Impartial Games (cont.)

Orson R. L. Peters, Universiteit Leiden.

Universiteit Leiden
The Netherlands

Impartial games recap

- Both Left and Right have the same options in any game state.
- Examples: CRAM, NIM.
- No outcome classes $\mathcal{L}$ and $\mathcal{R}$, only $\mathcal{N}$ and $\mathcal{P}$.
- G is its own negative $\implies G + G = 0$.
- G's canonical form must have equal left and right sides.
- Sums of impartial games are impartial.

Impartial games are infinitesimals

- $\boldsymbol{G}$ is all-small (either $\mathbf{0}$ or has both left and right options).
- Therefore $\boldsymbol{G}$ must be an infinitesimal (or $\mathbf{0}$).
- A small subset, e.g. $\uparrow > \mathbf{0}$ therefore $\uparrow \in \mathcal{L}$ and thus not impartial.

The values of the game Nim

- Let the value of an NIM heap of size n be the *nimber* $*n$.
- Define $*0 = 0$ and $*1 = *$.
- In general for $k > 0$:

$$*k = \{0, *, *2, \ldots, *(k-1) \mid 0, *, *2, \ldots, *(k-1)\}$$

- Why? By strong induction if we assume that $*k$ is the correct value of the heap of size k for all $k < n$ then the above definition works for n, as it perfectly shows the options you'd have when facing a heap of size n.

The canonical forms of the game Nim

- Our earlier definition is also immediately in canonical form.
- Why? Crucial lemma $i \neq j \implies *i \parallel *j$.
- Studying $*i - *j$ we find that
  - it can't be $\mathbf{0}$ ($*i \neq *j$),
  - $*i - *j = *i + *j$ is a sum of two impartial games,
  - thus it must be in $\mathcal{N}$ and $*i$ and $*j$ are incomparable.
- All options of $*n$ are mutually incomparable, thus no dominated options.
- All options of $*n$ are incomparable with $*n$ itself, thus no reversible options.

Values of nimber games

- If an impartial game G's options are all nimbers we can compute its value.
- Define *minimum excluded value* $\mathbf{mex}\, S = \min(\mathbb{N}_0 \setminus S)$.
- E.g. $\mathbf{mex}\{1,2\} = 0$, $\mathbf{mex}\{0,1,2,4,7\} = 3$.

Theorem 7.7

If impartial game $G = \{*l_1, *l_2, \ldots, *l_k \mid G^L\}$ and $\mathbf{mex}\{l_1, l_2, \ldots, l_k\} = n$ then the value of G is $*n$, and thus its canonical form is $\{0, *, \ldots, *(n-1) \mid 0, *, \ldots, *(n-1)\}$.

Proof of values of nimber games

1. We prove $G - *n = 0$. Note that $G - *n = G + *n$.
2. Also note that $*a$ is always an option of $*b$ when $a < b$.
3. $*k$ with $k < n$ is an option of both G and $*n$.
4. $*k$ with $k = n$ is never an option.
5. $*k$ with $k > n$ is only an option in G.
6. If the first player moves by playing $*k$ with $k < n$ in either game the second player can answer with $*k$ in the other game, giving $*k + *k = 0$.
7. If the first player moves by playing $*k$ with $k > n$ (which must be in G) the second player can answer locally by playing $*n$, giving $*n + *n = 0$.

All impartial games are nimber games

1. Let's assume all impartial games born on days $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n} - \mathbf{1}$ are nimber games.
2. All options of impartial games born on day $\boldsymbol{n}$ are impartial games born on some day $\boldsymbol{k} < \boldsymbol{n}$.
3. By our assumption this means all options of all impartial games born on day $\boldsymbol{n}$ are nimber games.
4. Therefore all impartial games born on day $\boldsymbol{n}$ are nimber games.
5. $\mathbf{0}$ is a nimber game, and is the only impartial game at day $\mathbf{0}$.
6. Therefore we have a base case and can apply strong induction.

