Lessons In Play

Canonical Form: Reducing Games

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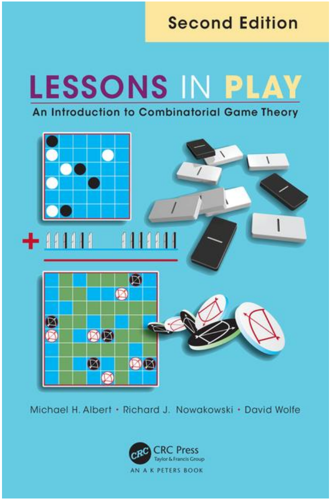
7-4-2020



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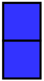
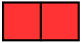
Domination in Domineering

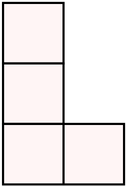
Left plays:



Right plays:

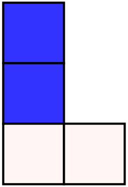


- Consider the board:
- Left moves first



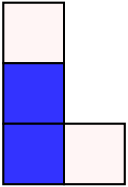
Domination in Domineering

Move A



-1

Move B



0

- Left always prefers B over A
- Move B ***dominates*** A
- Simplify a game through One-Hand-Tied principle

Theorem 4.33 Remove Dominated Option

- Given the games:

$$G = \{A, B, C \dots \mid H, I, J, \dots\}$$
$$G' = \{B, C \dots \mid H, I, J, \dots\}$$

- If $B \geq A$ then $G = G'$
- Idem for Right: if $H \leq I$ remove I

Recall

For games G and H
$G > H \iff G - H \in \mathcal{L}$
$\mathrm{G} < \mathrm{H} \iff G - H \in \mathcal{R}$

Recall

For games G and H
$G = H \iff G - H = 0$
$\iff G - H \in \mathcal{P}$

Theorem 4.33 Proof

Show: if $B \geq A$, then $G - G' = 0 \iff \boldsymbol{G} + (\text{-}\boldsymbol{G'}) = 0$

$$\overbrace{\{A, B, C,\ \ldots \mid H, I, J,\ \ldots\}}^{\boldsymbol{G}} \quad + \quad \overbrace{\{\text{-}H, \text{-}I, \text{-}J,\ \ldots \mid \text{-}B, \text{-}C,\ \ldots\}}^{\text{-}\boldsymbol{G'}}$$

Recall

For a game $G = \{\ \mathcal{G}^L \mid \mathcal{G}^R\ \}$:
$\text{-}G = \{\ \text{-}\mathcal{G}^R \mid \text{-}\mathcal{G}^L\ \}$

Theorem 4.33 Proof

Show: if $B \geq A$, then $G - G' = 0 \iff \boldsymbol{G} + (\text{-}\boldsymbol{G'}) = 0$

$$\overbrace{\{\overset{\checkmark}{A}, B, C, \ \dots \mid H, I, J, \ \dots\}}^{\boldsymbol{G}} \quad + \quad \overbrace{\{\text{-}H, \text{-}I, \text{-}J, \ \dots \mid \text{-}B, \text{-}C, \ \dots\}}^{\text{-}\boldsymbol{G'}}$$

- Left moves to A in G
  Right moves to $-B$ in G'
  $A - B \leq 0$

Theorem 4.33 Proof

Show: if $B \geq A$, then $G - G' = 0 \iff \boldsymbol{G} + (\text{-}\boldsymbol{G'}) = 0$

$$\overbrace{\{\check{A}, \check{B}, \check{C}, \ \ldots \mid \check{H}, \check{I}, \check{J}, \ \ldots\}}^{\boldsymbol{G}} \quad + \quad \overbrace{\{\text{-}\check{H}, \text{-}\check{I}, \text{-}\check{J}, \ \ldots \mid \text{-}\check{B}, \text{-}\check{C}, \ \ldots\}}^{\text{-}\boldsymbol{G'}}$$

- Left moves to A in G
  Right moves to $-B$ in G'
  $A - B \leq 0$
- First player moves to m in G,
  Second player moves to $-m$ in G'
  $m - m = 0$
- Idem for any m in G'

Theorem 4.33 Proof

Show: if $B \geq A$, then $G - G' = 0 \iff \boldsymbol{G} + (\text{-}\boldsymbol{G'}) = 0$

$$\overbrace{\{\overset{\checkmark}{A}, \overset{\checkmark}{B}, \overset{\checkmark}{C}, \ \dots \mid \overset{\checkmark}{H}, \overset{\checkmark}{I}, \overset{\checkmark}{J}, \ \dots\}}^{\boldsymbol{G}} \quad + \quad \overbrace{\{\text{-}\overset{\checkmark}{H}, \text{-}\overset{\checkmark}{I}, \text{-}\overset{\checkmark}{J}, \ \dots \mid \text{-}\overset{\checkmark}{B}, \text{-}\overset{\checkmark}{C}, \ \dots\}}^{\text{-}\boldsymbol{G'}}$$

