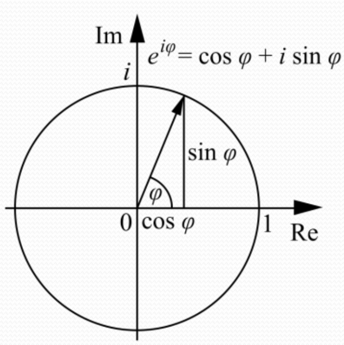


Complex Numbers

Define $e^{i\varphi} = \cos \varphi + i \sin \varphi$

Note: you can write any complex number $z = a + bi$ as:

$z = r\, e^{i\varphi}$, with $r = |z|$

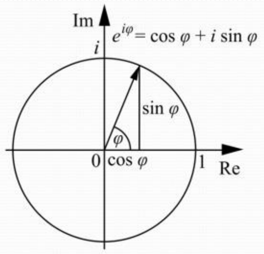


Complex Numbers and Functions

Let $z = r\,e^{i\varphi}$, then $\bar{z} = r\,e^{-i\varphi}$ (alternative notation: z^*)

Let $z_1 = r_1\, e^{-i\varphi_1}$, and $z_2 = r_2\, e^{-i\varphi_2}$, then

$$z_1 z_2 = r_1 r_2\, e^{-i(\varphi_1 + \varphi_2)}$$



Let f a given frequency.

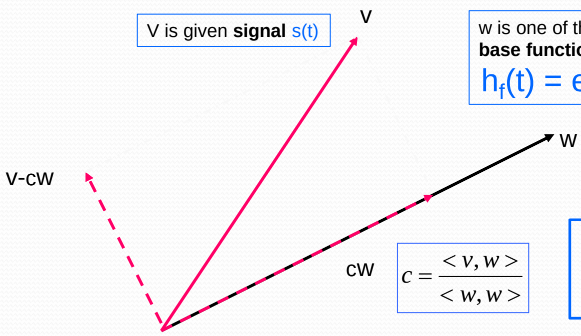
Let $h(t) = e^{i2\pi ft}$ then $h(t) = \cos 2\pi ft + i \sin 2\pi ft$, thus (360^0)

$h(t)$ is a function that is 'repeating' over time with frequency f

Fourier Coefficients

V is given **signal** s(t)

w is one of the generating **base functions**, e.g.:
$h_f(t) = e^{-i2\pi f t}$ with given frequency f



$$c = \frac{<v,w>}{<w,w>}$$

Fourier Coefficient

$$<v,w> = \int s(t).e^{-i2\pi f t}dt$$

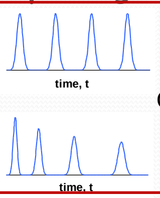
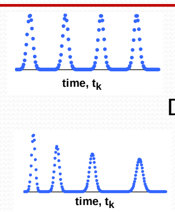
Note <w,w> = 1 for orthonormal basis functions.

Let <.,.> an in-product for our vector space V.
Then we calculate the Fourier coefficient **c** of v in V with respect to (basis) vector w by:

$$c = \frac{<v,w>}{<w,w>}$$

=> cw is the component of v along the direction of w.

Fourier Analysis – Different 'Flavours'

Input Signal in Time Domain				Frequency spectrum	
 Continuous Time	Periodic *(period T)*	FS		Discrete	$c_k = \frac{1}{T} \cdot \int_0^T s(t) \cdot e^{-ik\omega t} dt$
	Aperiodic	FT		Continuous	$S(f) = \int_{-\infty}^{+\infty} s(t) \cdot e^{-i2\pi f t} dt$
 Discrete Time	Periodic *(period T)*	DFS**		Discrete	$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-i\frac{2\pi k n}{N}}$
	Aperiodic	DTFT		Continuous	$S(f) = \sum_{n=-\infty}^{+\infty} s[n] \cdot e^{-i2\pi f n}$
	Aperiodic	DFT**		Discrete	$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-i\frac{2\pi k n}{N}}$

***Note:** $i = \sqrt{-1}$, $\omega = 2\pi/T$, $s[n]=s(t_n)$, N = No. of samples*

*** Calculated using FFT*

A Short History of Fourier Transform (1/2)

- **1669**: Newton: light spectra (*specter* = ghost) but no "frequency" concept (no waves).
- **18th century**: two important problems
  - → celestial bodies orbits: Lagrange, Euler & Clairaut approximate observation data with linear combination of periodic functions; Clairaut,1754(!) first DFT formula.
  - → vibrating strings: Euler describes vibrating string motion by sinusoids (wave equation).
  - → But consensus was: sum of sinusoids *only* represents smooth curves.
- **1807**: Fourier presents his work on heat conduction ⇒ Fourier analysis born.
  - → Diffusion equation ⇔ series (infinite) of sines & cosines.
  - → Strong criticism by peers blocks publication.
  - → **Work published, 1822** *("Theorie Analytique de la chaleur")*.

