Fundamentele Informatica 3

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10. Computable Functions
10.2. Quantification, Minimalization, and μ-Recursive Functions
10.3. Gödel Numbering

A slide from lecture 12:

Definition 10.1. Initial Functions

The initial functions are the following:

1. *Constant* functions: For each $k \geq 0$ and each $a \geq 0$, the constant function $C_a^k : \mathbb{N}^k \to \mathbb{N}$ is defined by the formula

$$C_a^k(X) = a \quad \text{for every } X \in \mathbb{N}^k$$

2. The *successor* function $s : \mathbb{N} \to \mathbb{N}$ is defined by the formula

$$s(x) = x + 1$$

3. *Projection* functions: For each $k \geq 1$ and each i with $1 \leq i \leq k$, the projection function $p_i^k : \mathbb{N}^k \to \mathbb{N}$ is defined by the formula

$$p_i^k(x_1, x_2, \ldots, x_k) = x_i$$

A slide from lecture 12:

Definition 10.2. The Operations of Composition and Primitive Recursion

1. Suppose f is a partial function from $\mathbb{N}^k$ to $\mathbb{N}$, and for each i with $1 \leq i \leq k$, g_i is a partial function from $\mathbb{N}^m$ to $\mathbb{N}$.
   The partial function obtained from f and $g_1, g_2, \ldots, g_k$ by composition is the partial function h from $\mathbb{N}^m$ to $\mathbb{N}$ defined by the formula

$$h(X) = f(g_1(X), g_2(X), \ldots, g_k(X)) \text{ for every } X \in \mathbb{N}^m$$

A slide from lecture 12:

Definition 10.2. The Operations of Composition and Primitive Recursion (continued)

2. Suppose $n \geq 0$ and g and h are functions of n and $n+2$ variables, respectively. (By "a function of 0 variables," we mean simply a constant.)
   The function obtained from g and h by the operation of *primitive recursion* is the function $f : \mathbb{N}^{n+1} \to \mathbb{N}$ defined by the formulas

$$\begin{aligned} f(X, 0) &= g(X) \\ f(X, k+1) &= h(X, k, f(X, k)) \end{aligned}$$

   for every $X \in \mathbb{N}^n$ and every $k \geq 0$.

A slide from lecture 12:

Theorem 10.4.

Every primitive recursive function is total and computable.

PR:
total and computable

Turing-computable functions:
not necessarily total

A slide from lecture 12:

n*-place predicate* P is function from $\mathbb{N}^n$ to $\{\text{true}, \text{false}\}$

characteristic function χ_P defined by

$$\chi_P(X) = \begin{cases} 1 & \text{if } P(X) \text{ is true} \\ 0 & \text{if } P(X) \text{ is false} \end{cases}$$

We say P is primitive recursive...

10.2. Quantification, Minimalization, and μ-Recursive Functions

A slide from lecture 13:

Definition 10.11. Bounded Minimalization

For an $(n+1)$-place predicate P, the *bounded minimalization* of P is the function $m_P : \mathbb{N}^{n+1} \to \mathbb{N}$ defined by

$$m_P(X,k) = \begin{cases} \min\{y \mid \ 0 \leq y \leq k \text{ and } P(X,y)\} & \text{if this set is not empty} \\ k+1 & \text{otherwise} \end{cases}$$

The symbol μ is often used for the minimalization operator, and we sometimes write

$$m_P(X,k) = \overset{k}{\mu}\, y[P(X,y)]$$

An important special case is that in which $P(X,y)$ is $(f(X,y)=0)$, for some $f : \mathbb{N}^{n+1} \to \mathbb{N}$. In this case m_P is written m_f and referred to as the bounded minimalization of f.

A slide from lecture 13:

Theorem 10.12.

If P is a primitive recursive $(n+1)$-place predicate,
its bounded minimalization m_P is a primitive recursive function.

Proof. . .

A slide from lecture 13:

Example 10.13. The nth Prime Number

$PrNo(0) = 2$
$PrNo(1) = 3$
$PrNo(2) = 5$

$$\begin{aligned} Prime(n) \;=\; & (n \geq 2) \wedge \neg(\text{there exists } y \text{ such that} \\ & \qquad y \geq 2 \wedge y \leq n-1 \wedge Mod(n,y) = 0) \end{aligned}$$

A slide from lecture 13:

Example 10.13. The nth Prime Number

Let

$$P(x, y) = \ (y > x \land \mathit{Prime}(y))$$

Then

$$\begin{aligned} \mathit{PrNo}(0) &= 2 \\ \mathit{PrNo}(k+1) &= m_P(\mathit{PrNo}(k), (\mathit{PrNo}(k))! + 1) \end{aligned}$$

is primitive recursive, with $h(x_1, x_2) = \ldots$

A slide from lecture 12:

Theorem 10.4.

