

## Seminar (Combinatorial) Algorithms



Universiteit  
Leiden

Hendrik Jan Hoogeboom & Walter Kosters

Spring 2022

Snellius 174

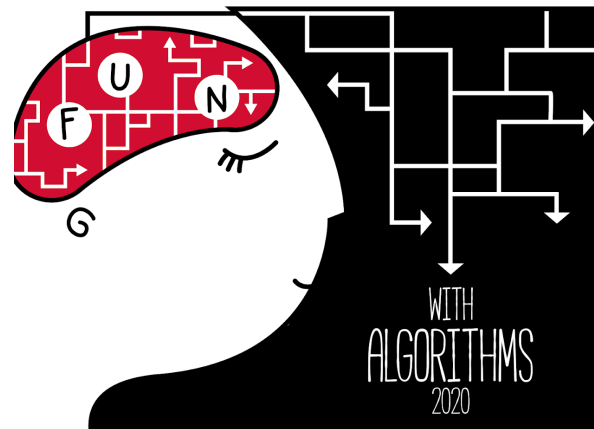
Tuesday February 8, 2022, 11:15–13:00

[www.liacs.leidenuniv.nl/~kosterswa/semalg/](http://www.liacs.leidenuniv.nl/~kosterswa/semalg/)

- this year's topic slide 3
- example presentation from 2020 slides 4–23
- want to participate? slide 24
  - decide y/n
  - try to solve problem (\*) from slide 6
  - look at this year's topic
  - send e-mail

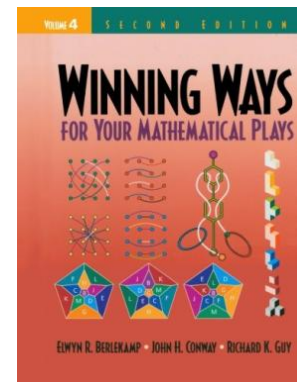
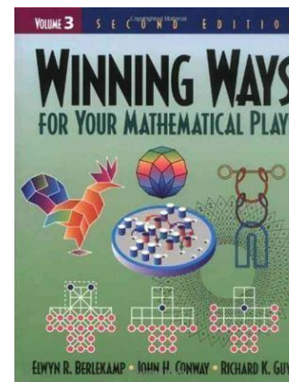
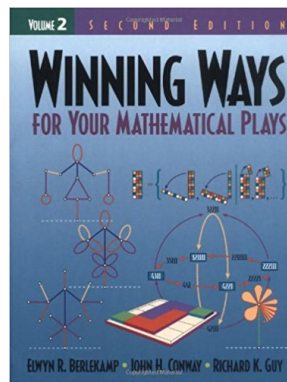
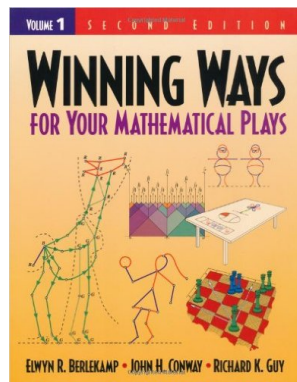
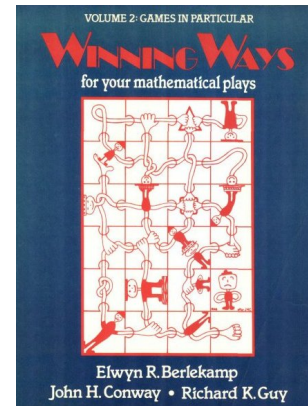
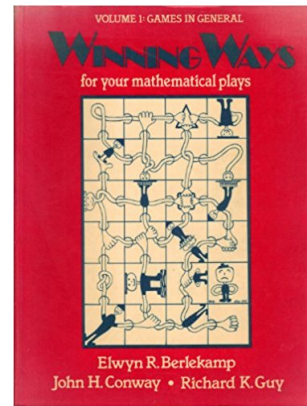
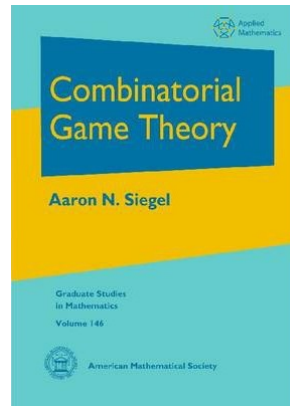
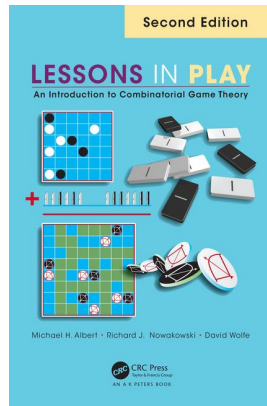
We discuss papers from **FUN 2016**, **FUN2018**, **FUN2021**, the 8/9/10th International Conference on Fun with Algorithms. See

