Fundamentele Informatica 3

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http://www.liacs.leidenuniv.nl/~vlietrvan1/fi3/

Rudy van Vliet
kamer 124 Snellius, tel. 071-527 5777
rvvliet(at)liacs(dot)nl

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9. Undecidable Problems
9.4. Post's Correspondence Problem
9.5. Undecidable Problems
Involving Context-Free Languages

Huiswerkopgave 3

Reducties en (on-)beslisbaarheid

A slide from lecture 10

9.4. Post's Correspondence Problem

Instance:

10	01	0	100	1
101	100	10	0	010

A slide from lecture 10

Instance:

10	01	0	100	1
101	100	10	0	010

Match:

10	1	01	0	100	100	0	100
101	010	100	10	0	0	10	0

A slide from lecture 10

Definition 9.14. Post's Correspondence Problem

An instance of Post's correspondence problem (*PCP*) is a set

$$\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\}$$

of pairs, where $n \geq 1$ and the α_i's and β_i's are all nonnull strings over an alphabet Σ.

The decision problem is this:

Given an instance of this type, do there exist a positive integer k and a sequence of integers $i_1, i_2, \ldots, i_k$, with each i_j satisfying $1 \leq i_j \leq n$, satisfying

$$\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k} = \beta_{i_1}\beta_{i_2}\ldots\beta_{i_k} \quad ?$$

$i_1, i_2, \ldots, i_k$ need not all be distinct.

A slide from lecture 10

Definition 9.14. Post's Correspondence Problem (continued)

An instance of the modified Post's correspondence problem (*MPCP*) looks exactly like an instance of *PCP*, but now the sequence of integers is required to start with 1. The question can be formulated this way:

Do there exist a positive integer k and a sequence $i_2, i_3, \ldots, i_k$ such that

$$\alpha_1 \alpha_{i_2} \ldots \alpha_{i_k} = \beta_1 \beta_{i_2} \ldots \beta_{i_k} \quad ?$$

(Modified) correspondence system, match.

A slide from lecture 10

Theorem 9.15. *MPCP* $\leq$ *PCP*

Proof.

For instance

$$I = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\}$$

of *MPCP*, construct instance $J = F(I)$ of *PCP*, such that I is yes-instance, if and only if J is yes-instance.

Theorem 9.16. *Accepts* $\leq$ *MPCP*

The technical details of the proof of this result do not have to be known for the exam. However, one must be able to carry out the construction below.

Proof. . .

For every instance (T, w) of *Accepts*, construct instance $F(T, w)$ of *MPCP*, such that . . .

A slide from lecture 3

Notation:

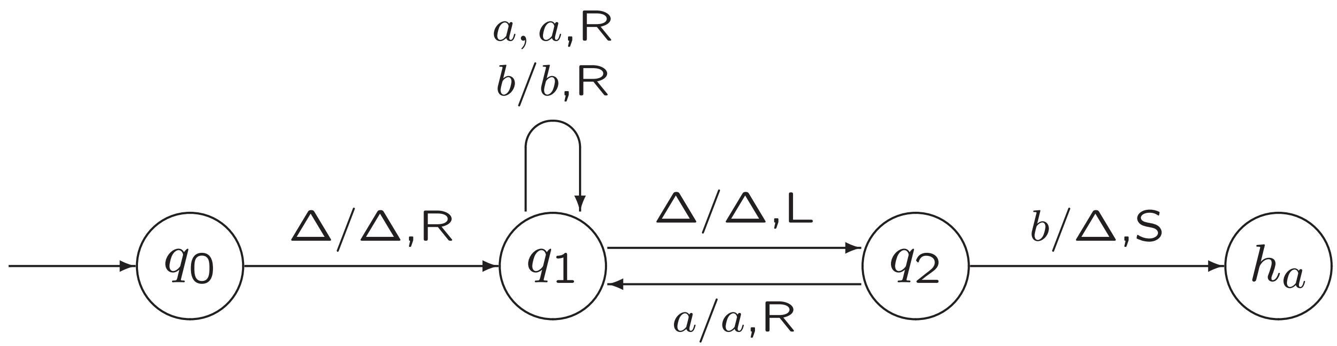
description of tape contents: $x\underline{\sigma}y$ or $x\underline{y}$

