Fundamentele Informatica 3

7. Turing Machines

Exercise 7.4.

For each of the following languages, draw a transition diagram for a Turing machine that accepts that language.

a. $\cdots$

b. $\{a^i b^j \mid i < j\}$

c. $\cdots$

d. $\cdots$

7.1. A General Model of Computation

Definition 7.1. Turing machines

A Turing machine (TM) is a 5-tuple $T = (Q, \Sigma, \Gamma, q_0, \delta)$, where

Q is a finite set of states. The two *halt* states h_a and h_r are not elements of Q.

Σ, the input alphabet, and Γ, the tape alphabet, are both finite sets, with $\Sigma \subseteq \Gamma$. The *blank* symbol Δ is not an element of Γ.

q_0, the initial state, is an element of Q.

δ is the transition function: $\cdots$

Turing machine

Turing machine has a finite alphabet of symbols. (actually two alphabets. . .)

Turing machine has a finite number of states.

Example 7.7. Accepting $L = \{a^i b a^j \mid 0 \leq i < j\}$

To illustrate that a Turing machine T may run forever for an input that is not in $L(T)$. No problem!

Assumptions about a human computer working with a pencil and paper:

1. $\cdots$

2. Each step taken by the computer depends only on the symbol he is currently examining and on his "state of mind" at the time;

3. $\cdots$

A move of a Turing machine consists of:

1. Changing from the current state to another, possibly different state;

2. Replacing the symbol in the current square by another, possibly different symbol;

3. Leaving the tape head on the current square, or moving it one square to the right, or moving it one square to the left if it is not already on the leftmost square.

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Q is a finite set of states. The two *halt* states h_a and h_r are not elements of Q.

Σ, the input alphabet, and Γ, the tape alphabet, are both finite sets, with $\Sigma \subseteq \Gamma$. The *blank* symbol Δ is not an element of Γ.

q_0, the initial state, is an element of Q.

δ is the transition function:

$$\delta : Q \times (\Gamma \cup \{\Delta\}) \to (Q \cup \{h_a, h_r\}) \times (\Gamma \cup \{\Delta\}) \times \{R, L, S\}$$

Interpretation of

$$\delta(p, X) = (q, Y, D)$$

If q is h_a or h_r, the move causes T to halt

What if $D = L$ and T is on square 0?

Normally, TM starts with
- input string starting in square 1 and all other squares blank,
- and its tape head on square 0.

Tape always contains finite number of non-blanks.

Notation:

configuration . . .

Notation:

description of tape contents: $x\underline{a}y$ or $x\underline{y}$

$x\underline{y} = x\underline{y}\Delta = x\underline{y}\Delta\Delta$
if $y = \Lambda$, then $x\underline{\Delta}$

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configuration $xqy = xqy\Delta = xqy\Delta\Delta$
if $y = \Lambda$, then $xq\Delta$

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move: $xqy \vdash_T zrw$ $xqy \vdash_T^* zrw$
      $xqy \vdash zrw$ $xqy \vdash^* zrw$

example: configuration $aabqa\Delta a$ and $\delta(q,a) = (r, \Delta, L)$

initial configuration corresponding to input x: . . .

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      $xqy \vdash zrw$ $xqy \vdash^* zrw$

example: configuration $aabqa\Delta a$ and $\delta(q,a) = (r, \Delta, L)$

initial configuration corresponding to input x: $q_0\Delta x$

In the third edition of the book, a configuration is denoted as $(q, x\underline{a}y)$ or (q, xqy) instead of $x\underline{a}y$ or xqy.
This old notation is also allowed for Fundamentele Informatica 3.

7.2. Turing Machines as Language Acceptors

Definition 7.2. Acceptance by a TM

If $T = (Q, \Sigma, \Gamma, q_0, \delta)$ is a TM and $x \in \Sigma^*$, x is accepted by T if

$$q_0\Delta x \vdash_T^* uh_a y$$

for some strings $u, y \in (\Gamma \cup \{\Delta\})^*$

(i.e., if, starting in the initial configuration corresponding to input x, T eventually halts in the accepting state).

N.B.: sequence of moves leading to h_a is unique

A language $L \subseteq \Sigma^*$ is accepted by T if $L = L(T)$, where

$$L(T) = \{x \in \Sigma^* \mid x \text{ is accepted by } T\}$$

7.3. Turing Machines That Compute Partial Functions

Example 7.10. The Reverse of a String

$$\Delta\,a\,a\,b\,a\,b$$
$$\Delta A\,a\,b\,a\,b$$
$$\Delta A\,a\,b\,a\,A$$
$$\Delta B\,a\,b\,a\,A$$
$$\Delta B\,A\,b\,A\,A$$
$$\Delta B\,A\,b\,A\,A$$
$$\Delta B\,A\,B\,A\,A$$
$$\Delta b\,a\,b\,a\,a$$

Simple version of:

Definition 7.9. A Turing Machine Computing a Function

Let $T = (Q, \Sigma, \Gamma, q_0, \delta)$ be a Turing machine, and f a partial function on Σ^* with values in Γ^*. We say that T computes f if for every x in the domain of f,

$$q_0\Delta x \vdash_T^* h_a\Delta f(x)$$

and no other input string is accepted by T.

Definition 7.9. A Turing Machine Computing a Function

Let $T = (Q, \Sigma, \Gamma, q_0, \delta)$ be a Turing machine, and f a partial function on $(\Sigma^*)^k$ with values in Γ^*. We say that T computes f if for every $(x_1, x_2, \ldots, x_k)$ in the domain of f,

$$q_0\Delta x_1\Delta x_2\Delta \ldots \Delta x_k \vdash_T^* h_a\Delta f(x_1, x_2, \ldots, x_k)$$

and no other input that is a k-tuple of strings is accepted by T.

A partial function $f : (\Sigma^*)^k \to \Gamma^*$ is Turing-computable, or simply computable, if there is a TM that computes f.

Example 7.12. The Quotient and Remainder Mod 2

Example 7.14. The Characteristic Function of a Set

$$\chi_L(x) = \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{if } x \notin L \end{cases}$$

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From computing χ_L to accepting L

From accepting L to computing χ_L

Een Intermezzo