A Short History of Fourier Transform (2/2)

- **19th / 20th century**: two paths for Fourier analysis - Continuous & Discrete.

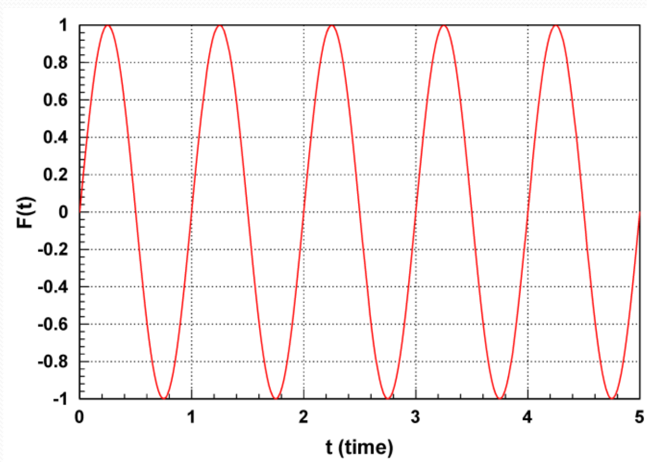
CONTINUOUS

→ Fourier extends the analysis to arbitrary functions (Fourier Transform).
→ Dirichlet, Poisson, Riemann, Lebesgue address Fourier Series convergence.
→ Other FT variants born from varied needs (ex.: Short Time FT - speech analysis).

DISCRETE: Fast calculation methods (FFT) **For us: $O(N^2) \rightarrow O(N\log N)$**

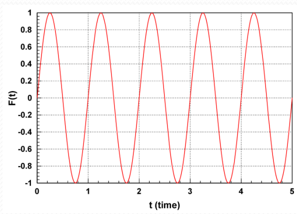
→ **1805** - Gauss, first usage of FFT (manuscript in Latin went unnoticed!!! Published 1866).
→ **1965** - IBM's Cooley & Tukey "rediscover" FFT algorithm (*"An algorithm for the machine calculation of complex Fourier series"*).
→ Other DFT variants for different applications (ex.: Warped DFT - filter design & signal compression).
→ FFT algorithm refined & modified for most computer platforms.
→ *Fastest Fourier Transform in the West (FFTW)*

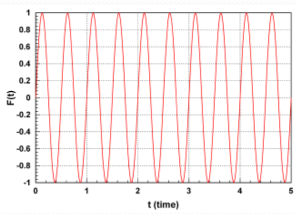
Another Space, Another Base

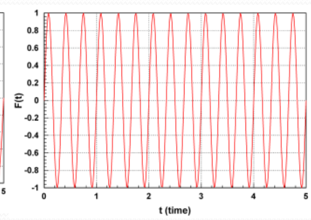


F(t) = sin(2π.t)

Another Space, Another Base





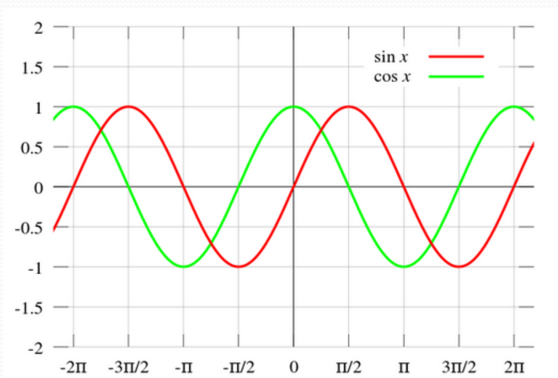


F(t) = sin(2π.t)   F(t) = sin(2π.2t)   F(t) = sin(2π.3t)

F(t) = cos(2π.t)   F(t) = cos(2π.2t)   F(t) = cos(2π.3t)

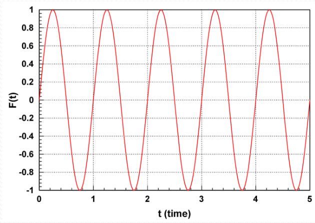
{ cos(2π.kt), sin(2π.kt) $\}_k$ forms an orthogonal basis