configuration $xqy = xqy\Delta = xqy\Delta\Delta$

initial configuration corresponding to input x: $q_0\Delta x$

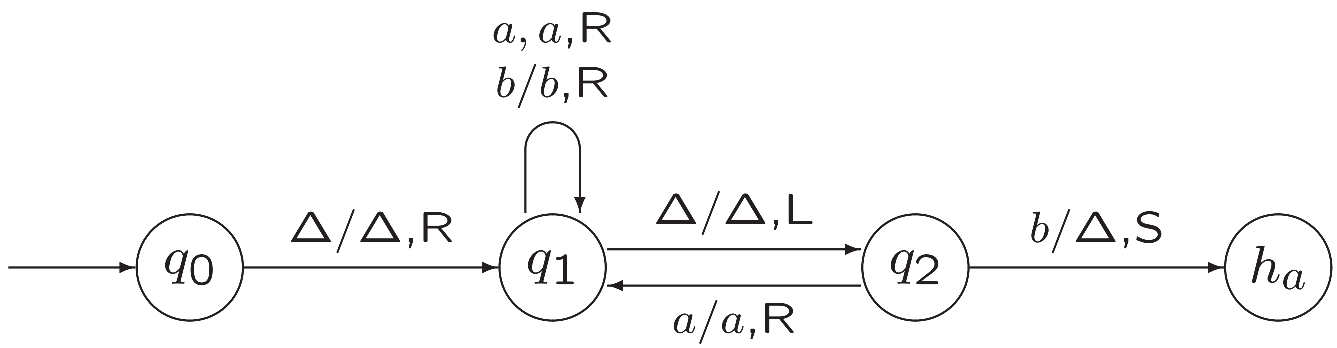
In the third edition of the book, a configuration is denoted as $(q, x\underline{y})$ or $(q, x\underline{\sigma}y)$ instead of xqy or $xq\sigma y$.
In one case, we still use this old notation.

Example 9.18. A Modified Correspondence System for a TM



T accepts . . .

Example 9.18. A Modified Correspondence System for a TM



T accepts all strings in $\{a, b\}^*$ ending with b.

Proof of Theorem 9.16. (continued)

Take

$(\alpha_1, \beta_1) = (\#, \#q_0\Delta w\#)$

Pairs of type 1: (a, a) for every $a \in \Gamma \cup \{\Delta\}$, and $(\#, \#)$

Pairs of type 2: corresponding to moves in T, e.g.,

(qa, bp), if $\delta(q, a) = (p, b, R)$

(cqa, pcb), if $\delta(q, a) = (p, b, L)$

Proof of Theorem 9.16. (continued)

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$(q\#, pa\#)$, if $\delta(q, \Delta) = (p, a, S)$

Proof of Theorem 9.16. (continued)

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Pairs of type 3: for every $a, b \in \Gamma \cup \{\Delta\}$, the pairs

$(h_a a, h_a)$, (ah_a, h_a), $(ah_a b, h_a)$

One pair of type 4:

$(h_a\#\#, \#)$

Proof of Theorem 9.16. (continued)

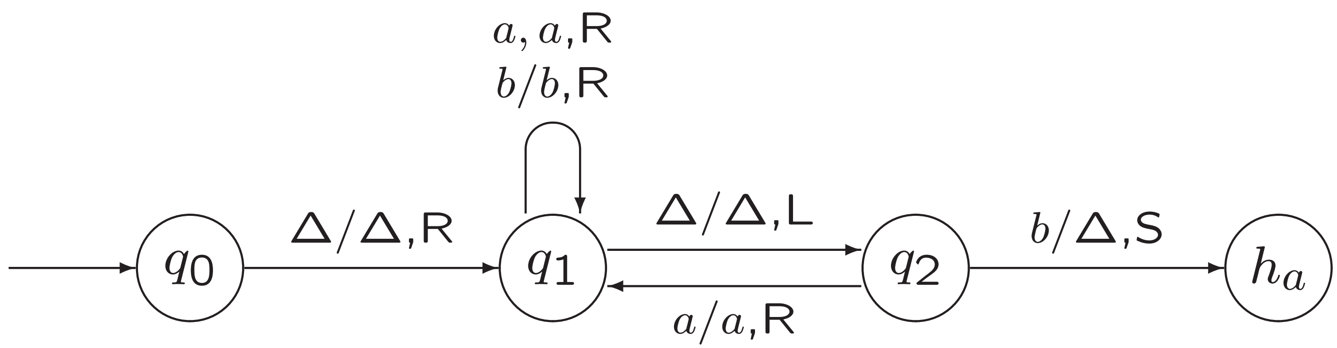
Two assumptions in book:

1. T never moves to h_r
2. $w \neq \Lambda$ (i.e., special initial pair if $w = \Lambda$)

These assumptions are not necessary...

