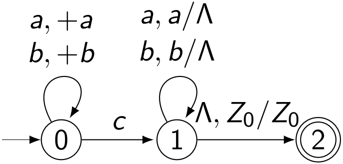
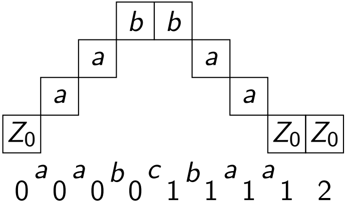
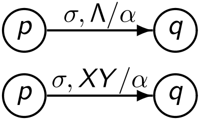
SimplePal =
$\{ \; xcx^r \mid x \in \{a, b\}^* \; \}$





[M] Fig 5.5

Incorrect notations:



top stack symbol required

remove/consider one stack symbol at a time

From lecture 10:

for each state and stack symbol
– on each symbol/Λ at most one transition
– not both symbol and Λ-transition

Definition

$\delta(q, \sigma, X) \cup \delta(q, \Lambda, X)$ at most one element for each $q \in Q, \sigma \in \Sigma, X \in \Gamma$

DPDA $\approx$ DCFL

[M] Def 5.10

$pre(L) = \{ x\#y \mid x \in L \text{ and } xy \in L \}$

$L = Pal = \{\Lambda, a, b, aa, bb, aaa, aba, bab, bbb, aaaa, abba, \ldots\}$
$pre(L) = \ldots$

$L = \{a^i b^j \mid i < j\} = \{b, bb, abb, bbb, abbb, bbbb, aabbb, abbbb, \ldots\}$
$pre(L) = \ldots$

$pre(L) = \{ x\#y \mid x \in L \text{ and } xy \in L \}$

CFL not closed under *pre*
DCFL *is* closed under *pre*

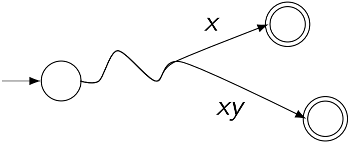
[M] Exercise 5.20 & 6.22

CFL not closed under complement
DCFL is closed under complement ⊠
(the obvious proof does not work)

CFL is closed under regular operations $\cup, \cdot, *$
DCFL is not closed under either of these ⊠

Non/determinism

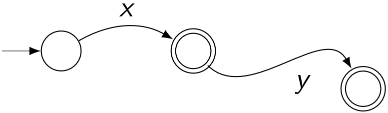
language L $\quad x \in L$, $xy \in L$



$K = \{\ a^n b^n \mid n \geq 1\ \} \cup \{\ a^n b^m c^n \mid m, n \geq 1\ \}$

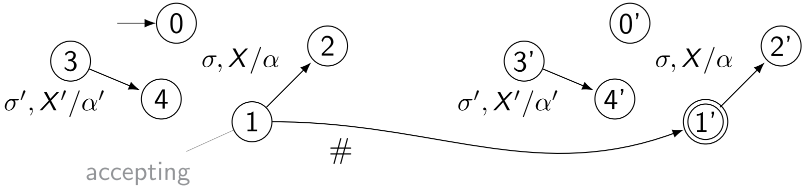
$\underline{a^n}\ \underline{b^n} \quad \underline{a^n}\ b^m\ \underline{c^n}$ $\quad$ different behaviour on b's

$pre(K) = \ldots$



DCFL is closed under *pre*

$pre(L) = \{ x\#y \mid x, xy \in L \}$



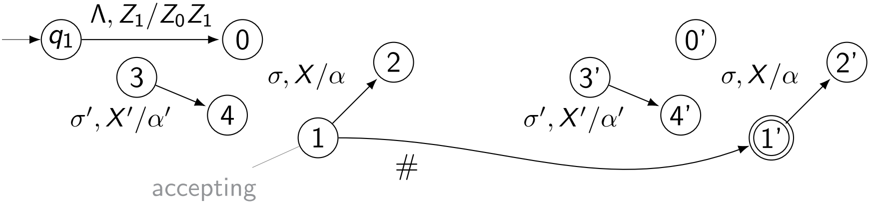
$M = (Q, \Sigma, \Gamma, q_0, Z_0, A, \delta)$ with $L = L(M)$
construct $M_1 = (Q_1, \Sigma \cup \{\#\}, \Gamma, q_1, Z_1, A_1, \delta_1)$ with $L(M_1) = pre(L)$

