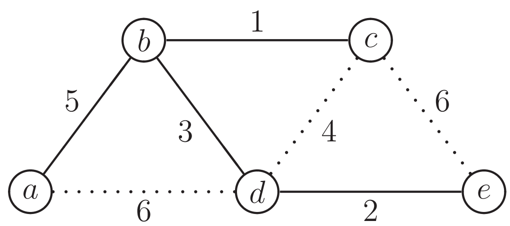
ALGORITMIEK: some solutions to exercise class 12

Exercise 1.

a. The algorithm's progress is described by the following table:

a	b	c	d	e	Action
0	∞	∞	∞	∞	Start with a
–	5	∞	6	∞	Choose b, from a
–	–	1	3	∞	Choose c, from b
–	–	–	3	6	Choose d, from b
–	–	–	–	2	Choose e, from d

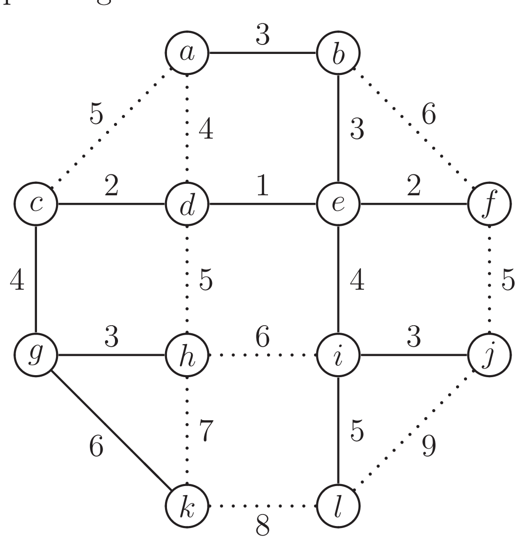
The resulting minimum spanning tree is:



b. The algorithm's progress is described by the following table:

a	b	c	d	e	f	g	h	i	j	k	l	Action
0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	Start with a
–	3	5	4	∞	∞	∞	∞	∞	∞	∞	∞	Choose b, from a
–	–	5	4	3	6	∞	∞	∞	∞	∞	∞	Choose e, from b
–	–	5	1	–	2	∞	∞	4	∞	∞	∞	Choose d, from e
–	–	2	–	–	2	∞	5	4	∞	∞	∞	Choose c, from d
–	–	–	–	–	2	4	5	4	∞	∞	∞	Choose f, from e
–	–	–	–	–	–	4	5	4	5	∞	∞	Choose g, from c
–	–	–	–	–	–	–	3	4	5	6	∞	Choose h, from g
–	–	–	–	–	–	–	–	4	5	6	∞	Choose i, from e
–	–	–	–	–	–	–	–	–	3	6	5	Choose j, from i
–	–	–	–	–	–	–	–	–	–	6	5	Choose l, from i
–	–	–	–	–	–	–	–	–	–	6	–	Choose k, from g

The resulting minimum spanning tree is:

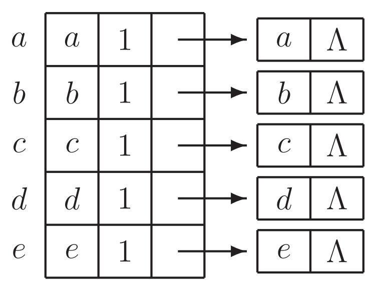


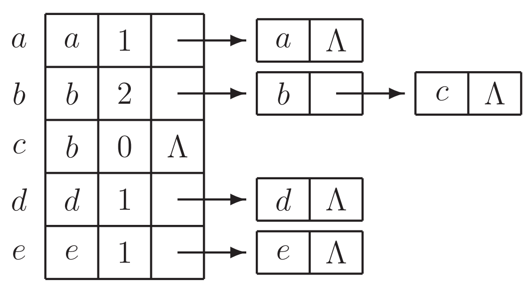
Exercise 2.

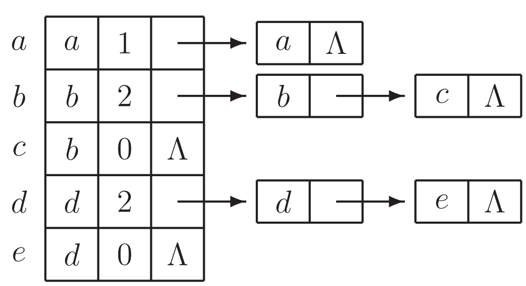
a. We subsequently consider
edge $b - c$, with weight 1, and add it to the tree (no cycle created)
edge $d - e$, with weight 2, and add it to the tree (no cycle created)
edge $b - d$, with weight 3, and add it to the tree (no cycle created)
edge $c - d$, with weight 4, and skip it (would create cycle $d - b - c - d$)
edge $a - b$, with weight 5, and add it to the tree (no cycle created)
Since the graph has five vertices, and by now, we have four edges, the minimum spanning tree is complete and Kruskal's algorithm terminates.