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  Right moves to $-B$ in G'
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- First player moves to m in G,
  Second player moves to $-m$ in G'
  $m - m = 0$

- Idem for any m in G'

- **All Second Player Wins**

Reversibility

Imagine:

- A game where Right has some advantage
- Left chooses some "good" move A
- But then Right has the move A^R, which reverses A and Right regains the advantage
- Left "actually" chooses A^R

Definition: Reversible Move

Let us formalize:

- Given a game $G = \{ A, \ldots \mid \ldots \}$
- $A = \{ \mathcal{A}^L \mid \mathcal{A}^R \}$
- We call A reversible $\iff \exists A^R \in \mathcal{A}^R : A^R \leq G$
- A^R is called a reversing move
- One-Hand-Tied: If Left moves to A, Right responds with A^R

Example: Reversible Move

We have a game $G = \{ \quad \{ 2 \mid \{ -8 \mid -9 \} \} \quad \mid -1 \}$

- $A = \{ 2 \mid \{ -8 \mid -9 \} \}$
- $A^R = \{ -8 \mid -9 \}$

Example: Reversible Move

We have a game $G = \{ \quad \{ 2 \mid \{ \text{-8} \mid \text{-9} \} \} \quad \mid \text{-1} \}$

- $A = \{ 2 \mid \{ \text{-8} \mid \text{-9} \} \}$
- $A^R = \{ \text{-8} \mid \text{-9} \}$

$A^R \leq G$?

- $G - \{ \text{-8} \mid \text{-9} \} = G + \{ 9 \mid 8 \}$

Example: Reversible Move

We have a game $G = \{ \quad \{ 2 \mid \{ \text{-}8 \mid \text{-}9 \} \} \quad \mid \text{-}1 \}$

- $A = \{ 2 \mid \{ \text{-}8 \mid \text{-}9 \} \}$
- $A^R = \{ \text{-}8 \mid \text{-}9 \}$

$A^R \leq G$?

- $G - \{ \text{-}8 \mid \text{-}9 \} = G + \{ 9 \mid 8 \}$
- Left can win moving first:
  $\overset{L}{\mapsto} \; G + 9 \; \overset{R}{\mapsto} \; \text{-}1 + 9 > 0$

Example: Reversible Move

We have a game $G = \{ \quad \{ 2 \mid \{ \text{-}8 \mid \text{-}9 \} \} \quad \mid \text{-}1 \}$

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$A^R \leq G$?

- $G - \{ \text{-}8 \mid \text{-}9 \} = G + \{ 9 \mid 8 \}$
- Left can win moving first:
  $\overset{L}{\mapsto} \; G + 9 \; \overset{R}{\mapsto} \; \text{-}1 + 9 > 0$
- Right cannot win moving first:
  $\overset{R}{\mapsto} \; \text{-}1 + \{ 9 \mid 8 \} \; \overset{L}{\mapsto} \; \text{-}1 + 9 > 0$
  and
  $\overset{R}{\mapsto} \; G + 8 \; \overset{L}{\mapsto} \; \{ 2 \mid \{ \text{-}8 \mid \text{-}9 \} \} + 8 \; \overset{R}{\mapsto} \; \{ \text{-}8 \mid \text{-}9 \} + 8$
  $\overset{L}{\mapsto} \; \text{-}8 + 8 = 0$

Example: Reversible Move

We have a game $G = \{ \quad \{ 2 \mid \{ \text{-}8 \mid \text{-}9 \} \} \quad \mid \text{-}1 \}$

- $G - \{ \text{-}8 \mid \text{-}9 \} \in \mathcal{L}$
- $G - \{ \text{-}8 \mid \text{-}9 \} > 0$
- $\{ \text{-}8 \mid \text{-}9 \} < G$
- $\{ 2 \mid \{ \text{-}8 \mid \text{-}9 \} \}$ is ***reversible***

Example: Reversible Move

We have a game $G = \{ \quad \{ 2 \mid \{ \text{-8} \mid \text{-9} \} \} \quad \mid \text{-1} \}$

- $G - \{ \text{-8} \mid \text{-9} \} \in \mathcal{L}$
- $G - \{ \text{-8} \mid \text{-9} \} > 0$
- $\{ \text{-8} \mid \text{-9} \} < G$
- $\{ 2 \mid \{ \text{-8} \mid \text{-9} \} \}$ is ***reversible***

We could then say

- If Left moves to $\{ 2 \mid \{ \text{-8} \mid \text{-9} \} \}$,
  Right will move to $\{ \text{-8} \mid \text{-9} \}$
- When considering $\{ 2 \mid \{ \text{-8} \mid \text{-9} \} \}$,
  we can replace it with -8
- $G = \{ \text{-8} \mid \ -1 \}$