- $Q_1 = Q \cup Q'$ where $Q' = \{ q' \mid q \in Q \}$ primed copy
- $q_1 = q_0$, $Z_1 = Z_0$
- $A_1 = A' = \{ q' \mid q \in A \}$ accepting states in copy
- $\delta_1(p', \sigma, X) = \{(q', \alpha) \mid (q, \alpha) \in \delta(p, \sigma, X)\}$ two copies

  for all $p \in A, X \in \Gamma$: $\delta_1(p, \#, X) = \{(p', X)\}$ move to primed copy

DCFL is closed under *pre*

$pre(L) = \{ x\#y \mid x, xy \in L \}$



$M = (Q, \Sigma, \Gamma, q_0, Z_0, A, \delta)$ with $L = L(M)$

construct $M_1 = (Q_1, \Sigma \cup \{\#\}, \Gamma \cup \{Z_1\}, q_1, Z_1, A_1, \delta_1)$ with $L(M_1) = pre(L)$

- $Q_1 = Q \cup Q' \cup \{q_1\}$ where $Q' = \{ q' \mid q \in Q \}$ — primed copy
- $A_1 = A' = \{ q' \mid q \in A \}$ — accepting states in copy
- $\delta_1(p', \sigma, X) = \{(q', \alpha) \mid (q, \alpha) \in \delta(p, \sigma, X)\}$ — two copies

  $\delta_1(q_1, \Lambda, Z_1) = \{(q_0, Z_0 Z_1)\}$ — Z_1 under Z_0

  for all $p \in A, X \in \Gamma_1$: $\delta_1(p, \#, X) = \{(p', X)\}$ — move to primed copy

ABOVE

For $K = \{ a^n b^n \mid n \geq 1 \} \cup \{ a^n b^m c^n \mid m, n \geq 1 \}$
we have $pre(K) = K\# \cup \{a^n b^n \# b^k c^n \mid n \geq 1, k \geq 0\}$.
This language is not context-free, but K is, and thus the context-free languages are not closed under pre.

Again, this construction works because (for deterministic automata) the computation on uv *must* extend the computation on u.

Note the resulting PDA might not be deterministic at accepting states in original Q (like node 1 in the diagram), if that node has an outgoing Λ-transition.
There is however a method that avoids Λ-transitions at accepting states. Whenever $(q, \alpha) \in \delta(p, \Lambda, A)$ for an accepting state p, just 'predict' the next letter σ read, add a new state (q, σ), add $((q, \sigma), \alpha)$ to $\delta(p, \sigma, A)$ (which was empty beforehand, why?). Do this for every σ, and remove the Λ-transition. Then keep simulating Λ-transitions, until σ is read.

$L = \{ a^i b^j \mid i \neq j \}$

$S \rightarrow X \mid Y$ (choice!)

$X \rightarrow aXb \mid aX \mid a$ $(i > j)$

$Y \rightarrow aYb \mid Yb \mid b$ $(i < j)$

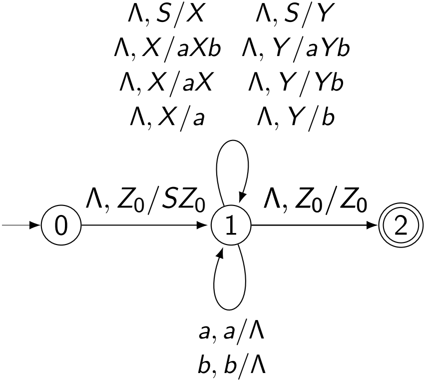
$S \Rightarrow X \Rightarrow aXb \Rightarrow aaXb \Rightarrow aaaXbb \Rightarrow aaaabb$

$L = \{ a^i b^j \mid i \neq j \}$

$S \to X \mid Y$ (choice!)