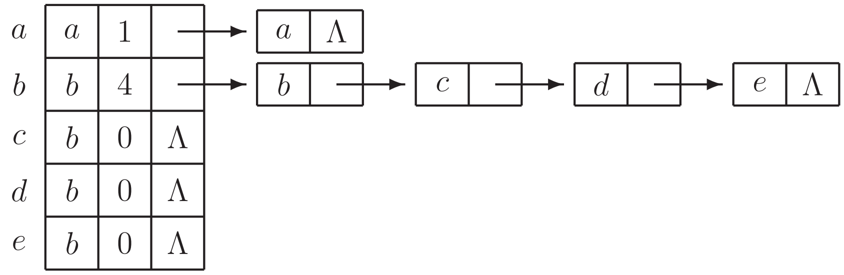
The subsets describing connected components are:
$\{a\}, \{b\}, \{c\}, \{d\}, \{e\}$ before the addition of the first edge
$\{a\}, \{b, c\}, \{d\}, \{e\}$ after adding edge $b - c$
$\{a\}, \{b, c\}, \{d, e\}$ after adding edge $d - e$
$\{a\}, \{b, c, d, e\}$ after adding edge $b - d$
$\{a, b, c, d, e\}$ after adding edge $a - b$

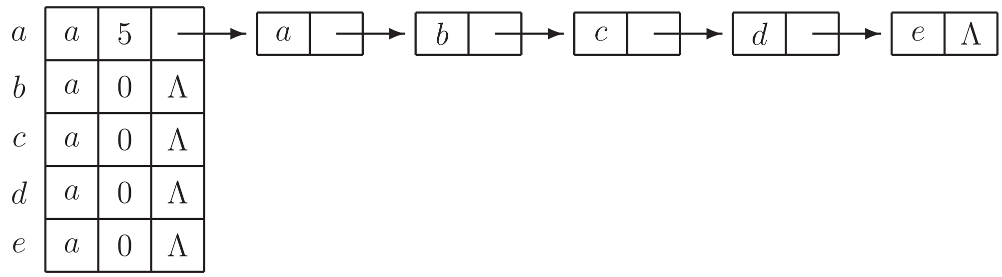
The linked-list representations of the collections of subsets at the same five moments are:





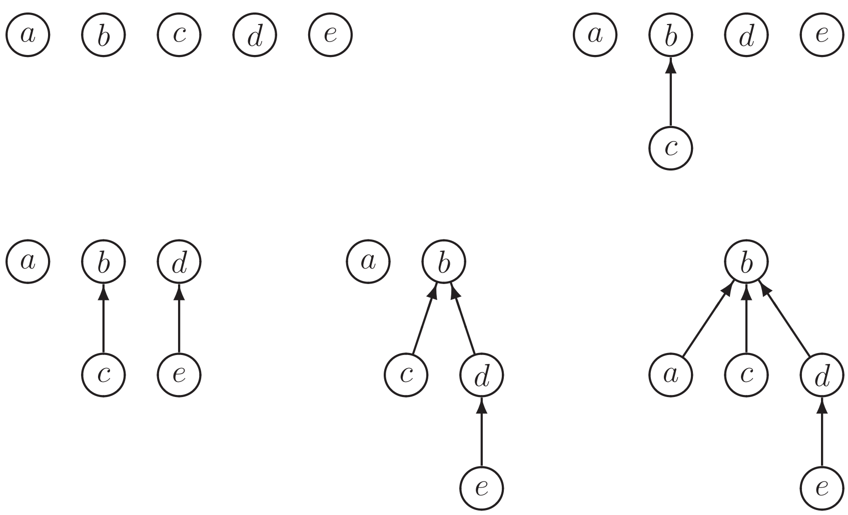






In these pictures, each array element has three fields: the name of (the representative of) the subset it is in, the size of the subset, and a pointer to the linked list with the elements of the subset. The last two fields make sense only for the representatives of the subsets. We arbitrarily chose the smallest element of a subsets to be the representative.

The tree representations of the respective collections of subsets are:



Here, when we performed a union of two trees, we attached the smaller tree (measured by its height) to the root of the larger tree. When the trees had the same height, the alphabetically smaller root became the root of the combined tree.

Exercise 3.

a. This statement is true. Kruskal's algorithm may start with the addition of edge e, and thus yield a minimum spanning tree including e.

b. This statement is false. Consider, e.g., a complete three-vertices graph, where each edge has the same (and thus minimum) weight, say weight 1. Let e be any of these edges. The subgraph containing the other two edges is a minimum spanning tree not containing e.

c. This statement is true. Suppose that a connected weighted graph with all distinct edge weights has two different minimum spanning trees, T_1 and T_2.

Let E^* be the set of all edges that occur in exactly one of the two trees. Because the trees are different, E^* cannot be empty. Let e_1 be the edge with minimum weight in E^*. Because all edge weights are distinct, e_1 is unique. Without loss of generality, assume that e_1 occurs in T_1, and not in T_2.

When we add edge e_1 to T_2, we create a cycle. This cycle must contain at least one edge e_2 that is not in T_1. Otherwise, T_1 would not be acyclic. Hence, e_2 is an element of E^*, and thus has a larger weight than e_1.

Now removing e_2 from T_2, after the addition of e_1, yields a spanning tree with a smaller total weight than T_2. This contradicts the assumption that T_2 is a minimum spanning tree.

d. This statement is false. Consider, e.g., a four-vertices tree, where all three edges have the same weight, say weight 1. Add to these tree the missing edges, all with the same, higher weight, say weight 2. The resulting graph has exactly one minimum spanning tree, the tree we started with, despite the fact that many edges have the same weight (weight 1 or weight 2).

Exercise 6. (Travelling Salesman Problem) This problem has been worked out in the slides of lecture 12.