Fundamentele Informatica 3

voorjaar 2016

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college 10, 11 april 2016

9. Undecidable Problems
 9.3. More Decision Problems Involving Turing Machines
 9.4. Post's Correspondence Problem

Definition 9.6. Reducing One Decision Problem to Another, and Reducing One Language to Another

Suppose P_1 and P_2 are decision problems. We say P_1 is reducible to P_2 ($P_1 \leq P_2$)

- if there is an algorithm
- that finds, for an arbitrary instance I of P_1 , an instance F(I) of P_2 ,
- such that

for every I the answers for the two instances are the same, or I is a yes-instance of P_1

if and only if F(I) is a yes-instance of P_2 .

Theorem 9.7. Suppose $L_1 \subseteq \Sigma_1^*$, $L_2 \subseteq \Sigma_2^*$, and $L_1 \leq L_2$. If L_2 is recursive, then L_1 is recursive.

Suppose P_1 and P_2 are decision problems, and $P_1 \leq P_2$. If P_2 is decidable, then P_1 is decidable.

Proof...

Theorem 9.8. Both Accepts and Halts are undecidable.

Proof.

- 1. Prove that Self-Accepting \leq Accepts ...
- 2. Prove that $Accepts \leq Halts \dots$

9.3. More Decision Problems Involving Turing Machines

Accepts: Given a TM T and a string x, is $x \in L(T)$? Instances are . . .

Halts: Given a TM T and a string x, does T halt on input x? Instances are . . .

Self-Accepting: Given a TM T, does T accept the string e(T)? Instances are ...

Now fix a TM T: T-Accepts: Given a string x, does T accept x ? Instances are ... Decidable or undecidable ? (cf. **Exercise 9.7.**)

Exercise 9.7.

As discussed at the beginning of Section 9.3, there is at least one TM T such that the decision problem

"Given w, does T accept w?"

is unsolvable.

Show that every TM accepting a nonrecursive language has this property.

1. Accepts-A: Given a TM T, is $\Lambda \in L(T)$?

Proof.

1. Prove that $Accepts \leq Accepts - \Lambda$. . .

Reduction from *Accepts* to *Accepts*- Λ .

Instance of *Accepts* is (T_1, x) for TM T_1 and string x. Instance of *Accepts*- Λ is TM T_2 .

 $T_2 = F(T_1, x) =$ $Write(x) \rightarrow T_1$

 T_2 accepts Λ , if and only if T_1 accepts x.

If we had an algorithm/TM A_2 to solve Accepts- Λ , then we would also have an algorithm/TM A_1 to solve Accepts, as follows:

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A<sub>1</sub>:
Given instance (T_1, x) of Accepts,
1. construct T_2 = F(T_1, x);
2. run A<sub>2</sub> on T<sub>2</sub>.
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A_1 answers 'yes' for (T_1, x),
if and only if A_2 answers 'yes' for T_2,
if and only T_2 accepts \Lambda,
if and only if T_1 accepts x.
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2. AcceptsEverything: Given a TM T with input alphabet Σ , is $L(T) = \Sigma^*$?

Proof.

2. Prove that Accepts- $\Lambda \leq AcceptsEverything \dots$

3. Subset: Given two TMs T_1 and T_2 , is $L(T_1) \subseteq L(T_2)$?

Proof.

3. Prove that $AcceptsEverything \leq Subset \dots$

4. Equivalent: Given two TMs T_1 and T_2 , is $L(T_1) = L(T_2)$

Proof.

4. Prove that $Subset \leq Equivalent \dots$

5. WritesSymbol:

Given a TM T and a symbol a in the tape alphabet of T, does T ever write a if it starts with an empty tape ?

Proof.

5. Prove that Accepts- $\Lambda \leq WritesSymbol \dots$

AtLeast10MovesOn- Λ : Given a TM T, does T make at least ten moves on input Λ ?

WritesNonblank: Given a TM T, does T ever write a nonblank symbol on input Λ ?

Theorem 9.10.

The decision problem *WritesNonblank* is decidable.

Proof...

Definition 9.11. A Language Property of TMs

A property R of Turing machines is called a *language property* if, for every Turing machine T having property R, and every other TM T_1 with $L(T_1) = L(T)$, T_1 also has property R.

A language property of TMs is *nontrivial* if there is at least one TM that has the property and at least one that doesn't.

In fact, a language property is a property of the languages accepted by TMs.

Theorem 9.12. Rice's Theorem

If R is a nontrivial language property of TMs, then the decision problem

 P_R : Given a TM T, does T have property R ?

is undecidable.

Proof...

Prove that Accepts- $\Lambda \leq P_R \ldots$

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(or that Accepts-\Lambda \leq P_{not-R} \dots)
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 T_2 highly unspecified...

Definition 9.6. Reducing One Decision Problem to Another, and Reducing One Language to Another

Suppose P_1 and P_2 are decision problems. We say P_1 is reducible to P_2 ($P_1 \leq P_2$)

- if there is an algorithm
- that finds, for an arbitrary instance I of P_1 , an instance F(I) of P_2 ,
- such that

for every I the answers for the two instances are the same, or I is a yes-instance of P_1

if and only if F(I) is a yes-instance of P_2 .

Examples of decision problems to which Rice's theorem can be applied:

- 1. Accepts-L: Given a TM T, is L(T) = L? (assuming ...)
- 2. AcceptsSomething: Given a TM T, is there at least one string in L(T) ?
- 3. AcceptsTwoOrMore: Given a TM T, does L(T) have at least two elements ?
- 4. AcceptsFinite: Given a TM T, is L(T) finite ?
- 5. *AcceptsRecursive*:

Given a TM T, is L(T) recursive ? (note that ...)

All these problems are undecidable.