$X \to aXb \mid aX \mid a$ $(i > j)$

$Y \to aYb \mid Yb \mid b$ $(i < j)$



CFG $G = (V, \Sigma, S, P)$

Definition (Nondeterministic Top-Down PDA)

$NT(G) = (Q, \Sigma, \Gamma, q_0, Z_0, A, \delta)$, as follows:

- $Q = \{q_0, q_1, q_2\}$
- $A = \{q_2\}$
- $\Gamma = V \cup \Sigma \cup \{Z_0\}$
- start $\quad \delta(q_0, \Lambda, Z_0) = \{(q_1, SZ_0)\}$
- *expand* $\quad \delta(q_1, \Lambda, A) = \{(q_1, \alpha) \mid A \to \alpha \text{ in } P\}$ $\quad$ for $A \in V$
- *match* $\quad \delta(q_1, \sigma, \sigma) = \{(q_1, \Lambda)\}$ $\quad$ for $\sigma \in \Sigma$
- finish $\quad \delta(q_1, \Lambda, Z_0) = \{(q_2, Z_0)\}$ $\quad$ check empty stack

[M] Def 5.17

From lecture 8:

$AeqB = \{\ x \in \{a,b\}^* \mid n_a(x) = n_b(x)\ \}$

$aaabbb, ababab, aababb, \ldots$

$$S \to \Lambda \mid aB \mid bA$$
$$A \to aS \mid bAA \qquad A \text{ generates } n_a(x) = n_b(x) + 1$$
$$B \to bS \mid aBB \qquad B \text{ generates } n_a(x) + 1 = n_b(x)$$

$S \Rightarrow aB \Rightarrow aaBB \Rightarrow aabSB \Rightarrow \ldots$ (different options)

(1) $aabB \Rightarrow aabaBB \Rightarrow aababSB \Rightarrow aababB \Rightarrow aababbS \Rightarrow aababb$

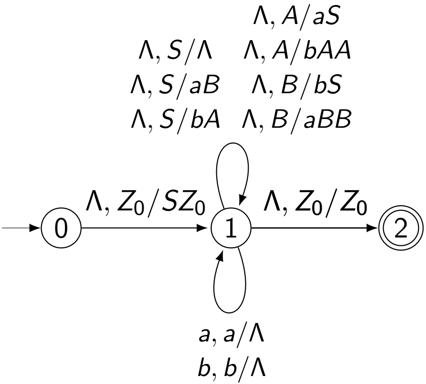
(2) $aaba\underline{B}B \Rightarrow aabab\underline{S}B \Rightarrow aabab\underline{B} \Rightarrow aababb\underline{S} \Rightarrow aababb$

(2') $aabaB\underline{B} \Rightarrow aaba\underline{B}bS \Rightarrow aababSb\underline{S} \Rightarrow aabab\underline{S}b \Rightarrow aababb$

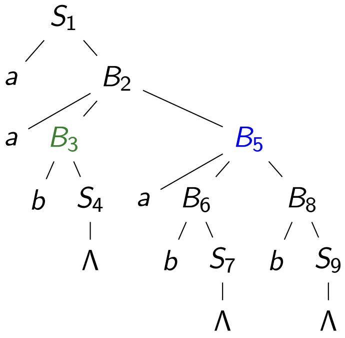
[M] E 4.8

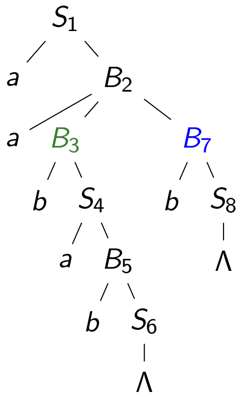
$$AeqB = \{\ x \in \{a, b\}^* \mid n_a(x) = n_b(x)\ \}$$

$$S \to \Lambda \mid aB \mid bA$$
$$A \to aS \mid bAA$$
$$B \to bS \mid aBB$$



From lecture 8:





$S \Rightarrow aB \Rightarrow aaBB \Rightarrow aabSB \Rightarrow aabB \Rightarrow aabaBB \Rightarrow aababSB \Rightarrow aababB \Rightarrow aababbS \Rightarrow aababb$

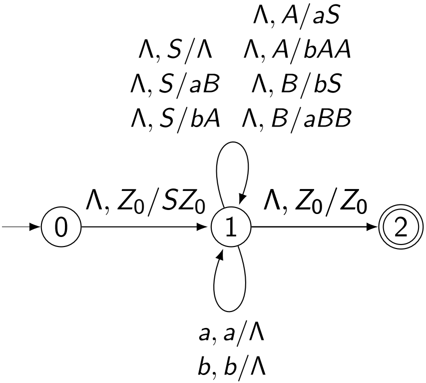
$S \Rightarrow aB \Rightarrow aaBB \Rightarrow aabSB \Rightarrow aabaBB \Rightarrow aababSB \Rightarrow aababB \Rightarrow aababbS \Rightarrow aababb$

