Fundamentele Informatica 3

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http://www.liacs.nl/home/rvvliet/fi3/

Rudy van Vliet
kamer 124 Snellius, tel. 071-527 5777
rvvliet(at)liacs(dot)nl

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9. Undecidable Problems
9.3. More Decision Problems Involving Turing Machines
9.4. Post's Correspondence Problem

A slide from lecture 9:

Definition 9.6. Reducing One Decision Problem to Another, and Reducing One Language to Another

Suppose P_1 and P_2 are decision problems. We say P_1 is reducible to P_2 ($P_1 \leq P_2$)
• if there is an algorithm
• that finds, for an arbitrary instance I of P_1, an instance $F(I)$ of P_2,
• such that
for every I the answers for the two instances are the same,
or I is a yes-instance of P_1
if and only if $F(I)$ is a yes-instance of P_2.

A slide from lecture 9:

Theorem 9.7. Suppose $L_1 \subseteq \Sigma_1^*$, $L_2 \subseteq \Sigma_2^*$, and $L_1 \leq L_2$. If L_2 is recursive, then L_1 is recursive.

Suppose P_1 and P_2 are decision problems, and $P_1 \leq P_2$. If P_2 is decidable, then P_1 is decidable.

Proof. . .

A slide from lecture 9:

Theorem 9.9. The following five decision problems are undecidable.

5. *WritesSymbol*:
Given a TM T and a symbol a in the tape alphabet of T, does T ever write a if it starts with an empty tape ?

Proof.

5. Prove that *Accepts*-$\Lambda \leq$ *WritesSymbol* . . .

AtLeast10MovesOn-Λ:
Given a TM T, does T make at least ten moves on input Λ ?

WritesNonblank: Given a TM T, does T ever write a nonblank symbol on input Λ ?

Theorem 9.10.
The decision problem *WritesNonblank* is decidable.

Proof. . .

Definition 9.11. A Language Property of TMs

A property R of Turing machines is called a *language property* if, for every Turing machine T having property R, and every other TM T_1 with $L(T_1) = L(T)$, T_1 also has property R.

A language property of TMs is *nontrivial* if there is at least one TM that has the property and at least one that doesn't.

In fact, a language property is a property *of the languages accepted by* TMs.

Theorem 9.12. Rice's Theorem
If R is a nontrivial language property of TMs, then the decision problem

P_R: Given a TM T, does T have property R ?

is undecidable.

Proof. . .

Prove that *Accepts*-$\Lambda \leq P_R$ . . .

(or that *Accepts*-$\Lambda \leq P_{\text{not}-R}$ . . .)

T_2 highly unspecified. . .

A slide from lecture 9:

Definition 9.6. Reducing One Decision Problem to Another, and Reducing One Language to Another

Suppose P_1 and P_2 are decision problems. We say P_1 is reducible to P_2 ($P_1 \leq P_2$)

- if there is an algorithm
- that finds, for an arbitrary instance I of P_1, an instance $F(I)$ of P_2,
- such that
  for every I the answers for the two instances are the same,
  or I is a yes-instance of P_1
  if and only if $F(I)$ is a yes-instance of P_2.

Examples of decision problems to which Rice's theorem can be applied:

1. *Accepts-L*: Given a TM T, is $L(T) = L$? (assuming . . .)
2. *AcceptsSomething*:
   Given a TM T, is there at least one string in $L(T)$?
3. *AcceptsTwoOrMore*:
   Given a TM T, does $L(T)$ have at least two elements ?
4. *AcceptsFinite*: Given a TM T, is $L(T)$ finite ?
5. *AcceptsRecursive*:
   Given a TM T, is $L(T)$ recursive ? (note that . . .)

All these problems are undecidable.

Rice's theorem cannot be applied (directly)

- if the decision problem does not involve just one TM
Equivalent: Given two TMs T_1 and T_2, is $L(T_1) = L(T_2)$

Rice's theorem cannot be applied (directly)

- if the decision problem does not involve just one TM
Equivalent: Given two TMs T_1 and T_2, is $L(T_1) = L(T_2)$

- if the decision problem involves the *operation* of the TM
WritesSymbol: Given a TM T and a symbol a in the tape alphabet of T, does T ever write a if it starts with an empty tape ?
WritesNonblank: Given a TM T, does T ever write a nonblank symbol on input Λ ?

- if the decision problem involves a *trivial* property
Accepts-NSA: Given a TM T, is $L(T) =$ *NSA* ?

9.4. Post's Correspondence Problem

Instance:

10	01	0	100	1
101	100	10	0	010

Instance:

10	01	0	100	1
101	100	10	0	010

Match:

10	1	01	0	100	100	0	100
101	010	100	10	0	0	10	0

Definition 9.14. Post's Correspondence Problem

An instance of Post's correspondence problem (*PCP*) is a set

$$\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\}$$

of pairs, where $n \geq 1$ and the α_i's and β_i's are all nonnull strings over an alphabet Σ.

The decision problem is this:

Given an instance of this type, do there exist a positive integer k and a sequence of integers $i_1, i_2, \ldots, i_k$, with each i_j satisfying $1 \leq i_j \leq n$, satisfying

$$\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k} = \beta_{i_1}\beta_{i_2}\ldots\beta_{i_k} \quad ?$$

$i_1, i_2, \ldots, i_k$ need not all be distinct.