Rice's theorem cannot be applied (directly)

• if the decision problem does not involve just one TM *Equivalent*: Given two TMs T_1 and T_2 , is $L(T_1) = L(T_2)$ Rice's theorem cannot be applied (directly)

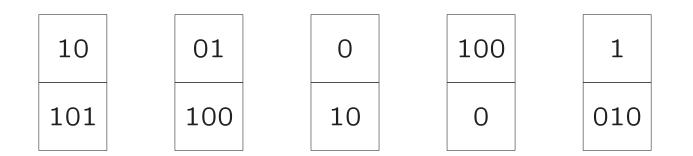
• if the decision problem does not involve just one TM Equivalent: Given two TMs T_1 and T_2 , is $L(T_1) = L(T_2)$

• if the decision problem involves the *operation* of the TM *WritesSymbol*: Given a TM T and a symbol a in the tape alphabet of T, does T ever write a if it starts with an empty tape ? *WritesNonblank*: Given a TM T, does T ever write a nonblank symbol on input Λ ?

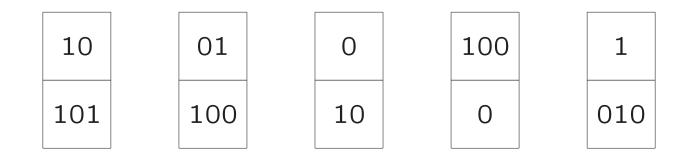
• if the decision problem involves a *trivial* property Accepts-NSA: Given a TM T, is L(T) = NSA ?

9.4. Post's Correspondence Problem

Instance:



Instance:



Match:

10	1	01	0	100	100	0	100
101	010	100	10	0	0	10	0

Definition 9.14. Post's Correspondence Problem

An instance of Post's correspondence problem (PCP) is a set

 $\{(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_n,\beta_n)\}$

of pairs, where $n \ge 1$ and the α_i 's and β_i 's are all nonnull strings over an alphabet Σ .

The decision problem is this:

Given an instance of this type, do there exist a positive integer k and a sequence of integers i_1, i_2, \ldots, i_k , with each i_j satisfying $1 \le i_j \le n$, satisfying

$$\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k}=\beta_{i_1}\beta_{i_2}\ldots\beta_{i_k}$$
?

 i_1, i_2, \ldots, i_k need not all be distinct.

Definition 9.14. Post's Correspondence Problem (continued)

An instance of the modified Post's correspondence problem (*MPCP*) looks exactly like an instance of *PCP*, but now the sequence of integers is required to start with 1. The question can be formulated this way:

Do there exist a positive integer k and a sequence i_2,i_3,\ldots,i_k such that

$$\alpha_1 \alpha_{i_2} \dots \alpha_{i_k} = \beta_1 \beta_{i_2} \dots \beta_{i_k} \quad ?$$

(Modified) correspondence system, match.

Theorem 9.15. $MPCP \leq PCP$

Proof.

For instance

$$I = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_n, \beta_n)\}$$

of *MPCP*, construct instance J = F(I) of *PCP*, such that *I* is yes-instance, if and only if *J* is yes-instance.

For
$$1 \leq i \leq n$$
, if
 $(\alpha_i, \beta_i) = (a_1 a_2 \dots a_r, b_1 b_2 \dots b_s)$

we let

$$(\alpha'_i, \beta'_i) = (a_1 \# a_2 \# \dots a_r \#, \ \# b_1 \# b_2 \dots \# b_s)$$

1	0	01	0	100	1
1(01	100	10	0	010

1#0#	0#1#	0#	1#0#0#	1#
#1#0#1	#1#0#0	#1#0	#0	#0#1#0

10	1	01	0	100	100	0	100
101	010	100	10	0	0	10	0

	1#0#	1#	0#1#	0#	1#0#0#	1#0#0#	0#	1#(
-	#1#0#1	#0#1#0	#1#0#0	#1#0	#0	#0	#1#0	Ξ

10	1	01	0	100	100	0	100
101	010	100	10	0	0	10	0

1#	0#1#	0#	1#0#0#	1#0#0#	0#	1#0#0#
≠0#1#0	#1#0#0	#1#0	#0	#0	#1#0	#0

10	1	01	0	100	100	0	100
101	010	100	10	0	0	10	0

1#	0#1#	0#	1#0#0#	1#0#0#	0#	1#0#0#	\$
≠0#1#0	#1#0#0	#1#0	#0	#0	#1#0	#0	#\$

10	1	01	0	100	100	0	100
101	010	100	10	0	0	10	0

	1#0#	1#	0#1#	0#	1#0#0#	1#0#0#	0#	1#(
-	#1#0#1	#0#1#0	#1#0#0	#1#0	#0	#0	#1#0	Ξ

10	1	01	0	100	100	0	100
101	010	100	10	0	0	10	0

Match PCP:

#1#0#	1#	0#1#	0#	1#0#0#	1#0#0#	0#	1#0
#1#0#1	#0#1#0	#1#0#0	#1#0	#0	#0	#1#0	=

For $1 \leq i \leq n$, if

$$(\alpha_i,\beta_i)=(a_1a_2\ldots a_r, b_1b_2\ldots b_s)$$

we let

$$(\alpha'_i, \beta'_i) = (a_1 \# a_2 \# \dots a_r \#, \ \# b_1 \# b_2 \dots \# b_s)$$

If

$$(\alpha_1,\beta_1)=(a_1a_2\ldots a_r, b_1b_2\ldots b_s)$$

add

$$(\alpha_1'', \beta_1'') = (\#a_1 \# a_2 \# \dots a_r \#, \ \#b_1 \# b_2 \dots \# b_s)$$

Finally, add

$$(\alpha'_{n+1},\beta'_{n+1}) = (\$,\#\$)$$

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