Sine vs Cosine Graphs

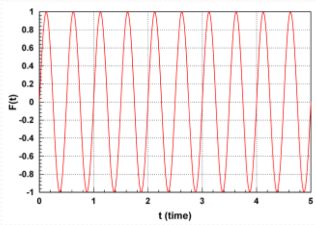


$$\sin(\varphi + \pi/2) = \cos(\varphi)$$

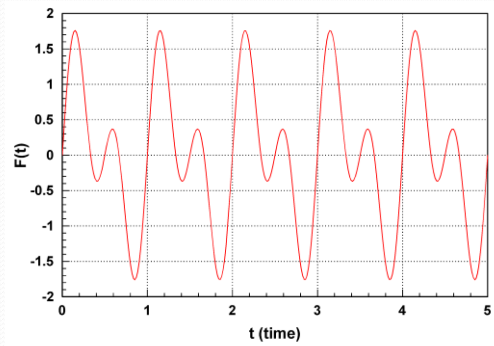
Linear Combination of Functions



F(t) = sin(2π.t)

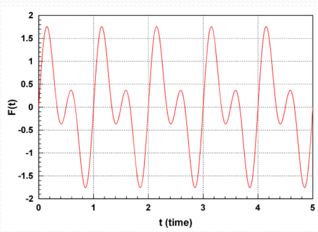


F(t) = sin(2π.2t)

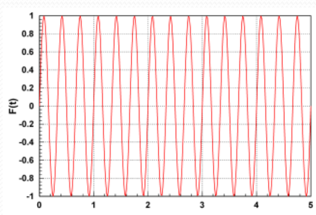


F(t) = sin(2π.t) + sin(2π.2t)

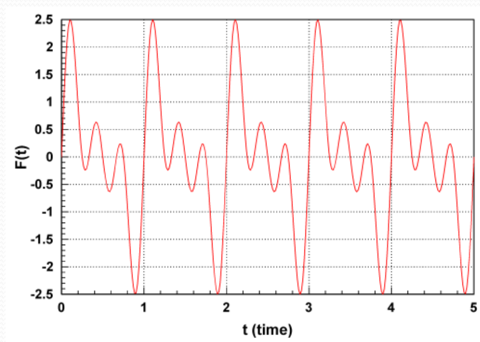
Linear Combination of Functions



F(t) = sin(2π.t) + sin(2π.2t)

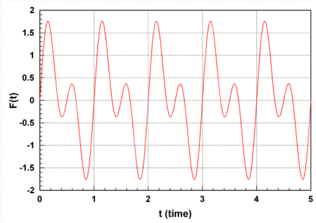


F(t) = sin(2π.3t)

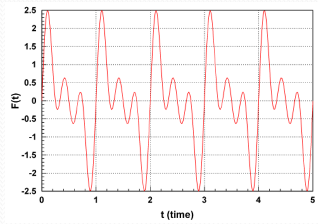


F(t) = sin(2π.t) + sin(2π.2t) + sin(2π.3t)

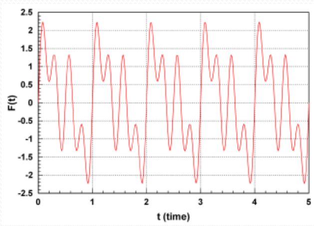
Linear Combination of Functions



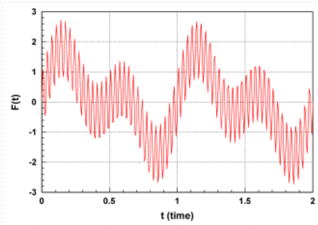
F(t) = sin(2π.t) + sin(2π.2t)



F(t) = sin(2π.t) + sin(2π.2t) + sin(2π.3t)



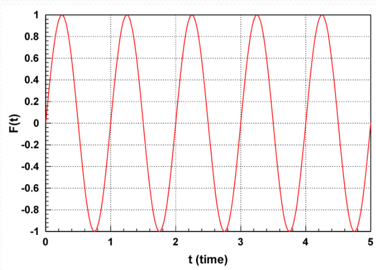
F(t) = sin(2π.t) + sin(2π.2t) + sin(2π.4t)



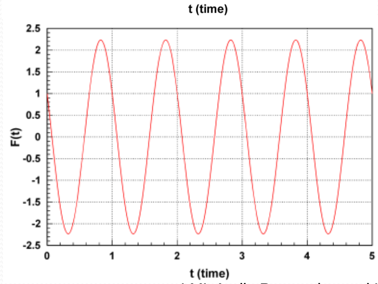
F(t) = sin(2π.t) + sin(2π.2t) + sin(2π.30t)

Linear Combination of Functions

Phase Shift:

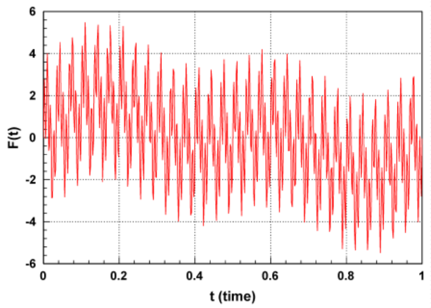


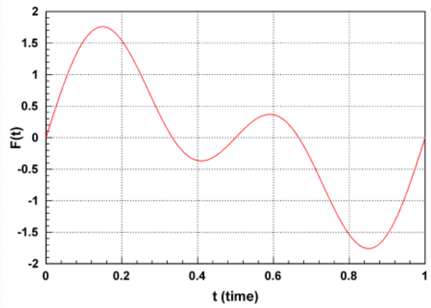
F(t) = sin(2π.t)



F(t) = cos(2π.t) – 2.sin(2π.t)

Low Band Pass Filters

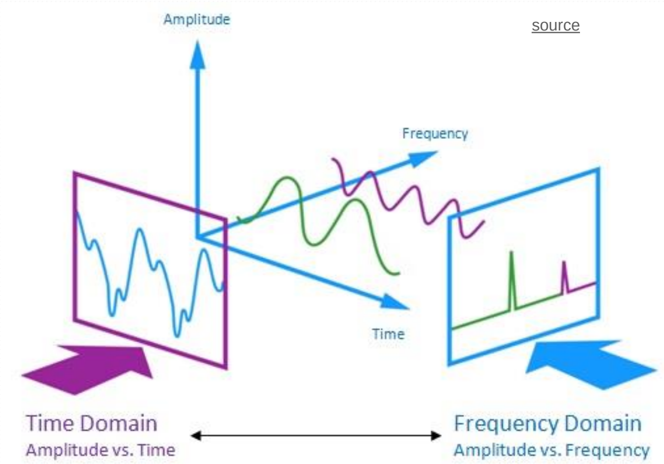




F(t) = sin(2π.t) + sin(2π.2t) + <- low freq.
sin(2π.30t) + sin(2π.120t) <- high freq.

F(t) = sin(2π.t) + sin(2π.2t)

Fourier Series



Fourier Series (FS)

** see next slide*

A periodic function s(t) satisfying Dirichlet's conditions * can be expressed as a Fourier series, with harmonically related sine/cosine terms.

synthesis — Inverse Fourier Transform

$$s(t) = a_0 + \sum_{k=1}^{+\infty} [a_k \cdot \cos(k\omega t) - b_k \cdot \sin(k\omega t)]$$

For all t but discontinuities

t: ~ time
a_0, a_k, b_k : Fourier coefficients.
k: ~ **frequency**, harmonic number
T: period, $\omega = 2\pi/T$

analysis — Fourier Transform

$$a_0 = \frac{1}{T} \cdot \int_0^T s(t)dt$$

(a_0 is signal average over a period, i.e. Direct Current (DC) term & zero-frequency component.)

$$a_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \cos(k\omega t)dt$$

$$-b_k = \frac{2}{T} \cdot \int_0^T s(t) \cdot \sin(k\omega t)dt$$

Note: $\{\cos(k\omega t), \sin(k\omega t)\}_k$ form orthogonal base of function space.

$s(t) \leftrightarrow S(k) = (a_k, b_k)$

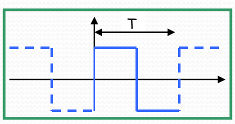
Fourier Series Convergence

Dirichlet conditions

In any period:

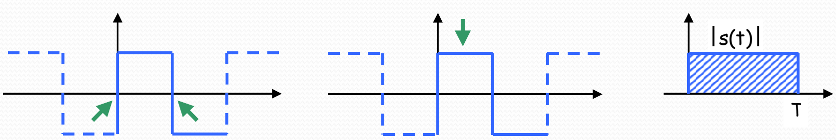
- **(a)** s(t) piecewise-continuous;
- **(b)** s(t) piecewise-monotonic;
- **(c)** s(t) absolutely integrable , $\int_0^T |s(t)|\, dt < \infty$

Example:
square wave



Rate of convergence

if s(t) discontinuous then $|a_k| < M/k$ for large k (M>0)



(a) ☑ (b) ☑ (c) ☑

Fourier Series Analysis - 1

$$a_0 = \frac{1}{T}\cdot\int_0^T s(t)dt$$

$$a_k = \frac{2}{T}\cdot\int_0^T s(t)\cdot\cos(k\omega t)dt$$

$$-b_k = \frac{2}{T}\cdot\int_0^T s(t)\cdot\sin(k\omega t)dt$$

Fourier series of square wave sw(t):