Theorem 4.34 Bypass Reversible Option

Given

- A game $G = \{\ A, B, C,\ ...\mid H, I, J,\ ...\ \}$
- A reversible option A
- Some right option of A: $A^R = \{\ W, X, Y,\ ...\mid ...\ \}$

Recall

$A = \{\ \mathcal{A}^L \mid \mathcal{A}^R\ \}$
$A^R \in \mathcal{A}^R :\ A^R \leq G$

Theorem 4.34 — Bypass Reversible Option

Given

- A game $G = \{\ A, B, C,\ \dots \mid H, I, J,\ \dots\ \}$
- A reversible option A
- Some right option of A: $A^R = \{\ W, X, Y,\ \dots \mid \dots\ \}$

Then $G = G'$
With $G' = \{\ W, X, Y, \dots, B, C,\ \dots \mid H, I, J,\ \dots\ \}$

Recall

$A = \{\ \mathcal{A}^L \mid \mathcal{A}^R\ \}$
$A^R \in \mathcal{A}^R :\ A^R \leq G$

Theorem 4.34 Proof

$$\overbrace{\{A, \overset{\checkmark}{B}, \overset{\checkmark}{C}, \; .. |\; \overset{\checkmark}{H}, \overset{\checkmark}{I}, \overset{\checkmark}{J}, \; ..\}}^{G} + \overbrace{\{\text{-}\overset{\checkmark}{H}, \text{-}\overset{\checkmark}{I}, \text{-}\overset{\checkmark}{J}, \; .. | \text{-}W, \text{-}X, \text{-}Y, \; .., \text{-}\overset{\checkmark}{B}, \text{-}\overset{\checkmark}{C}, \; ..\}}^{\text{-}G'}$$

$$\xrightarrow{R}$$

$$G - W$$

L wins:

$$\boldsymbol{G \geq A^R}$$

For the first move:

- Respond to all double moves with their counterpart
- If Right starts with -W, Left wins going second

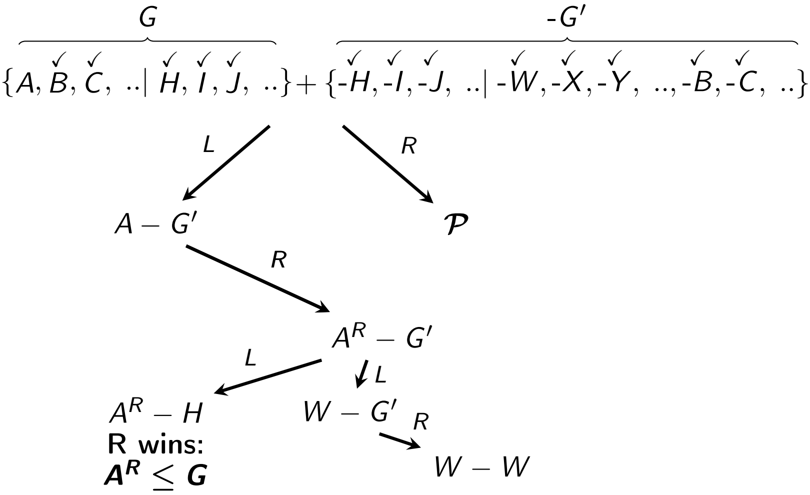
Recall

$A^R = \{\; W, X, Y, \; ... \mid ... \;\}$

Note

(-)W represents (-)$\{\; W, X, Y, ... \;\}$

Theorem 4.34 Proof



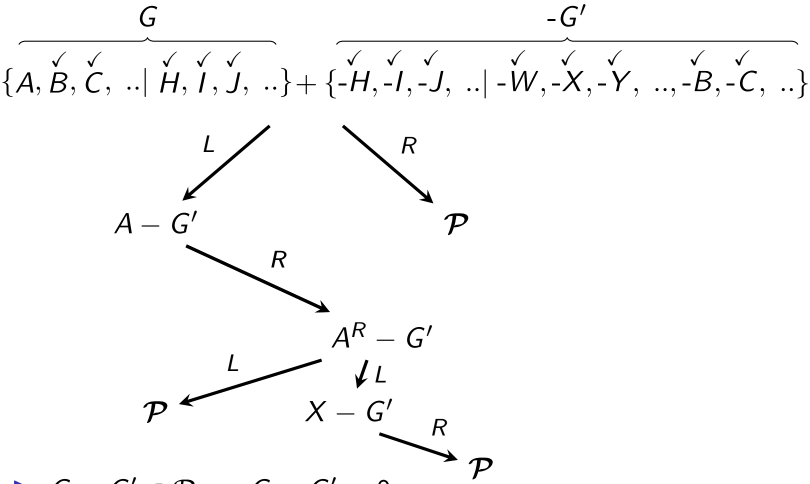
Recall

$A^R = \{\ W, X, Y, \ ... \mid ... \ \}$

Note

(-)H represents (-)$\{\ H, I, J, ... \ \}$

Theorem 4.34 Proof



- $G - G' \in \mathcal{P}$, so $G - G' = 0$
- $\{\, A, B, C, \ \dots \mid H, I, J, \ \dots \,\} =$
  $\{\, W, X, Y, \dots, B, C, \ \dots \mid H, I, J, \ \dots \,\}$