- [link 2016](#) & [another link](#)
- [link 2018](#) & [another link](#)
- [link 2021](#) & [another link](#)

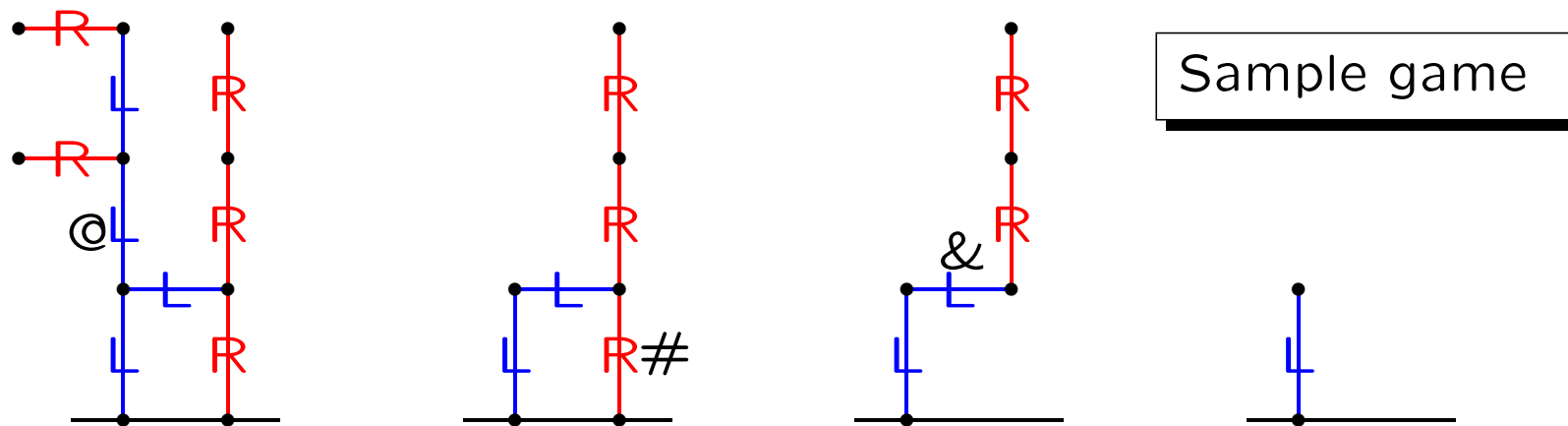


[www.liacs.leidenuniv.nl/~kosterswa/semalg/](http://www.liacs.leidenuniv.nl/~kosterswa/semalg/)

Overview: Hackenbush — Surreal numbers — Nim — +



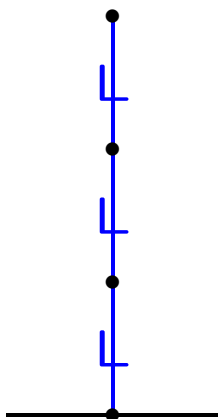
In the game (Blue-Red-)Hackenbush the two players **Left** = she and **Right** = he alternately remove a **blue** or a **Red** edge. All edges that are no longer connected to the ground, are also removed. *If you cannot move, you lose!*



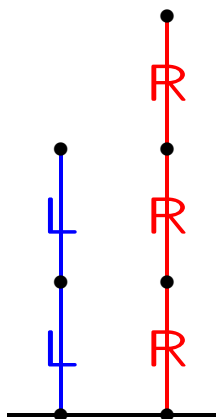
**Left** chooses @, **Right** # (stupid), **Left** &. Now **Left** wins because **Right** cannot move.

