Logica (I&E)

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`http://liacs.leidenuniv.nl/~vlietrvan1/logica/`

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2. Predicate logic
2.1. The need for a richer language
2.2. Predicate logic as a formal language

Als je niet kan winnen, moet je zorgen dat je niet verliest.

2. Predicate logic = *first-order logic*

2.1. The need for a richer language

Every student is younger than some instructor.

Predicate: ‘function of one or more objects, with values in $\{\text{true}, \text{false}\}$’

$S(\mathit{andy})$, $I(\mathit{paul})$, $Y(\mathit{andy}, \mathit{paul})$

How to express ‘every’ and ‘some’?

With variables:

$$\begin{aligned} S(x) &: \ x \text{ is a student} \\ I(x) &: \ x \text{ is an instructor} \\ Y(x,y) &: \ x \text{ is younger than } y \end{aligned}$$

And $\forall$ and $\exists$:

$$\forall x(S(x) \rightarrow (\exists y(I(y) \wedge Y(x,y))))$$

Not all birds can fly.

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi$$

Sound and complete

Example.

No books are gaseous.
Dictionaries are books.
Therefore, no dictionary is gaseous.

Example.

Every child is younger than its mother.

Andy and Paul have the same maternal grandmother.

Example.

Andy and Paul have the same maternal grandmother.

Special binary predicate equality:
$x = u$ instead of $= (x, y)$

Function symbol

Function of zero or more objects, with value an object

The grade obtained by student x in course y

Example.

Ann likes Mary's brother

$g(x, y)$

2.2. Predicate logic as a formal language

Terms and formulas

Terms: $a, p,\ x, y,\ m(a), g(x, y)$

Formulas: $Y(x, m(x))$

Vocabulary:
Predicate symbols $\mathcal{P}$
Function symbols (including constants) $\mathcal{F}$

2.2.1. Terms

Definition 2.1. Terms over $\mathcal{F}$ are defined as follows.

- Any variable is a term.
- If $c \in \mathcal{F}$ is a nullary function, then c is a term.
- If $t_1, t_2, \ldots, t_n$ are terms and $f \in \mathcal{F}$ has arity $n > 0$, then $f(t_1, t_2, \ldots, t_n)$ is a term.
- Nothing else is a term.

Dependent on set $\mathcal{F}$

$$t ::= x \quad | \quad c \quad | \quad f(t, \ldots, t)$$

Example 2.2.

Suppose:
n nullary
f unary
g binary

$g(f(n), n)$: OK
$f(g(n, f(n)))$: OK
$g(n)$: not OK
$f(f(n), n)$: not OK

$*(-(2, +(s(x), y)), x)$

2.2.2. Formulas

Definition 2.3. Formulas over $(\mathcal{F}, \mathcal{P})$ are defined as follows.

- If $P \in \mathcal{P}$ is a predicate symbol of arity $n \geq 1$, and if $t_1, t_2, \ldots, t_n$ are terms over $\mathcal{F}$, then $P(t_1, t_2, \ldots, t_n)$ is a formula.
- If ϕ is a formula, then so is $(\neg\phi)$
- If ϕ and ψ are formulas, then so are $\phi \wedge \psi$, $\phi \vee \psi$ and $\phi \rightarrow \psi$.
- If ϕ is a formula and x is a variable, then $(\forall x \phi)$ and $(\exists x \phi)$ are formulas.
- Nothing else is a formula.

$$\phi \;::=\; P(t_1, t_2, \ldots, t_n) \;\mid\; (\neg\phi) \;\mid\; (\phi \wedge \phi) \;\mid\; (\phi \vee \phi) \;\mid\; (\phi \rightarrow \phi) \;\mid\; (\forall x \phi) \;\mid\; (\exists x \phi)$$

Convention 2.4. Binding priorities

- $\neg$, $\forall y$ and $\exists y$ bind most tightly,
- then $\vee$ and $\wedge$
- then $\rightarrow$, which is right associative.

Example 2.5. Translate

Every son of my father is my brother.

into predicate logic. With ‘father’ either as predicate or as function symbol:

1. Predicate. . .

2. Function symbol. . .

2.2.3. Free and bound variables

Two kinds of truth:
A formula can be true in a particular model or for all models:

$$\forall x(S(x, f(m)) \to B(x, m) \vee x = m)$$

$$P(c) \wedge \forall y(P(y) \to Q(y)) \to Q(c)$$

Parse tree of

$$\forall x((P(x) \to Q(x)) \wedge S(x, y))$$

N.B.: function symbols and predicate symbols may have $n > 2$ children in parse tree.

Variables occur next to $\forall$ or $\exists$, or as leafs.

Definition 2.6. Let ϕ be a formula in predicate logic.
An occurrence of x in ϕ is free in ϕ if it is a leaf node in the parse tree of ϕ such that there is no path upwards from that node x to a node $\forall x$ or $\exists x$.
Otherwise, that occurrence of x is called bound.
For $\forall x \phi$ or $\exists x \phi$, we say that ϕ – minus any of ϕ's subformulas $\exists x \psi$ or $\forall x \psi$ – is the scope of $\forall x$, respectively $\exists x$.

Three occurrences of x...
One occurrence of y...

Example.

Parse tree of

$$(\forall x(P(x) \wedge Q(x))) \rightarrow (\neg P(x) \vee Q(y))$$

Free and bound variables. . .

Substitution

Variables are placeholders

Definition 2.7.
Given a variable x, a term t and a formula ϕ, we define $\phi[t/x]$ to be the formula obtained by replacing each free occurrence of variable x in ϕ with t.

Example.

$$\phi = \forall x((P(x) \to Q(x)) \wedge S(x, y))$$

$$\phi[f(x, y)/x] = \ldots$$

Example.

$$\phi = (\forall x(P(x) \wedge Q(x))) \to (\neg P(x) \vee Q(y))$$

$$\phi[f(x,y)/x] = \ldots$$