Every primitive recursive function is total and computable.

PR:
total and computable

Turing-computable functions:
not necessarily total

Unbounded minimalization

Total?

Unbounded minimalization

Total?

A possible definition:

$$M(X) = \begin{cases} (\text{min}\{y \mid \; P(X,y) \text{ is true}\}) + 1 & \text{if this set is not empty} \\ 0 & \text{otherwise} \end{cases}$$

Computable?

A slide from lecture 13:

Unbounded quantification

$\mathit{Sq}(x, y) = \quad (y^2 = x)$

$H(x, y) = \quad T_u$ stopt na precies y stappen voor invoer s_x

Definition 10.14. Unbounded Minimalization

If P is an $(n+1)$-place predicate, the *unbounded minimalization* of P is the partial function $M_P : \mathbb{N}^n \to \mathbb{N}$ defined by

$$M_P(X) = \min\{y \mid \ P(X, y) \text{ is true}\}$$

$M_P(X)$ is undefined at any $X \in \mathbb{N}^n$ for which there is no y satisfying $P(X, y)$.

Definition 10.14. Unbounded Minimalization

If P is an $(n+1)$-place predicate, the *unbounded minimalization* of P is the partial function $M_P : \mathbb{N}^n \to \mathbb{N}$ defined by

$$M_P(X) = \min\{y \mid \ P(X, y) \text{ is true}\}$$

$M_P(X)$ is undefined at any $X \in \mathbb{N}^n$ for which there is no y satisfying $P(X, y)$.

The notation $\mu\, y[P(X, y)]$ is also used for $M_P(X)$.
In the special case in which $P(X, y) = (f(X, y) = 0)$, we write $M_P = M_f$ and refer to this function as the unbounded minimalization of f.

Definition 10.15. μ-Recursive Functions

The set $\mathcal{M}$ of μ-recursive, or simply *recursive*, partial functions is defined as follows.

1. Every initial function is an element of $\mathcal{M}$.

2. Every function obtained from elements of $\mathcal{M}$ by composition or primitive recursion is an element of $\mathcal{M}$.

3. For every $n \geq 0$ and every total function $f : \mathbb{N}^{n+1} \to \mathbb{N}$ in $\mathcal{M}$, the function $M_f : \mathbb{N}^n \to \mathbb{N}$ defined by

$$M_f(X) = \mu\, y[f(X, y) = 0]$$

   is an element of $\mathcal{M}$.

Example.

Let

$$f(x,k) = p_1^2(x,k) \dot{-} C_1^2(x,k)$$

$M_f(x)$ . . .

Exercise.

a. Give an example of a non-total function f and another function g, such that the composition of f and g is total.

b. Can you also find an example of a non-total function f and another function g, such that the composition of g and f is total?

Theorem 10.16.

All μ-recursive partial functions are computable.

Proof. . .

10.3. Gödel Numbering

Definition 10.17.
The Gödel Number of a Sequence of Natural Numbers

For every $n \geq 1$ and every finite sequence $x_0, x_1, \ldots, x_{n-1}$ of n natural numbers, the *Gödel number* of the sequence is the number

$$\mathit{gn}(x_0, x_1, \ldots, x_{n-1}) = 2^{x_0} 3^{x_1} 5^{x_2} \ldots (\mathit{PrNo}(n-1))^{x_{n-1}}$$

where *PrNo*(i) is the ith prime (Example 10.13).

Example 10.18.

The Power to Which a Prime is Raised in the Factorization of x

Function *Exponent* $: \mathbb{N}^2 \to \mathbb{N}$ defined as follows:

$$\mathit{Exponent}(i,x) = \begin{cases} \text{the exp. of } \mathit{PrNo}(i) \text{ in } x\text{'s prime fact.} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Theorem 10.19.

Suppose that $g : \mathbb{N}^n \to \mathbb{N}$ and $h : \mathbb{N}^{n+2} \to \mathbb{N}$ are primitive recursive functions, and $f : \mathbb{N}^{n+1} \to \mathbb{N}$ is obtained from g and h by course-of-values recursion; that is

$$\begin{aligned} f(X,0) &= g(X) \\ f(X,k+1) &= h(X,k,\mathsf{gn}(f(X,0),\ldots,f(X,k))) \end{aligned}$$

Then f is primitive recursive.

Proof. . .

Example.

Fibonacci

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f(n-1) + f(n-2) & \text{if } n \geq 2 \end{cases}$$