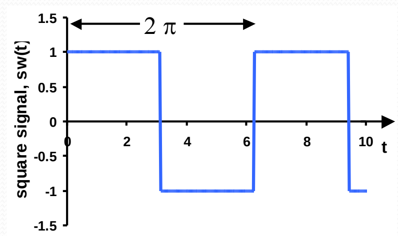
$$a_0 = \frac{1}{2\pi}\cdot\left\{\int_0^{\pi} dt + \int_{\pi}^{2\pi}(-1)dt\right\} = 0$$ *(zero average)*

$$a_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi} \cos kt\, dt - \int_{\pi}^{2\pi} \cos kt\, dt\right\} = 0$$

$$-b_k = \frac{1}{\pi}\cdot\left\{\int_0^{\pi} \sin kt\, dt - \int_{\pi}^{2\pi} \sin kt\, dt\right\} = \ldots = \frac{2}{k\cdot\pi}\cdot(1-\cos k\pi) = $$

$$= \begin{cases} \frac{4}{k\cdot\pi}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

sw(t)



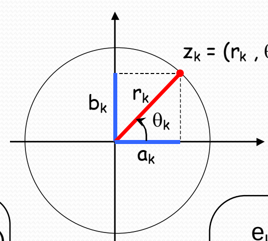
$T = 2\pi \Rightarrow \omega = 1$

$$sw(t) = \frac{4}{\pi}\cdot\sin t + \frac{4}{3\cdot\pi}\cdot\sin 3\cdot t + \frac{4}{5\cdot\pi}\cdot\sin 5\cdot t + \ldots$$

Fourier Series Analysis - 2

Fourier spectrum representations (in **k**)

$$s(t) = \sum_{k=0}^{\infty} v_k(t)$$

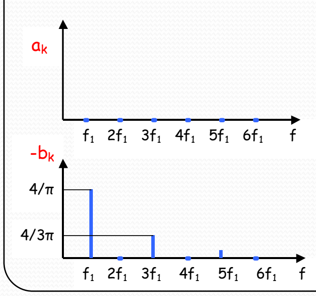


$r_k = \sqrt{a_k^2 + b_k^2}$

$\theta_k = \arctan(b_k/a_k)$

Rectangular

$e_k(t) = a_k\cos(\omega k\, t) - b_k\sin(\omega k\, t)$



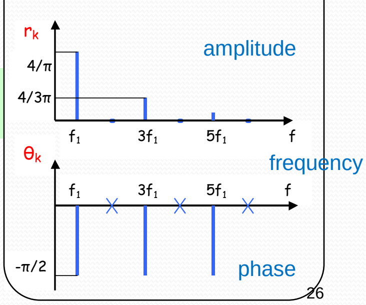
r_K = amplitude,
θ_K = phase

$f_k = k\,\omega/2\pi$

Fourier spectrum of square-wave.

Polar

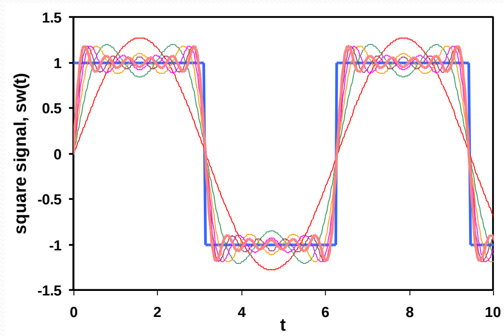
$e_k(t) = r_k \cos(\omega k\, t + \theta_k)$



Fourier Series Synthesis

Square wave reconstruction from spectral terms

$$sw_9(t) = \sum_{k=1}^{9} [-b_k \cdot \sin(kt)]$$

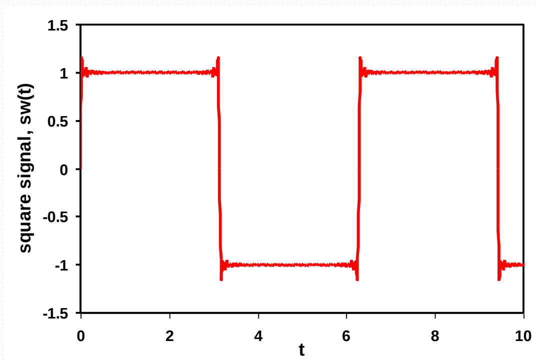


Convergence may be slow (~1/k) - ideally need infinite terms.
Practically, series truncated when remainder below computer tolerance (⇒ *error*). ***BUT*** … Gibbs' Phenomenon.