Theorem 9.17.
Post's correspondence problem is undecidable.

Example 9.18. A Modified Correspondence System for a TM



T accepts all strings in $\{a, b\}^*$ ending with b.

Pairs of type 2:

$$(q_0\Delta, \Delta q_1) \quad (q_0\#, \Delta q_1\#) \quad (q_1 a, a q_1) \quad (q_1 b, b q_1)$$
$$(a q_1 \Delta, q_2 a \Delta) \quad (b q_1 \Delta, q_2 b \Delta) \quad \ldots$$

Study this example yourself.

9.5. Undecidable Problems Involving Context-Free Languages

For an instance

$$\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\}$$

of *PCP*, let. . .

CFG G_α be defined by productions. . .

For an instance

$$\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\}$$

of PCP, let. . .

CFG G_α be defined by productions

$$S_\alpha \to \alpha_i S_\alpha c_i \mid \alpha_i c_i \quad (1 \le i \le n)$$

Example derivation:

$S_\alpha \Rightarrow \alpha_2 S_\alpha c_2 \Rightarrow \alpha_2 \alpha_5 S_\alpha c_5 c_2 \Rightarrow \alpha_2 \alpha_5 \alpha_1 S_\alpha c_1 c_5 c_2 \Rightarrow \alpha_2 \alpha_5 \alpha_1 \alpha_3 c_3 c_1 c_5 c_2$

Unambiguous

For an instance

$$\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\}$$

of *PCP*, let. . .

CFG G_α be defined by productions

$$S_\alpha \to \alpha_i S_\alpha c_i \mid \alpha_i c_i \quad (1 \le i \le n)$$

CFG G_β be defined by productions

$$S_\beta \to \beta_i S_\beta c_i \mid \beta_i c_i \quad (1 \le i \le n)$$

Example.

Let I be the following instance of PCP:

10	01	0	100	1
101	100	10	0	010

G_α and G_β. . .

Theorem 9.20.
These two problems are undecidable:

1. *CFGNonEmptyIntersection*:
   Given two CFGs G_1 and G_2, is $L(G_1) \cap L(G_2)$ nonempty?

2. *IsAmbiguous*:
   Given a CFG G, is G ambiguous?

Proof. . .

Theorem 9.20.
This problem is undecidable:

1. *CFGNonEmptyIntersection*:
   Given two CFGs G_1 and G_2, is $L(G_1) \cap L(G_2)$ nonempty?

Alternative proof. . .

Let CFG G_1 be defined by productions

$$S_1 \to \alpha_i S_1 \beta_i^r \mid \alpha_i \# \beta_i^r \quad (1 \le i \le n)$$

Let CFG G_2 be defined by productions

$$S_2 \to a S_2 a \mid b S_2 b \mid a \# a \mid b \# b$$

Let T be TM, let x be string accepted by T, and let

$$z_0 \vdash z_1 \vdash z_2 \vdash z_3 \ldots \vdash z_n$$

be ‘succesful computation’ of T for x,
i.e., $z_0 = q_0 \Delta x$
and z_n is accepting configuration.

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and z_n is accepting configuration.

Successive configurations z_i and z_{i+1} are almost identical; hence the language

$$\{z\#z'\# \mid \ z \text{ and } z' \text{ are config's of } T \text{ for which } z \vdash z'\}$$

cannot be described by CFG,
cf. $XX = \{xx \mid x \in \{a,b\}^*\}$.

Let T be TM, let x be string accepted by T, and let

$$z_0 \vdash z_1 \vdash z_2 \vdash z_3 \ldots \vdash z_n$$

be ‘succesful computation’ of T for x,
i.e., $z_0 = q_0 \Delta x$
and z_n is accepting configuration.

On the other hand, $z_i \# z_{i+1}^r$ is almost a palindrome, and palindromes *can* be described by CFG.

Lemma.