By the way, **Right** can win here, whoever starts!

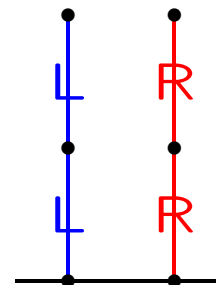
When playing Hackenbush, what is the **value of a position**?



value 3



value  $2 - 3 = -1$



value  $2 - 2 = 0$

value  $> 0$ : **Left** wins (whoever starts)

$\mathcal{L}$

value  $= 0$ : first player loses

$\mathcal{P}$

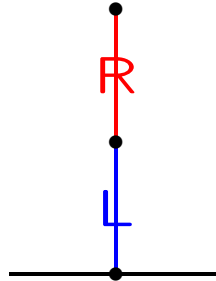
value  $< 0$ : **Right** wins (whoever starts)

$\mathcal{R}$

(\*) Hackenbush has no “first player wins”!

$\mathcal{N}$

But what is the value of this position?



If **Left** begins, she wins immediately. If **Right** begins, **Left** can still move, and also wins. So **Left** always wins. Therefore, the value is  $> 0$ .

Is the value equal to 1?

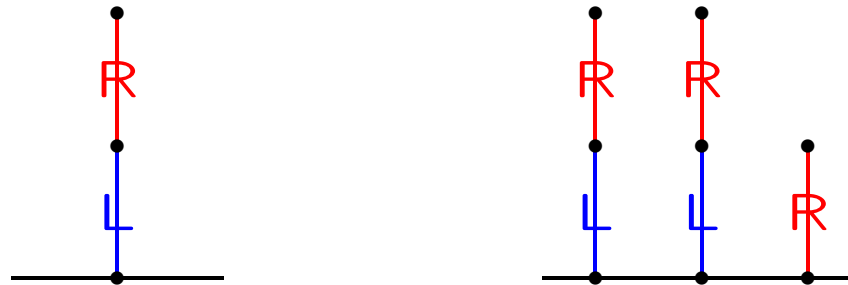
If the value in the left hand side position  $x$  would be 1, the value of the right hand side position would be  $1 + (-1) = 0$ , and the first player should lose. Is this true?



No! If **Left** begins, **Left** loses, and if **Right** begins **Right** can also win. So **Right** always wins (i.e., can always win), and therefore the right hand side position is  $< 0$ , and  $x + (-1) < 0$ , and the left one is between 0 and 1.



The left hand side position is denoted by  $\{ 0 \mid 1 \}$ .



Note that the right hand side position does have value 0: the first player loses. And so we have:

$$\{ 0 \mid 1 \} + \{ 0 \mid 1 \} + (-1) = 0,$$

and “apparently”  $\{ 0 \mid 1 \} = 1/2$ .





Donald E. (Ervin) Knuth

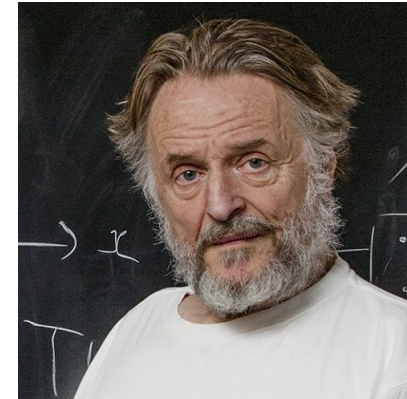
1938, US

NP; KMP

$\text{T}_{\text{E}}\text{X}$

change-ringing; 3:16

The Art of Computer  
Programming



John H. (Horton) Conway

1937–2020, UK → US

$C_{01}$ ,  $C_{02}$ ,  $C_{03}$

Doomsday algorithm

game of Life; Angel problem

Winning Ways for your  
Mathematical Plays

**Surreal numbers**

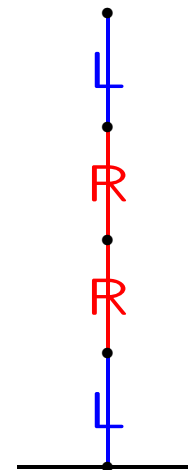
In this way we define **surreal numbers**: “decent” pairs of sets of previously defined surreal numbers: all elements from the left set are smaller than those from the right set.