Reduction Rules

Reduction Rule: Domination

If we have some game ***G*** *where a move* ***A*** *is* ***dominated*** *by another move from the same player.*
We may remove ***A*** *from* ***G***.

Reduction Rule: Reversible Moves

If we have some game ***G*** *where a player has a move* ***A*** *that is* ***reversible****, where* $\boldsymbol{A^R}$ *is a* ***reversing move****.*
A *may be replaced by the moves that this player would have in* $\boldsymbol{A^R}$.

Example 4.42

$G = \{ \quad \{ 4 \mid 1 \}, 0 \quad \mid 2 \}$

Domination:

- $\{ 4 \mid 1 \} \in \mathcal{L}$, so $\{ 4 \mid 1 \} > 0$
- $\{ 4 \mid 1 \}$ dominates 0
- $G = \{ \quad \{ 4 \mid 1 \} \quad \mid 2 \}$

Example 4.42

$$G = \{ \quad \{ 4 \mid 1 \} \quad \mid 2 \}$$

Reversing option for Left:

- $A = \{ 4 \mid 1 \}$, $A^R = 1 = \{ 0 \mid \}$
- $G - A^R = \{ \quad \{ 4 \mid 1 \} \quad \mid 2 \} + \{ \mid 0 \}$

Example 4.42

$$G = \{\ \ \{4 \mid 1\}\ \ \mid 2\}$$

Reversing option for Left:

- $A = \{4 \mid 1\}$, $A^R = 1 = \{0 \mid \}$
- $G - A^R = \{\ \ \{4 \mid 1\}\ \ \mid 2\} + \{\mid 0\}$
- $\overset{R}{\mapsto} \{\ \ \{4 \mid 1\}\ \ \mid 2\} + 0 \Rightarrow$ Left wins going second
- $\overset{R}{\mapsto} 2 + \{\mid 0\} = 1 \Rightarrow$ Left wins going second

Example 4.42

$$G = \{ \quad \{ 4 \mid 1 \} \quad \mid 2 \}$$

Reversing option for Left:

- $A = \{ 4 \mid 1 \}$, $A^R = 1 = \{ 0 \mid \}$
- $G - A^R = \{ \quad \{ 4 \mid 1 \} \quad \mid 2 \} + \{ \mid 0 \}$
- $\overset{\mathsf{R}}{\mapsto} \{ \quad \{ 4 \mid 1 \} \quad \mid 2 \} + 0 \Rightarrow$ Left wins going second
- $\overset{\mathsf{R}}{\mapsto} 2 + \{ \mid 0 \} = 1 \Rightarrow$ Left wins going second
- Left wins going second at the least
- $\{ 0 \mid \} \leq G$, so $G = \{ 0 \mid 2 \}$

Example 4.42

$G = \{\, 0 \mid 2 \,\}$

Reversing option, now for Right:

- $G = \{\, 0 \mid \quad \{\, 1 \mid \,\} \quad \}$
- $A = \{\, 1 \mid \,\}, A^R = 1 = \{\, 0 \mid \,\}$

Example 4.42

$G = \{\ 0 \mid 2\ \}$

Reversing option, now for Right:

- $G = \{\ 0 \mid \quad \{\ 1 \mid\ \} \quad \}$
- $A = \{\ 1 \mid\ \}, A^R = 1 = \{\ 0 \mid\ \}$
- Remember: $\{\ 0 \mid\ \} \leq G$
- $G = \{\ 0 \mid\ \} = 1$

Conclusion

- Through the application of these rules, we can reduce the expressions of a game.
- More complex game expressions might now actually be readable

Reduction Rule: Domination

If we have some game $\boldsymbol{G}$ *where a move* $\boldsymbol{A}$ *is* ***dominated*** *by another move from the same player.*
We may remove $\boldsymbol{A}$ *from* $\boldsymbol{G}$*.*

Reduction Rule: Reversible Moves

If we have some game $\boldsymbol{G}$ *where a player has a move* $\boldsymbol{A}$ *that is* ***reversible****, where* $\boldsymbol{A^R}$ *is a* ***reversing move****.*
$\boldsymbol{A}$ *may be replaced by the moves that this player would have in* $\boldsymbol{A^R}$*.*