Lessons In Play

Stops, All-Smalls & Infinitesimals

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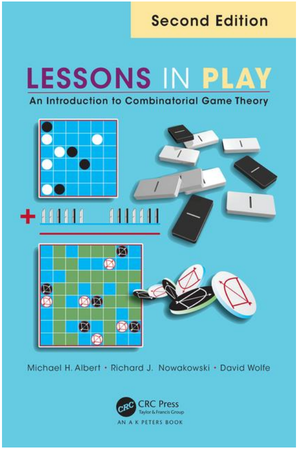
21-4-2020


Second Edition

LESSONS IN PLAY

An Introduction to Combinatorial Game Theory

Michael H. Albert • Richard J. Nowakowski • David Wolfe

CRC Press
Taylor & Francis Group
AN A K PETERS BOOK

Contents

Stops

- Players promise to stop playing the moment a game is a number
- This moment is the **Stopping Position**
- Left wants to maximize
- Right wants to minimize

Definition 5.32

- Given a game G, the stopping point is recursively defined
- The *Left Stop* is the number we get when Left starts

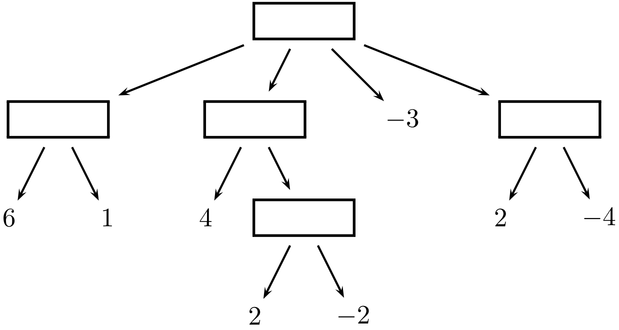
$$\mathbf{LS}(G) = \begin{cases} G & \text{if } G \text{ is a number} \\ max(\mathbf{RS}(G^L)) & \text{if } G \text{ is not a number} \end{cases}$$

- The *Right Stop* when Right starts

$$\mathbf{RS}(G) = \begin{cases} G & \text{if } G \text{ is a number} \\ min(\mathbf{LS}(G^R)) & \text{if } G \text{ is not a number} \end{cases}$$

Exercise 5.33

$$\left\{ \{6 \mid 1\}, \{4 \mid \{2 \mid -2\}\} \mid -3, \{2 \mid -4\} \right\}$$



Definition 5.32

$$\mathbf{LS}(G) = \begin{cases} G \\ max(\mathbf{RS}(G^L)) \end{cases} \qquad \mathbf{RS}(G) = \begin{cases} G & G \in \mathbb{Q} \\ min(\mathbf{LS}(G^R)) & G \notin \mathbb{Q} \end{cases}$$

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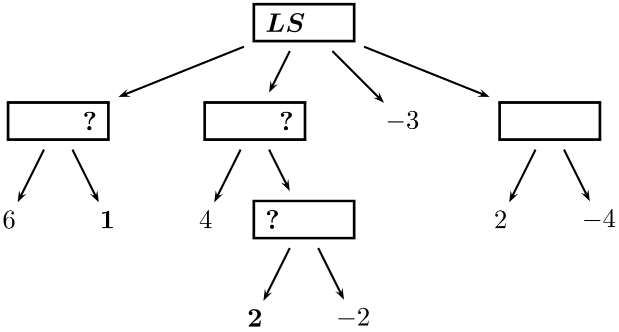

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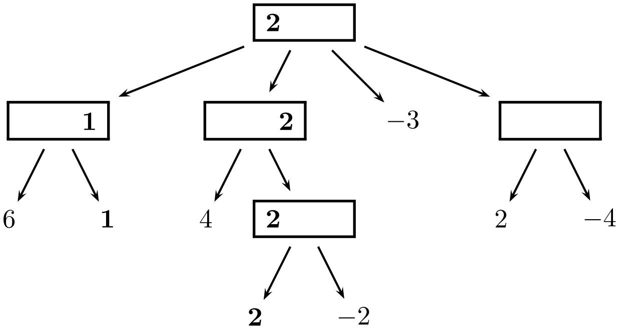