Start with  $0 = \{ \emptyset \mid \emptyset \} = \{ \text{nothing} \mid \text{nothing} \} = \{ \mid \}$ : the game where both **Left** and **Right** have no moves at all, and therefore the first player loses: born on day 0.

And then  $1 = \{ 0 \mid \}$  en  $-1 = \{ \mid 0 \}$ , born on day 1.

And  $42 = \{ 41 \mid \}$ , born on day 42.

And  $\frac{3}{8} = \{ \frac{1}{4} \mid \frac{1}{2} \}$ , born on day 4.



Sets can be infinite:  $\pi = \{ 3, 3\frac{1}{8}, 3\frac{9}{64}, \dots \mid 4, 3\frac{1}{2}, 3\frac{1}{4}, 3\frac{3}{16}, \dots \}$ .

We define:  $\varepsilon = \{ 0 \mid \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots \}$ , an “incredibly small number”, and  $\omega = \{ 0, 1, 2, 3, \dots \mid \} = \{ \mathbf{N} \mid \emptyset \}$ , a “terribly large number, some sort of  $\infty$ ”.

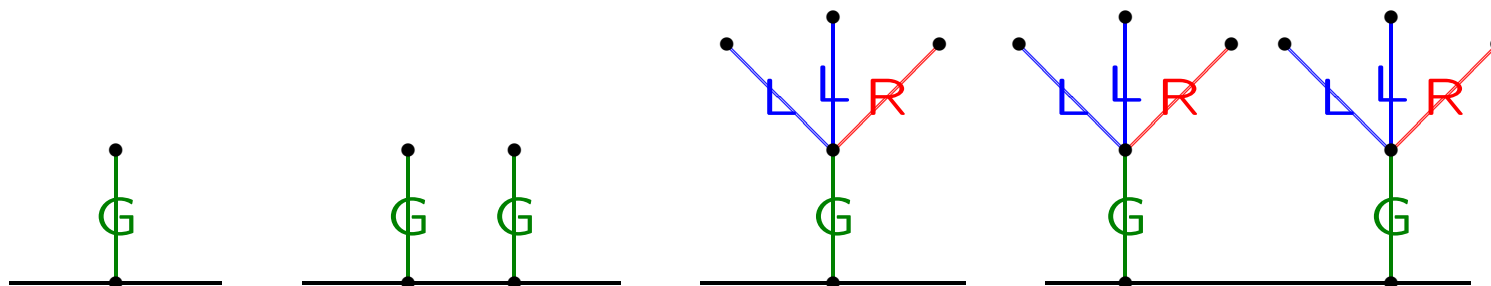
Then we have  $\varepsilon \cdot \omega = 1$  — if you know how to multiply.

And then  $\omega + 1$ ,  $\sqrt{\omega}$ ,  $\omega^\omega$ ,  $\varepsilon/2$ ,  
and so on!

But we will stick to  
“games” like  $\{ 1 \mid -1 \}$ .



In **Red-Green-Blue-Hackenbush** we also have **Green** edges, that can be removed by both players.



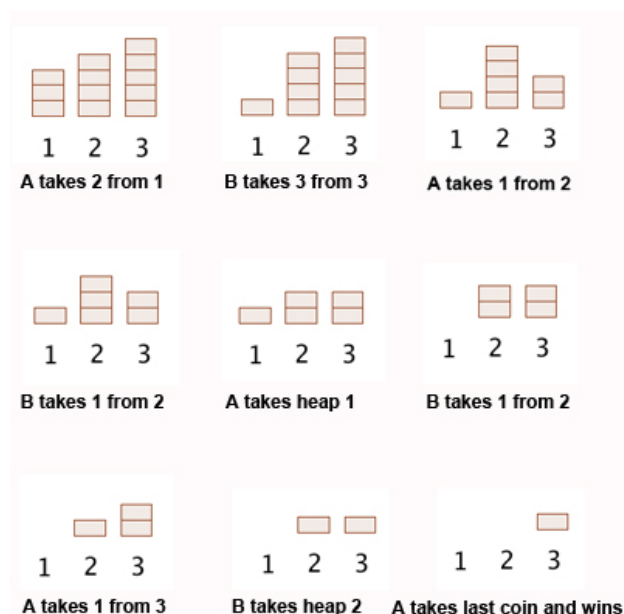
The first position has value  $*$   $= \{ 0 \mid 0 \}$  (not a surreal number), because the player to move can win.