A massive corollary

- All impartial games are nimber games.
- All impartial games thus have a value of $*n$ for some n.
- **All impartial games are equivalent to some Nim heap of size n.**
- We can find out n by applying Theorem 7.7 in a bottom-up style.
- This value is also called the *nim-value* or *Grundy-value* of impartial game G.
- This is notated $\mathcal{G}(G) = n$ where $G = *n$.
- But what about the value of $G + H$?

Exclusive-or

The exclusive-OR (notated $\oplus$) is an operation on non-negative integers that

- is commutative, $a \oplus b = b \oplus a$,
- is associative, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$,
- has zero as its additive identity, $a \oplus 0 = a$ and
- is its own inverse, $a \oplus a = 0$.

It's defined by creating a new number that has 1 exactly where the binary digits of a and b differ and 0 everywhere else.

Adding Nim heaps

- Let $G = \textsc{Nim}(a, b, \ldots, k)$. Then G is a $\mathcal{P}$-position if and only if $q = a \oplus b \oplus \cdots \oplus k = 0$.
- Why? We provide a winning strategy.
- If $q = 0$ you either already lost (all heaps are 0) or you are forced to increase the value above zero.
- When $q > 0$ the game is never over, and we can always bring q back to 0 in one move.
- Thus $q > 0$ are the $\mathcal{N}$-positions and $q = 0$ are the $\mathcal{P}$-positions.

Forced increase of XOR-sum

If $a \oplus b \oplus \cdots \oplus k = 0$ and it's your turn WLOG you must reduce a to a' but now the XOR-sum has value

$$\begin{aligned} a' \oplus b \oplus \cdots \oplus k &= \\ 0 \oplus a' \oplus b \oplus \cdots \oplus k &= \\ (a \oplus a) \oplus a' \oplus b \oplus \cdots \oplus k &= \\ a \oplus a' \oplus (a \oplus b \oplus \cdots \oplus k) &= \\ a \oplus a' \oplus 0 &= \\ a \oplus a' & \end{aligned}$$

which is bigger than zero for any legal move $a \neq a'$.

Reducing the XOR-sum to zero in one move

- Let $q = a \oplus b \oplus \cdots \oplus k > 0$.
- Note that there must exist a $m \in \{a, b, \ldots, k\}$ such that the leftmost bit set of q is also set in m.
- Therefore that leftmost bit becomes 0 in $m \oplus q$, and thus $m \oplus q < m$.
- This means that reducing the heap of size m to $m \oplus q$ is a legal move, and that our new value is (taking into account commutativity):

$$(a \oplus b \oplus \cdots \oplus k) \oplus q = q \oplus q = 0$$

Adding games

- We know that the game NIM$(a, b, a \oplus b)$ is always a $\mathcal{P}$-position since $a \oplus b \oplus (a \oplus b) = 0$.
- Going back to Nimbers this allows us to derive

$$\begin{aligned} *a + *b + *(a \oplus b) &= 0 \\ *a + *b + *(a \oplus b) + *(a \oplus b) &= *(a \oplus b) \\ *a + *b &= *(a \oplus b). \end{aligned}$$

- Which generalizes to

$$*a + *b + *c + \cdots + *n = *(a \oplus b \oplus c \oplus \cdots \oplus n).$$

Wrapping up

- Impartial games have a direct equivalence with NIM games.
- An impartial game has *nim-value* or *Grundy-value* $\mathcal{G}(G) = k$ iff $G = *k$.
- $\mathcal{G}(G) = \text{mex}\{\mathcal{G}(H) \mid H \in G^L\}$.
- Through adding NIM games we can see that

$$*a + *b + \cdots + *n = *(a \oplus b \oplus \cdots \oplus n).$$

- If G, H, and J are impartial games, then $G = H + J$ iff $\mathcal{G}(G) = \mathcal{G}(H) \oplus \mathcal{G}(J)$.