$AeqB = \{\ x \in \{a, b\}^* \mid n_a(x) = n_b(x)\ \}$

$S \to \Lambda \mid aB \mid bA$

$A \to aS \mid bAA$

$B \to bS \mid aBB$



q_0 $aababb$ Z_0

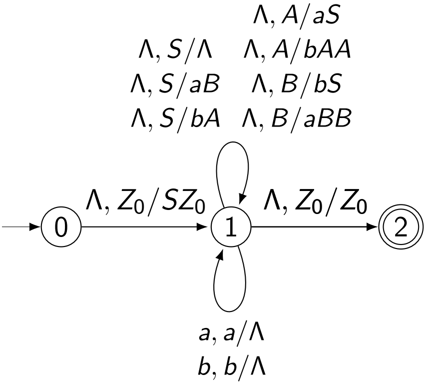
Top-down = expand-match

$AeqB = \{\ x \in \{a, b\}^* \mid n_a(x) = n_b(x)\ \}$

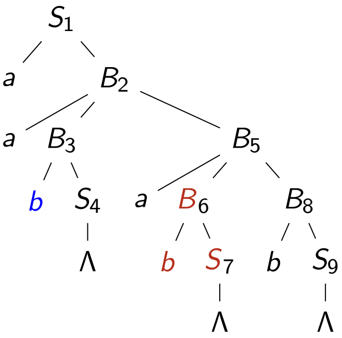
$S \to \Lambda \mid aB \mid bA$

$A \to aS \mid bAA$

$B \to bS \mid aBB$



q_0	$aababb$	Z_0	
q_1	$aababb$	$S\ Z_0$	$1 : S \to aB$
q_1	$aababb$	$aB\ Z_0$	match a
q_1	$a\ ababb$	$B\ Z_0$	$2 : B \to aBB$
q_1	$a\ ababb$	$aBB\ Z_0$	match a
q_1	$aa\ babb$	$BB\ Z_0$	$3 : B \to bS$
q_1	$aa\ babb$	$bSB\ Z_0$	match b
q_1	$aab\ abb$	$SB\ Z_0$	$4 : S \to \Lambda$
q_1	$aab\ abb$	$B\ Z_0$	$5 : B \to aBB$
q_1	$aab\ abb$	$aBB\ Z_0$	match a
q_1	$aaba\ bb$	$BB\ Z_0$	$6 : B \to bS$
q_1	$aaba\ bb$	$bSB\ Z_0$	match b
q_1	$aabab\ b$	$SB\ Z_0$	$7 : S \to \Lambda$
q_1	$aabab\ b$	$B\ Z_0$	$8 : B \to bS$
q_1	$aabab\ b$	$bS\ Z_0$	match b
q_1	$aababb$	$S\ Z_0$	$9 : S \to \Lambda$
q_1	$aababb$	Z_0	
q_2	$aababb$	Z_0	

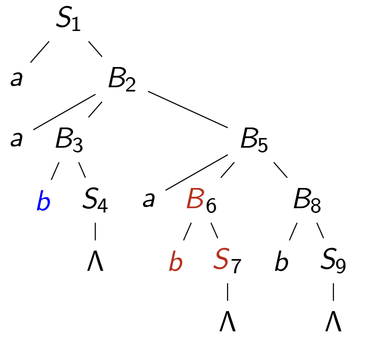


preorder: leftmost

$S \overset{\ell}{\Rightarrow} aB \Rightarrow aaBB \Rightarrow aabSB \Rightarrow$
$aabB \Rightarrow aabaBB \Rightarrow aababSB \Rightarrow$
$aababB \Rightarrow aababbS \Rightarrow aababb$

q_0	$aababb$	Z_0	
q_1	$aababb$	$S\ Z_0$	$1 : S \to aB$
q_1	$aababb$	$aB\ Z_0$	match a
q_1	$a\ ababb$	$B\ Z_0$	$2 : B \to aBB$
q_1	$a\ ababb$	$aBB\ Z_0$	match a
q_1	$aa\ babb$	$BB\ Z_0$	$3 : B \to bS$
q_1	$aa\ babb$	$bSB\ Z_0$	match b
q_1	$aab\ abb$	$SB\ Z_0$	$4 : S \to \Lambda$
q_1	$aab\ abb$	$B\ Z_0$	$5 : B \to aBB$
q_1	$aab\ abb$	$aBB\ Z_0$	match a
q_1	$aaba\ bb$	$BB\ Z_0$	$6 : B \to bS$
q_1	$aaba\ bb$	$bSB\ Z_0$	match b
q_1	$aabab\ b$	$SB\ Z_0$	$7 : S \to \Lambda$
q_1	$aabab\ b$	$B\ Z_0$	$8 : B \to bS$
q_1	$aabab\ b$	$bS\ Z_0$	match b
q_1	$aababb$	$S\ Z_0$	$9 : S \to \Lambda$
q_1	$aababb$	Z_0	
q_2	$aababb$	Z_0	