The second position is  $* + * = 0$  (player to move loses).

The third position is a first player win.

The fourth position is a win for **Left** (whoever begins), and is therefore  $> 0$ .

In the **Nim** game we have several stacks of tokens = coins = matches. A move consists of taking a nonzero number of tokens from one of the stacks. If you cannot move, you lose (“normal play”).



The game is **impartial**: both players have the same moves. (And for the **misère** version: if you cannot move, you win.)

For Nim we have **Bouton's analysis** from 1901.

We define the **nim-sum**  $x \oplus y$  of two positive integers  $x$  and  $y$  as the bitwise XOR of their binary representations: addition without carry. With two stacks of equal size the first player loses ( $x \oplus x = 0$ ): use the “mirror strategy”.

A nim game with stacks of sizes  $a_1, a_2, \dots, a_k$  is a first player loss exactly if  $a_1 \oplus a_2 \oplus \dots \oplus a_k = 0$ . And this sum is the **Sprague-Grundy value**.

We denote a nim game with value  $m$  by  $*m$  (the same as a stalk of  $m$  green Hackenbush edges; not a surreal number). And  $*1 = *$ . So if  $m \neq 0$  the first player loses.



The **Sprague-Grundy Theorem** roughly says that every impartial game is a Nim game.

With stacks of sizes 29, 21 and 11, we get  $29 \oplus 21 \oplus 11 = 3$ :

11101	29
10101	21
1011	11
-----	--
00011	3

So a first player win, with unique winning move  $11 \rightarrow 8$ .

Why this move, and why is it unique?

How to add these “games” (we already did)? Like this:

$$a + b = \{ A_L + b, a + B_L \mid A_R + b, a + B_R \}$$

if  $a = \{ A_L \mid A_R \}$  and  $b = \{ B_L \mid B_R \}$ .

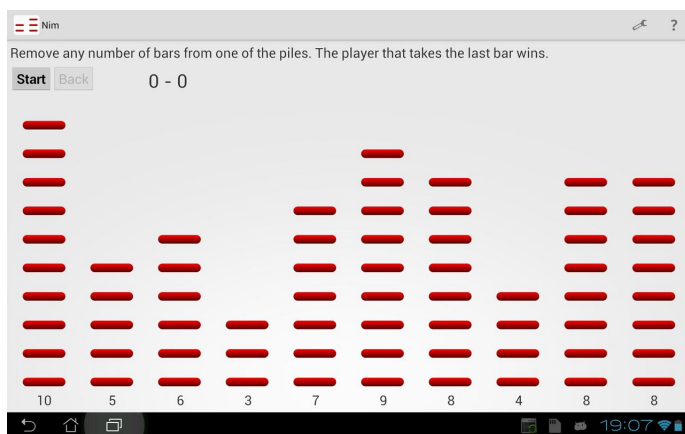
We put  $u + \emptyset = \emptyset$  and (more general)  $u + V = \{ u + v : v \in V \}$ .

This corresponds with the following: you play two games in parallel, and in every move you must play in exactly one of these two games: the **disjunctive sum**.

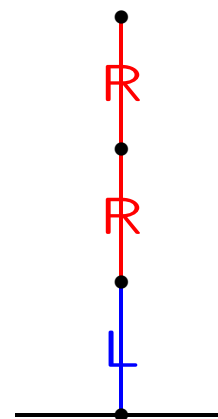
$$\text{Verify that } 1 + \frac{1}{2} = \{ 1 \mid 2 \} = \frac{3}{2}.$$

See [Claus Tøndering's paper](#).