Gibbs Phenomenon

Overshoot exist at each discontinuity

$$sw_{79}(t) = \sum_{k=1}^{79} [-b_k \cdot \sin(kt)]$$



- First observed by Michelson, 1898. Explained by Gibbs.
- Max overshoot pk-to-pk = 8.95% of discontinuity magnitude.
- FS converges to (-1+1)/2 = 0 at discontinuities, *in this case*.

Complex Fourier Series

Euler's notation:

$e^{-jt} = (e^{jt})^* = \cos(t) - i \cdot \sin(t)$ → "phasor" $\cos(t) = \frac{e^{it} + e^{-it}}{2}$ $\sin(t) = \frac{e^{it} - e^{-it}}{2 \cdot i}$

analysis

$$c_k = \frac{1}{T} \cdot \int_0^T s(t) \cdot e^{-ik\omega t} dt$$

synthesis

$$s(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{ik\omega t}$$

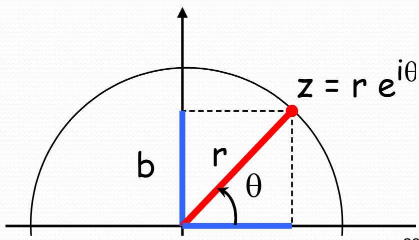
Complex form of FS (Laplace 1782). Harmonics c_k separated by $\Delta f = 1/T$ on frequency plot.

Note: $c_{-k} = (c_k)^*$

Link to FS real coeffs.

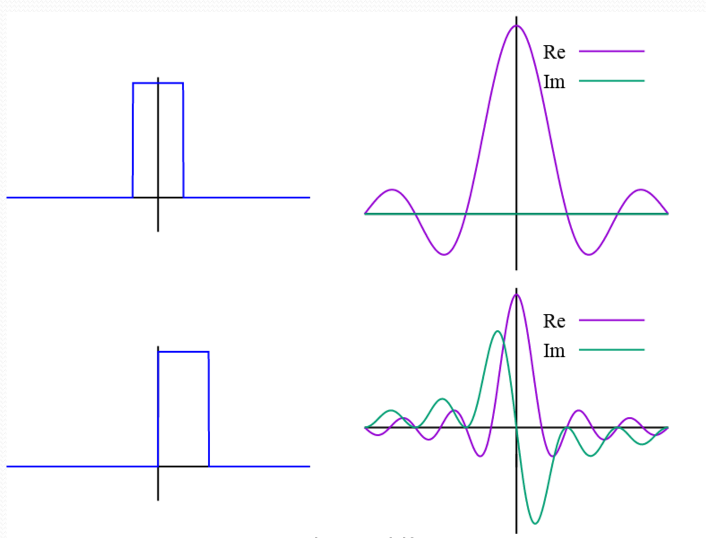
$c_0 = a_0$

$$c_k = \frac{1}{2} \cdot (a_k + i \cdot b_k) = \frac{1}{2} \cdot (a_{-k} - i \cdot b_{-k})$$



Signal

Fourier Transformed Signal

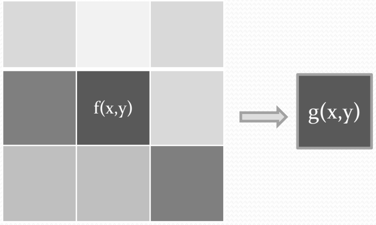


Phase Shift

Fourier Series Properties

	Time (t)	Frequency (f)
Homogeneity	$a \cdot s(t)$	$a \cdot S(f)$
Additivity	$s(t) + u(t)$	$S(f)+U(f)$
Linearity	$a \cdot s(t) + b \cdot u(t)$	$a \cdot S(f) + b \cdot U(f)$
Time reversal	$s(-t)$	$S(-f)$
Multiplication	$s(t) \cdot u(t)$	$\frac{1}{T} \cdot \int_0^T S(f-t) \cdot U(t)\,dt$
Convolution	$\sum_{m=-\infty}^{\infty} s(m)u(t-m)$	$S(f) \cdot U(f)$
Time shifting	$s(t-\bar{t})$	$e^{-i\frac{2\pi f \cdot t}{T}} \cdot S(f)$
Frequency shifting	$e^{+i\frac{2\pi m t}{T}} \cdot s(t)$	$S(f - m)$

Image Processing: Convolutional Filters and Kernels



$$g(x,y) = \omega * f(x,y) = \sum_{dx=-a}^{a} \sum_{dy=-b}^{b} \omega(dx,dy) f(x+dx, y+dy)$$

Operation	Kernel ω	Image result g(x,y)
Identity	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	
Edge detection	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$	
	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$	
	$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$	
Sharpen	$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 0 \end{bmatrix}$	

https://en.wikipedia.org/wiki/Kernel_(image_processing)

Discrete Fourier Series (DFS)

Band-limited signal s[n], period = N.