Top-down = expand-match



preorder: leftmost

$S \overset{\ell}{\Rightarrow} aB \Rightarrow aaBB \Rightarrow aabSB \Rightarrow$
$aabB \Rightarrow aabaBB \Rightarrow aababSB \Rightarrow$
$aababB \Rightarrow aababbS \Rightarrow aababb$

q_0	$aababb$	Z_0	
q_1	$aababb$	$S\ Z_0$	$1 : S \to aB$
q_1	$aababb$	$aB\ Z_0$	match a
q_1	$a\ ababb$	$B\ Z_0$	$2 : B \to aBB$
q_1	$a\ ababb$	$aBB\ Z_0$	match a
q_1	$aa\ babb$	$BB\ Z_0$	$3 : B \to bS$
q_1	$aa\ babb$	$bSB\ Z_0$	match b
q_1	$aab\ abb$	$SB\ Z_0$	$4 : S \to \Lambda$
q_1	$aab\ abb$	$B\ Z_0$	$5 : B \to aBB$
q_1	$aab\ abb$	$aBB\ Z_0$	match a
q_1	$aaba\ bb$	$BB\ Z_0$	$6 : B \to bS$
q_1	$aaba\ bb$	$bSB\ Z_0$	match b
q_1	$aabab\ b$	$SB\ Z_0$	$7 : S \to \Lambda$
q_1	$aabab\ b$	$B\ Z_0$	$8 : B \to bS$
q_1	$aabab\ b$	$bS\ Z_0$	match b
q_1	$aababb$	$S\ Z_0$	$9 : S \to \Lambda$
q_1	$aababb$	Z_0	
q_2	$aababb$	Z_0	

Theorem

If G is a context-free grammar, then the nondeterministic top-down PDA $\mathrm{NT}(G)$ *accepts the language* $L(G)$.

Proof: $L(G) \subseteq L(NT(G))\ldots$

The details of the proof in the other direction do not have to be known for the exam.

[M] Th 5.18

One leftmost derivation step:

$$y_i A_i \alpha_i \Rightarrow y_i \beta_i \alpha_i = y_i x_{i+1} A_{i+1} \alpha_{i+1} \qquad \text{with } y_i, x_{i+1} \in \Sigma^*$$

With $y_i = x_0 x_1 \ldots x_i$:

$$x_0 x_1 \ldots x_i A_i \alpha_i \Rightarrow x_0 x_1 \ldots x_i \beta_i \alpha_i = x_0 x_1 \ldots x_i x_{i+1} A_{i+1} \alpha_{i+1}$$

Complete leftmost derivation:

$$\begin{aligned} S &= x_0 A_0 \alpha_0 \\ &\Rightarrow x_0 x_1 A_1 \alpha_1 \\ &\Rightarrow x_0 x_1 x_2 A_2 \alpha_2 \\ &\Rightarrow \ldots \\ &\Rightarrow x_0 x_1 x_2 \ldots x_m A_m \alpha_m \\ &\Rightarrow x_0 x_1 x_2 \ldots x_m \beta_m \alpha_m = x \end{aligned}$$

Use induction on i to prove that for $i = 0, 1, \ldots, m$, in $NT(G)$,

$$(q_0, x, Z_0) = (q_0, x_0 x_1 \ldots x_m \beta_m \alpha_m) \vdash^* (q_1, x_{i+1} \ldots x_m \beta_m \alpha_m, A_i \alpha_i Z_0)$$

That is, $NT(G)$ can perform steps that read as input $x_0 x_1 \ldots x_i$ and leave $A_i \alpha_i Z_0$ on stack.

Then prove that

$$(q_1, \beta_m \alpha_m, A_m \alpha_m Z_0) \vdash^* (q_2, \Lambda, Z_0)$$