The language

$$L_1 = \{z\#(z')^r\# \mid z \text{ and } z' \text{ are config's of } T \text{ for which } z \vdash z'\}$$

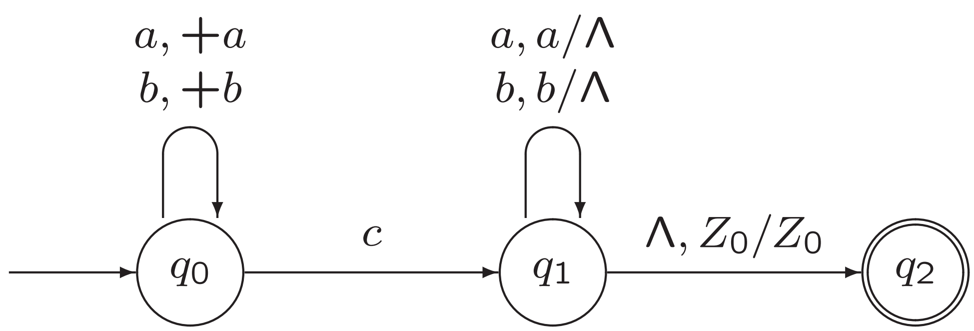
is context-free.

Proof. . .

A slide from lecture 1

Example 5.3. A Pushdown Automaton Accepting *SimplePal*

SimplePal $= \{xcx^r \mid x \in \{a,b\}^*\}$



Definition 9.21. Valid Computations of a TM

Let $T = (Q, \Sigma, \Gamma, q_0, \delta)$ be a Turing machine.

A *valid computation* of T is a string of the form

$$z_0 \# z_1^r \# z_2 \# z_3^r \ldots \# z_n \#$$

if n is even, or

$$z_0 \# z_1^r \# z_2 \# z_3^r \ldots \# z_n^r \#$$

if n is odd,
where in either case, $\#$ is a symbol not in Γ,
and the strings z_i represent successive configurations of T on some input string x, starting with the initial configuration z_0 and ending with an accepting configuration.

The set of valid computations of T will be denoted by C_T.

Theorem 9.22.

For a TM $T = (Q, \Sigma, \Gamma, q_0, \delta)$,
• the set C_T of valid computations of T is the intersection of two context-free languages,
• and its complement C'_T is a context-free language.

Proof. . .

Theorem 9.22.

For a TM $T = (Q, \Sigma, \Gamma, q_0, \delta)$,
• the set C_T of valid computations of T is the intersection of two context-free languages,
• and its complement C'_T is a context-free language.

Proof. Let

$$\begin{aligned}
L_1 &= \{z\#(z')^r\# \mid \ z \text{ and } z' \text{ are config's of } T \text{ for which } z \vdash z'\} \\
L_2 &= \{z^r\#z'\# \mid \ z \text{ and } z' \text{ are config's of } T \text{ for which } z \vdash z'\} \\
I &= \{z\# \mid \ z \text{ is initial configuration of } T\} \\
A &= \{z\# \mid \ z \text{ is accepting configuration of } T\} \\
A_1 &= \{z^r\# \mid \ z \text{ is accepting configuration of } T\}
\end{aligned}$$

$$C_T = L_3 \cap L_4$$

where

$$\begin{aligned} L_3 &= I L_2^*(A_1 \cup \{\Lambda\}) \\ L_4 &= L_1^*(A \cup \{\Lambda\}) \end{aligned}$$

for each of which we can algorithmically construct a CFG

If $x \in C'_T$ (i.e., $x \notin C_T$), then. . .

If $x \in C'_T$ (i.e., $x \notin C_T$), then

1. Either, x does not end with #

Otherwise, let $x = z_0 \# z_1 \# \ldots \# z_k \#$

(no reversed strings in this partitioning)

2. Or, for some even i, z_i is not configuration of T

3. Or, for some odd i, z_i^r is not configuration of T

4. Or z_0 is not initial configuration of T

5. Or z_k is neither accepting configuration, nor the reverse of one

6. Or, for some even i, $z_i \not\vdash z_{i+1}^r$

7. Or, for some odd i, $z_i^r \not\vdash z_{i+1}$

If $x \in C'_T$ (i.e., $x \notin C_T$), then

1. Either, x does not end with #

Otherwise, let $x = z_0 \# z_1 \# \ldots \# z_k \#$

2. Or, for some even i, z_i is not configuration of T

3. Or, for some odd i, z_i^r is not configuration of T

4. Or z_0 is not initial configuration of T

5. Or z_k is neither accepting configuration, nor the reverse of one

6. Or, for some even i, $z_i \not\vdash z_{i+1}^r$

7. Or, for some odd i, $z_i^r \not\vdash z_{i+1}$

Hence, C'_T is union of seven context-free languages,
for each of which we can algorithmically construct a CFG