DFS generate periodic c_k with same signal period

DFS defined as:

FT: analysis

$$\tilde{c}_k = \frac{1}{N}\sum_{n=0}^{N-1} s[n]\cdot e^{-i\frac{2\pi k n}{N}}$$

Note: $\tilde{c}_{k+N} = \tilde{c}_k \Leftrightarrow$ **same period N**
i.e. time periodicity propagates to frequencies!

Orthogonality in DFS:

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{i\frac{2\pi\, n(k-m)}{N}} = \delta_{k,m}$$

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Kronecker's delta

IFT: synthesis

$$s[n] = \sum_{k=0}^{N-1} \tilde{c}_k \cdot e^{i\frac{2\pi k n}{N}}$$

N consecutive samples of s[n] completely describe s in time or frequency domains.

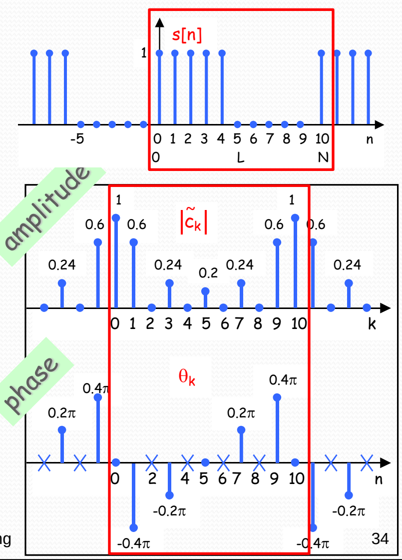
Synthesis: finite sum $\Leftarrow$ band-limited s[n]

Discrete Fourier Series Analysis

DFS of periodic discrete 1-Volt square-wave

s[n]: period N, duty factor L/N

$$\tilde{c}_k = \begin{cases} \dfrac{L}{N}, & k = 0, +N, \pm 2N, \ldots \\[2ex] \dfrac{e^{-i\frac{\pi k (L-1)}{N}}}{N} \cdot \dfrac{\sin\left(\dfrac{\pi k L}{N}\right)}{\sin\left(\dfrac{\pi k}{N}\right)}, & \text{otherwise} \end{cases}$$



Discrete signals ⇒ periodic frequency spectra.
Compare to continuous rectangular function (slide # 20)

Some Discrete Fourier Series Properties

	Time (n)	Frequency (k)
Homogeneity	$a \cdot s[n]$	$a \cdot S(k)$
Additivity	$s[n] + u[n]$	$S(k)+U(k)$
Linearity	$a \cdot s[n] + b \cdot u[n]$	$a \cdot S(k)+b \cdot U(k)$
Multiplication	$s[n] \cdot u[n]$	$\frac{1}{N} \cdot \sum_{h=0}^{N-1} S(h)U(k-h)$
Convolution	$\sum_{m=0}^{N-1} s[m] \cdot u[n-m]$	$S(k) \cdot U(k)$
Time shifting	$s[n - m]$	$e^{-i\frac{2\pi\, k \cdot m}{T}} \cdot S(k)$
Frequency shifting	$e^{+i\frac{2\pi\, h\, n}{T}} \cdot s[n]$	$S(k - h)$

References

1. Serge Lang, *Linear Algebra*, Springer Verlag New York Inc, 3rd Edition 1987.
2. Dr M.E. Angoletta at DISP2003, a DSP course given by CERN and University of Lausanne (UNIL)

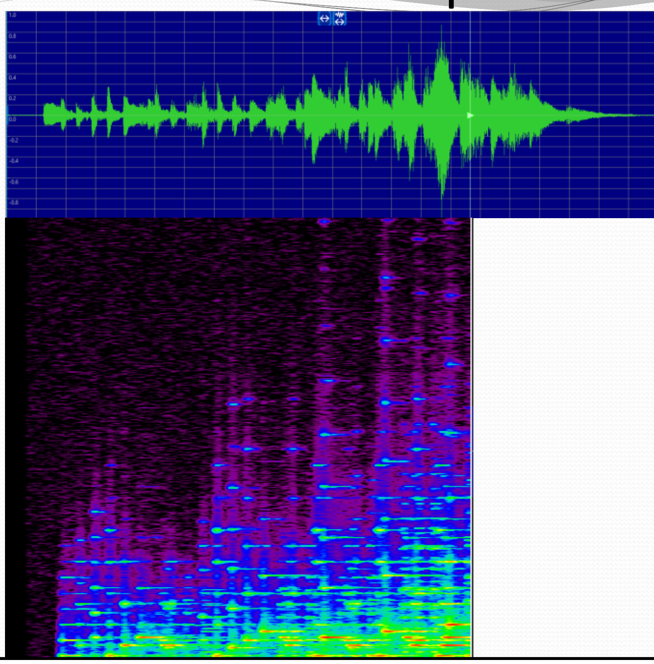
Schedule (tentative, visit regularly):

