Doo doo doo doo doo, doo doo doo doo doo doo
Doo doo doo doo doo, doo doo doo doo

– Crosby, Stills and Nash

Lessons in Play

Values of Games

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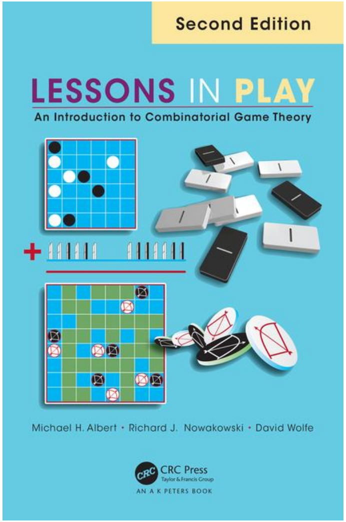
14–04–2020



Integers

$$0 \stackrel{\text{def}}{=} \{\,|\,\}$$

$$0 = \square$$

$$n \stackrel{\text{def}}{=} \{n-1\,|\,\}$$

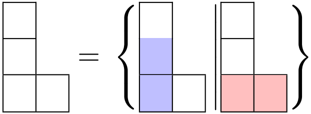
$$1 = \{\square\,|\,\}$$

$$-n = \{\,|\,1-n\}$$

$$-1 = \{\,|\,\square\}$$

(Fractional) Numbers

$$\frac{m}{2^j} = \left\{ \frac{m-1}{2^j} \,\middle|\, \frac{m+1}{2^j} \right\}$$



$$= \{0 \mid 1\} = \{\frac{1-1}{2^1} \mid \frac{1+1}{2^1}\} = \frac{1}{2^1}$$

$\frac{1}{2} + \frac{1}{2} = 1?$

$\frac{1}{2} = \{0 \mid 1\}, \quad 1 = \{0 \mid \}$

$\frac{1}{2} + \frac{1}{2} - 1 = \{0 \mid 1\} + \{0 \mid 1\} - 1 = 0$, and thus a second player win?

If one player moves on one the $\frac{1}{2}$, the other player moves on the other:
$0 + 1 - 1 = 0$ ✓

If Right moves -1 to 0:
$\frac{1}{2} + \frac{1}{2}$

Left moves $\frac{1}{2}$ to 0:
$\frac{1}{2}$: Loss for Right, second player win: 0

Weak number avoidance

Suppose that x is a number and G is not. If Left can win moving first on $x + G$, then she can do so, by moving on G (in the game $x + G$).

Some $x^L + G \geq 0$

Some $x^L + G > 0$ ($G \neq -x^L$)

Left wins moving first on $x^L + G$

Some $x^L + G^L \geq 0$

Some $x + G^L \geq 0$ ($x > x^L$)

Game-Number correspondence

$$a + b + c = 0 \quad \Leftrightarrow \quad A + B + C = 0$$
$$a + b + c < 0 \quad \Leftrightarrow \quad A + B + C < 0$$
$$a + b + c > 0 \quad \Leftrightarrow \quad A + B + C > 0$$

Game-Number correspondence(2)

$$a + b + c = 0 \quad \Leftrightarrow \quad A + B + C = 0$$

Suppose $a + b + c \geq 0$

Right moves to some $A^R + B + C$.

$a^R + b + c > 0$ ($A^R > A$).

$A^R + B + C > 0$.

$A + B + C \geq 0$.

$A + B + C \leq 0$ (symmetric).

$A + B + C = 0$.

The Simplest Number

For numbers $x^L < x^R$, the **simplest number** between x^L and x^R is defined by the unique number with the smallest birthday strictly between them.

For numbers $x^L < x^R$, the **simplest number** x between them is given by the following:

- If there are integer(s) n such that $x^L < n < x^R$, then x is the one that is smallest in absolute value.
- Otherwise, x is the number of the form $\frac{i}{2^j}$ between x^L and x^R for which j is minimal.

The Simplest Number (2)

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x^L	x^R	x
$\frac{1}{2}$	2	
$\frac{1}{8}$	$\frac{5}{8}$	
$-1\frac{27}{64}$	$-1\frac{9}{32} = -1\frac{18}{64}$	

The Simplest Number (2)

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x^L	x^R	x
$\frac{1}{2}$	2	1
$\frac{1}{8}$	$\frac{5}{8}$	$\frac{1}{2}$
$-1\frac{27}{64}$	$-1\frac{9}{32} = -1\frac{18}{64}$	$-1\frac{3}{8}(= -1\frac{24}{64})$

The Simplest Number (3)

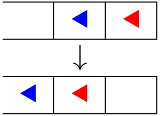
If all options of a game G are numbers and every left option G^L is strictly less than every right option G^R, then G is also a number:

G is the simplest number lying stritly between every G^L and every G^R.

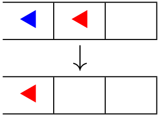
Push

Line of squares, with an open square on the left.

Blue and Red tokens that can only move left, and push each other.

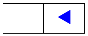
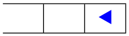


Tokens can be pushed off.

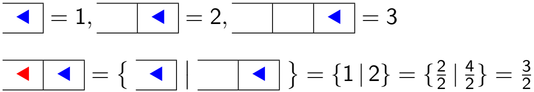


Who wins?

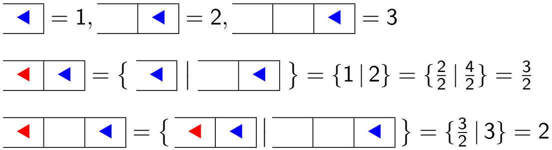
Who wins?

 $= 1$,  $= 2$,  $= 3$

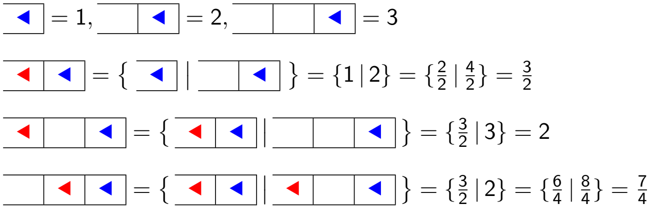
Who wins?



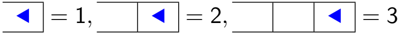
Who wins?

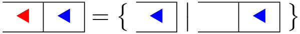


Who wins?

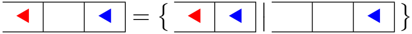


Who wins?

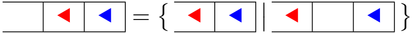




$= \{1 \mid 2\} = \{\tfrac{2}{2} \mid \tfrac{4}{2}\} = \tfrac{3}{2}$



$= \{\tfrac{3}{2} \mid 3\} = 2$



$= \{\tfrac{3}{2} \mid 2\} = \{\tfrac{6}{4} \mid \tfrac{8}{4}\} = \tfrac{7}{4}$

$= \{\tfrac{3}{2} \mid \tfrac{7}{4}, 2\} = \{\tfrac{3}{2} \mid \tfrac{7}{4}\} = \{\tfrac{12}{8} \mid \tfrac{14}{8}\} = \tfrac{13}{8}$

Who wins?

$$-\frac{3}{2} \quad + \quad -\frac{7}{4} \quad + \quad -2 \quad + \quad -\frac{13}{8}$$

$$= -\frac{1}{8}$$