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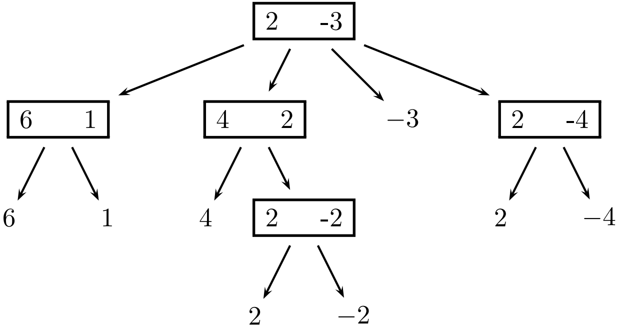
Exercise 5.33 Solution

$$\left\{ \{6 \mid 1\}, \{4 \mid \{2 \mid -2\}\} \mid -3, \{2 \mid -4\} \right\}$$



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Infinitesimal Games and $*$

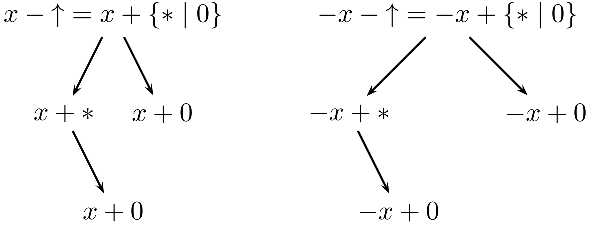
Definition

A game $\boldsymbol{G}$ is called *infinitesimal* $\iff -x < \boldsymbol{G} < x$ for all positive numbers x

- $*$ is infinitesimal
- $* = \{0 \mid 0\}$, and $-* = \{0 \mid 0\}$
- We only need to show: $* < x$ for any positive number x (since this implies: $-* > -x \quad \Rightarrow \quad -x < *$)
- Consider $x - * = x + *$
- Left can move to $x + 0 = x$ and wins
- By *Weak-Number-Avoidance*: if Right can win going first, it is with a move in $*$. But $x + 0 \in \mathcal{L}$
- $\boldsymbol{* < x}$

Up and Down

- $\uparrow \overset{\text{def}}{=} \{0 \mid *\}$, and $\downarrow \overset{\text{def}}{=} \{* \mid 0\}$
- Take some number $x > 0$



$x - \uparrow = x + \{* \mid 0\}$

$x + *$ $\quad$ $x + 0$

$x + 0$

$-x - \uparrow = -x + \{* \mid 0\}$

$-x + *$ $\quad$ $-x + 0$

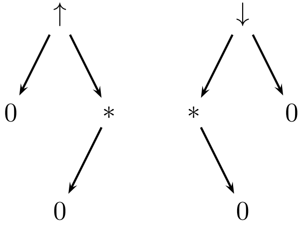
$-x + 0$

$x > \uparrow$ $\qquad$ $-x < \uparrow$

- $-x < \uparrow < x \iff \uparrow$ is *infinitesimal*
- Idem for $\downarrow$

Positive and Negative Infinitesimals

- In fact, $\uparrow$ is a positive and $\downarrow$ a negative *infinitesimal*



$\uparrow > 0$ $\quad$ $\downarrow < 0$

- So now we know:
  $0 < \uparrow < x$ and $-x < \downarrow < 0$, for all positive numbers x

Multiple Ups and Downs

We write sums of ups and downs using *double*, *triple*, and *quadruple* arrows:

$$\begin{array}{llll} \Uparrow & = \uparrow + \uparrow & \Downarrow & = \downarrow + \downarrow \\ ⤊ & = \uparrow + \uparrow + \uparrow & ⤋ & = \downarrow + \downarrow + \downarrow \\ ⟰ & = \uparrow + \uparrow + \uparrow + \uparrow & ⟱ & = \downarrow + \downarrow + \downarrow + \downarrow \end{array}$$