Definition 9.14. Post's Correspondence Problem (continued)

An instance of the modified Post's correspondence problem (*MPCP*) looks exactly like an instance of *PCP*, but now the sequence of integers is required to start with 1. The question can be formulated this way:

Do there exist a positive integer k and a sequence $i_2, i_3, \ldots, i_k$ such that

$$\alpha_1 \alpha_{i_2} \ldots \alpha_{i_k} = \beta_1 \beta_{i_2} \ldots \beta_{i_k} \quad ?$$

(Modified) correspondence system, match.

Theorem 9.15. *MPCP* $\leq$ *PCP*

Proof.

For instance

$$I = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\}$$

of *MPCP*, construct instance $J = F(I)$ of *PCP*, such that I is yes-instance, if and only if J is yes-instance.

For $1 \leq i \leq n$, if

$$(\alpha_i, \beta_i) = (a_1 a_2 \ldots a_r,\ b_1 b_2 \ldots b_s)$$

we let

$$(\alpha_i', \beta_i') = (a_1\#a_2\#\ldots a_r\#,\ \#b_1\#b_2\ldots\#b_s)$$

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we let

$$(\alpha_i', \beta_i') = (a_1\#a_2\#\ldots a_r\#,\ \#b_1\#b_2\ldots\#b_s)$$

If

$$(\alpha_1, \beta_1) = (a_1 a_2 \ldots a_r,\ b_1 b_2 \ldots b_s)$$

add

$$(\alpha_1'', \beta_1'') = (\#a_1\#a_2\#\ldots a_r\#,\ \#b_1\#b_2\ldots\#b_s)$$

Finally, add

$$(\alpha_{n+1}', \beta_{n+1}') = (\$, \#\$)$$

Theorem 9.16. *Accepts* $\leq$ *MPCP*

The technical details of the proof of this result do not have to be known for the exam. However, one must be able to carry out the construction below.

Proof...

For every instance (T, w) of *Accepts*, construct instance $F(T, w)$ of *MPCP*, such that ...

A slide from lecture 3

Notation:

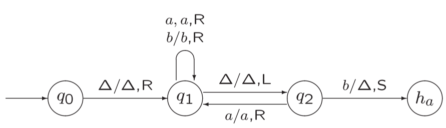
description of tape contents: $x\underline{\sigma}y$ or $x\underline{y}$

configuration $xqy = xqy\Delta = xqy\Delta\Delta$

initial configuration corresponding to input x: $q_0\Delta x$

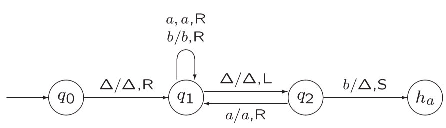
In the third edition of the book, a configuration is denoted as $(q, x\underline{y})$ or $(q, x\underline{\sigma}y)$ instead of xqy or $xq\sigma y$.
This old notation is also allowed for Fundamentele Informatica 3.

Example 9.18. A Modified Correspondence System for a TM



T accepts ...

Example 9.18. A Modified Correspondence System for a TM



T accepts all strings in $\{a, b\}^*$ ending with b.

Proof of Theorem 9.16. (continued)

Take
$(\alpha_1, \beta_1) = (\#, \#q_0\Delta w\#)$

Pairs of type 1: (a, a) for every $a \in \Gamma \cup \{\Delta\}$, and $(\#, \#)$

Pairs of type 2: corresponding to moves in T, e.g.,
(qa, bp), if $\delta(q, a) = (p, b, R)$
(cqa, pcb), if $\delta(q, a) = (p, b, L)$

Proof of Theorem 9.16. (continued)

Take
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$(q\#, pa\#)$, if $\delta(q, \Delta) = (p, a, S)$

Proof of Theorem 9.16. (continued)

Take
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Pairs of type 1: (a, a) for every $a \in \Gamma \cup \{\Delta\}$, and $(\#, \#)$

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$(q\#, pa\#)$, if $\delta(q, \Delta) = (p, a, S)$

Pairs of type 3: for every $a, b \in \Gamma \cup \{\Delta\}$, the pairs
$(h_a a, h_a)$, (ah_a, h_a), $(ah_a b, h_a)$

One pair of type 4:
$(h_a\#\#, \#)$

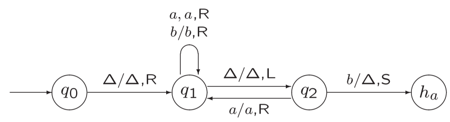
Proof of Theorem 9.16. (continued)

Two assumptions in book:
1. T never moves to h_r
2. $w \neq \Lambda$ (i.e., special initial pair if $w = \Lambda$)

These assumptions are not necessary. . .

Theorem 9.17.
Post's correspondence problem is undecidable.

Example 9.18. A Modified Correspondence System for a TM



T accepts all strings in $\{a, b\}^*$ ending with b.

Pairs of type 2:
$(q_0\Delta, \Delta q_1)$ $(q_0\#, \Delta q_1\#)$ $(q_1 a, a q_1)$ $(q_1 b, b q_1)$
$(a q_1 \Delta, q_2 a \Delta)$ $(b q_1 \Delta, q_2 b \Delta)$ . . .

Study this example yourself.

Huiswerkopgave 3

Reducties en (on-)beslisbaarheid