5-9	Organization and Introduction
12-9	Audio Production and Processing
19-9	ADC and an Algebraic Introduction to FT
26-9	FFT
3-10	No class: Leidens Ontzet.
10-10	Project Proposals (presentations by students)
17-10	Audio Features & student paper selection
24-10	Machine Learning
31-10	Student Paper Presentations I
7-11	Student Paper Presentations II
14-11	Student Paper Presentations III
21-11	Student Paper Presentations IV
28-11	**TBA**
5-12	Final Project Presentations Demo's
19-12	Project Deliverables: - Final Project - Scientific/technical paper (4-8 pages) - Code - Web Site (or github)

Assignments (Workshops@Home):

1. Vocal Tract Workshop. Due: 21-9 2023,
2. FFT Workshop and audio_data. Due: 9-10 2023.
3. Audio Features Workshop. Due TBA
4. Machine Learning Workshop. Due TBA.

FFT Workshop



API Project Proposals
(October 10th 2023)

5 minute Presentations (4 slides) addressing:

- Title + group members (1 – 4 members)
- Problem description
- Challenges
- What will be the goal for the Final Project Presentation/Demo
- Note: If the group consists of more than 1 member, add a 5th slide with an initial global division of the work between project members. This slide does not have to be presented.

Each API Project member should submit a copy of the pdf with the slides of the API Project Proposal Presentation on Bright space before October 9th 2023.

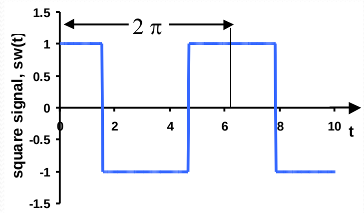
API Project Proposals
(October 10th 2023)

For inspiration:

- See previous projects on https://www.liacs.nl/~erwin/api
- International Society for Music Information Retrieval (ISMIR) http://www.ismir.net/conferences/
- INTERSPEECH https://www.isca-speech.org/iscaweb/index.php/online-archive
- Online proceedings:
  - https://dblp.org/db/conf/index.html
  - https://dblp.org/db/conf/interspeech/index.html
  - https://dblp.org/search?q=eurasip
  - Etc.

Fourier Series Time Shifting

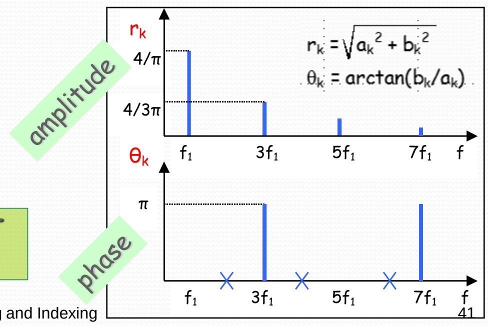
FS of even function:
$\pi/2$-advanced square-wave



$a_0 = 0$ *(zero average)*

$$a_k = \begin{cases} \frac{4}{k \cdot \pi}, & k \text{ odd}, \; k = 1,5,9\ldots \\ -\frac{4}{k \cdot \pi}, & k \text{ odd}, \; k = 3,7,11\ldots \\ 0, & k \text{ even}. \end{cases}$$

$-b_k = 0$ *(even function: $s(-x) = s(x)$)*



Note: amplitudes unchanged ***BUT***
phases advance by $k \cdot \pi/2$.

Fourier Transforms

Let s(.) a signal in the time domain: s(t) values as a function of time t $(-\infty < t < \infty)$

The same signal can be described as amplitudes and phases (complex values)
S(.) in the frequency domain: S(f) values as a function of frequency f $(-\infty < f < \infty)$

One can transform the representation s(t) in the time domain to the representation S(f) in the frequency domain by using the Fourier Transform equation:

$$S(f) = \int_{-\infty}^{\infty} s(t).e^{-2\pi ift} dt$$

And back, using the inverse FT-equation:

$$s(t) = \int_{-\infty}^{\infty} S(f).e^{2\pi ift} df$$