All-Small

We call a game $\boldsymbol{G}$ *all-small* if for every position $\boldsymbol{H}$ in $\boldsymbol{G}$:
Left has a move from $\boldsymbol{H} \iff$ Right has a move from $\boldsymbol{H}$

Alternatively, $\boldsymbol{G}$ is *all-small* $\iff$:

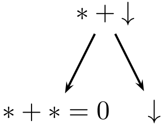
- $G = 0$, or
- $\mathcal{G}^L$ and $\mathcal{G}^R$ are non-empty and every element is *all-small*

All-Small Examples

- 0 is the only *all-small* number.
- For example: $1 = \{0 \mid \}$ is not *all-small*, Left has move 0 but Right has nothing.
- $* = \{0 \mid 0\}$ is *all-small*
- $\uparrow = \{0 \mid *\} = \big\{0 \mid \{0 \mid 0\}\big\}$ is *all-small*

Some Up & Down comparisons

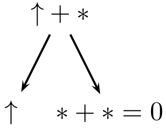
- $\downarrow \parallel * \parallel \uparrow$



$* + \downarrow$

$* + * = 0$ $\quad \downarrow$

$\mathcal{N}$

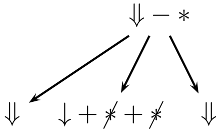


$\uparrow + *$

$\uparrow$ $\quad * + * = 0$

$\mathcal{N}$

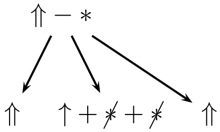
- $\Downarrow < * < \Uparrow$



$\Downarrow - *$

$\Downarrow$ $\quad \downarrow + \cancel{*} + \cancel{*}$ $\quad \Downarrow$

$\mathcal{R}$



$\Uparrow - *$

$\Uparrow$ $\quad \uparrow + \cancel{*} + \cancel{*}$ $\quad \Uparrow$

$\mathcal{L}$

Recall

$\uparrow = \{0 \mid *\}$

Recall

$\downarrow = \{* \mid 0\}$

Recall

$* = -*$

$= \{0 \mid 0\}$

Recall

$* + * = 0$

Notation: Addition

Instead of writing down entire sums, we concatenate the summands:

- Numbers, then
- $\uparrow$s and $\downarrow$s, then
- $*$

So $2\ +\uparrow+\uparrow+\ * = 2\Uparrow *$

Notation: $\uparrow$ and $\uparrow *$ Multiplication

The *canonical form* of $\uparrow, \Uparrow, ⤊$ ... and $\uparrow *, \Uparrow *, ⤊ *$ ... :

$$\begin{array}{lll@{\qquad\qquad}lll}
\uparrow & = & \{0 \mid *\} & \uparrow * & = & \{0, * \mid 0\} \\
\Uparrow & = & \{0 \mid \uparrow *\} & \Uparrow * & = & \{0 \mid \uparrow\} \\
⤊ & = & \{0 \mid \Uparrow *\} & ⤊ * & = & \{0 \mid \Uparrow\} \\
⟰ & = & \{0 \mid ⤊ *\} & ⟰ * & = & \{0 \mid ⤊\}
\end{array}$$

To further investigate this pattern, we denote the multiplication of game $\boldsymbol{g}$, by a scalar $\boldsymbol{n}$:

$$\boldsymbol{n} \cdot \boldsymbol{g} = \begin{cases} 0 & \textit{if } \boldsymbol{n} = 0 \\ \overbrace{\boldsymbol{g} + \boldsymbol{g} + \ldots + \boldsymbol{g}}^{\boldsymbol{n} \textit{ times}} & \textit{if } \boldsymbol{n} > 0 \\ (-\boldsymbol{n}) \cdot (-\boldsymbol{g}) & \textit{if } \boldsymbol{n} < 0 \end{cases}$$

For example:

- $3 \cdot \uparrow = ⤊$
- $-3 \cdot \uparrow = ⤋$

Theorem 5.48

For $n \geq 1$, the canonical forms of $\boldsymbol{n} \cdot \uparrow$ and $\boldsymbol{n} \cdot \uparrow\!*$ $= (n \cdot \uparrow) + *$ are given by:

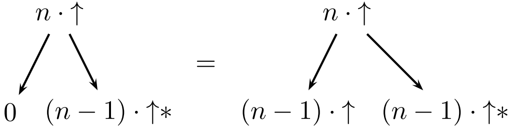
$$\boldsymbol{n} \cdot \uparrow \; = \{0 \mid (\boldsymbol{n} - 1) \;\; \cdot \uparrow *\} \quad \textit{if}\, \boldsymbol{n} \geq 1 \tag{1}$$

$$\boldsymbol{n} \cdot \uparrow * = \begin{cases} \{0 \mid (\boldsymbol{n} - 1) \cdot \uparrow\} & \textit{if}\, \boldsymbol{n} > 1 \\ \{0, * \mid 0\} & \textit{if}\, \boldsymbol{n} = 1 \end{cases} \tag{2}$$

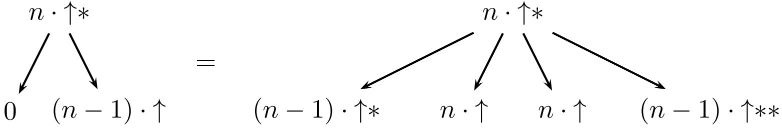
Theorem 5.48 Proof — Assumption

We assume the provided definitions to hold,
so the given canonical form equals the naive representation:

- (1): For $n > 0$:

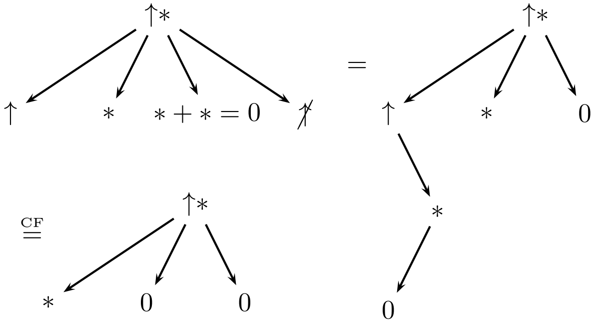


- (2): For $n > 1$:



Theorem 5.48 Proof — (2) for $n = 1$

This is an easily proven special case for (2)



- 0 *dominates* ↑ for Right
- ↑ is *reversible* in Left:
  $\uparrow * - * = \uparrow > 0 \quad \Rightarrow \quad * < \uparrow *$
- $\uparrow * = \{0, * \mid 0\}$

Recall

$\uparrow = \{0 \mid *\}$

Recall

$\downarrow < 0 < \uparrow$

Recall

$* = -*$

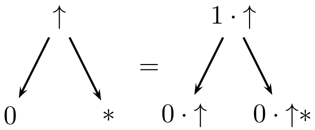
$= \{0 \mid 0\}$

Recall

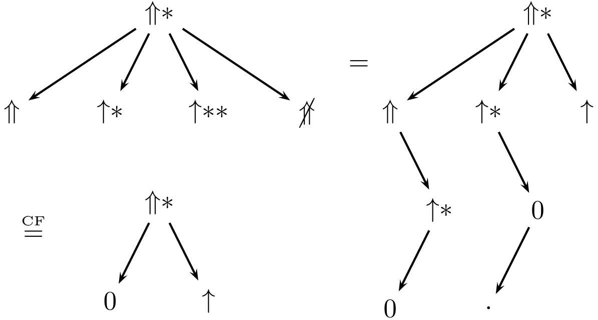
$* + * = 0$

Theorem 5.48 Proof — Base Cases

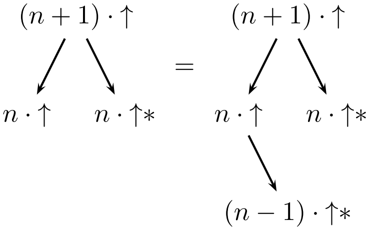
- (1) $n = 1$:



- (2) $n = 2$:



Theorem 5.48



$(n+1)\cdot\uparrow$

$n\cdot\uparrow$ $\quad$ $n\cdot\uparrow*$

$=$

$(n+1)\cdot\uparrow$

$n\cdot\uparrow$ $\quad$ $n\cdot\uparrow*$

$(n-1)\cdot\uparrow*$

Check $(n-1)\cdot\uparrow* \leq (n+1)\cdot\uparrow$:

- $(n+1)\cdot\uparrow - (n-1)\cdot\uparrow*$
- $2\cdot\uparrow - * = \Uparrow - * > 0$
- $(n-1)\cdot\uparrow* < (n+1)\cdot\uparrow$
- $n\cdot\uparrow$ is *Reversible*, replace with 0:

Proof — (1) for $(n+1)$

Recall

$\uparrow = \{0 \mid *\}$

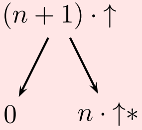
Recall

$\Downarrow < * < \Uparrow$

From (1) (2)

$\boldsymbol{n}\cdot\uparrow = \{0 \mid (\boldsymbol{n}-1)\cdot\uparrow\}$
$\boldsymbol{n}\cdot\uparrow* = \{0 \mid (\boldsymbol{n}-1)\cdot\uparrow\}$

Canonical $n+1$ case



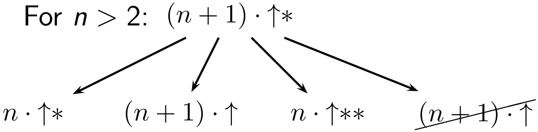
$(n+1)\cdot\uparrow$

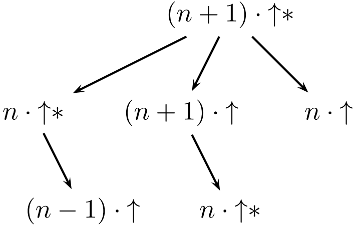
0 $\quad$ $n\cdot\uparrow*$

Theorem 5.48

Proof — (2) for $(n+1)$

For $n > 2$:





- $n \cdot \uparrow * < (n+1) \cdot \uparrow *$
  $(n-1) \cdot \uparrow < (n+1) \cdot \uparrow *$
- $n < (n+1)$
  $0 < \Uparrow * \iff * < \Uparrow$

Recall

$\uparrow = \{0 \mid *\}$

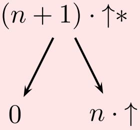
Recall

$\Downarrow < * < \Uparrow$

From (1) (2)

$\boldsymbol{n} \cdot \uparrow = \{0 \mid (\boldsymbol{n} - 1) \cdot \uparrow\}$
$\boldsymbol{n} \cdot \uparrow * = \{0 \mid (\boldsymbol{n} - 1) \cdot \uparrow\}$

Canonical $n + 1$ case



Theorem 5.48 Proof — Conclusion

- Assuming the following holds for $n \geq 1$:
  $n \cdot \uparrow \ = \{0 \mid (n-1) \ \cdot \uparrow *\}$

$$n \cdot \uparrow * = \begin{cases} \{0 \mid (n-1) \cdot \uparrow\} & if\, n > 1 \\ \{0, * \mid 0\} & if\, n = 1 \end{cases}$$

- We have proven the former holds for $n = 1$
- We have proven the latter holds for $n = 1$ and $n = 2$
- We have proven both then hold for $n + 1$

Proof by Induction

Theorem 5.48

A similar situation for ↓, with a similar proof.

$$n \cdot {\downarrow} = \{(n-1) \cdot {\downarrow}{*} \mid 0\}$$

$$n \cdot {\downarrow}{*} = \begin{cases} \{(n-1) \cdot {\downarrow} \mid 0\} & \textit{if } n > 1 \\ \{0 \mid 0, *\} & \textit{if } n = 1 \end{cases}$$

Conclusion

- Use *stop points* to give a value to any game
- Definition of *all-small* games
- *Infinitesimals*: Up, Down, Star
- *Infinitesimal* addition and multiplication