Now consider this addition of two game positions, with on the left a Nim position and on the right a Hackenbush position:



+



Then this sum is  $> 0$ , it is a win for **Left**!

And we even have:  $*m + 1/2^{10000} > 0$ , and therefore  $-1/2^{10000} < * < 1/2^{10000}$ , but  $*$  is not comparable to 0.

We finally play **Clobber**, on an  $m$  times  $n$  board, with white (Right) and black (Left) stones. A stone can capture = “clobber” a directly horizontally/vertically adjacent stone from the other color. If you cannot move, you lose.

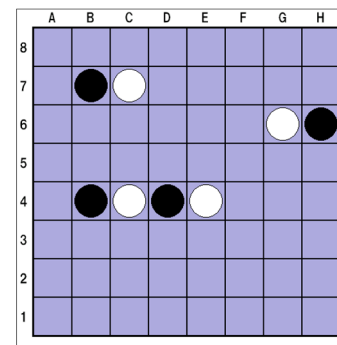
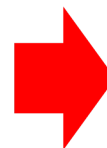
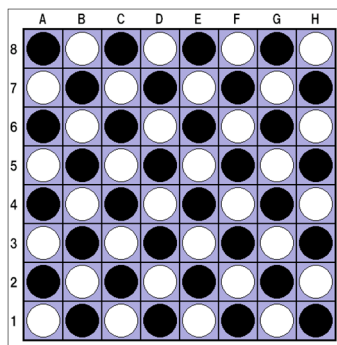
Some examples:

$$\boxed{\bullet} \boxed{\circ} = \{ 0 \mid 0 \} = *$$

$$\boxed{\bullet} \boxed{\bullet} \boxed{\circ} = \{ 0 \mid * \} = \uparrow > 0$$

$$\boxed{\bullet} \boxed{\circ} \boxed{\bullet} \boxed{\circ} \boxed{\bullet} \boxed{\circ} = 0$$

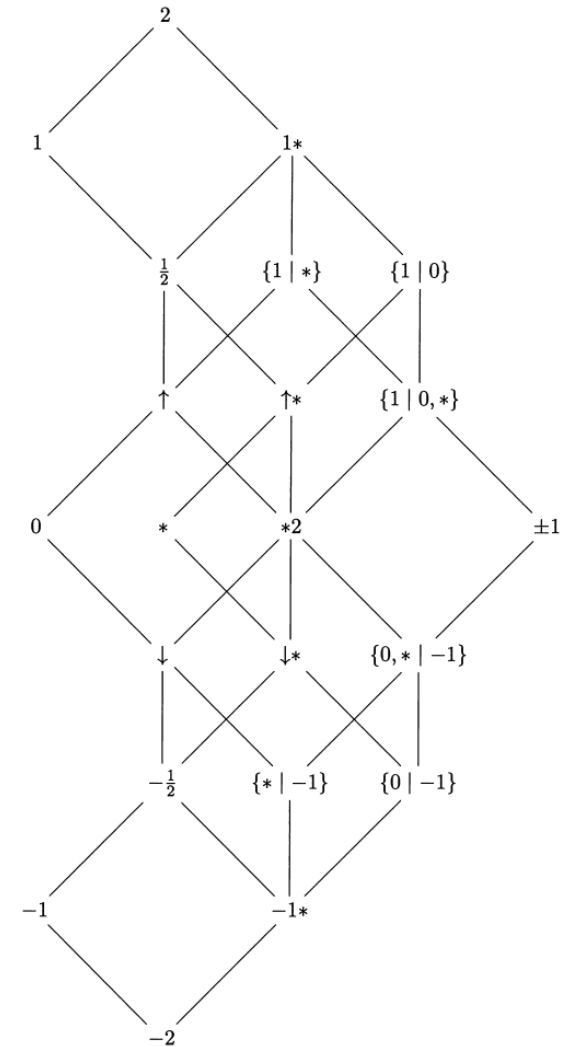
$$\boxed{\bullet} \boxed{\circ} \boxed{\bullet} \boxed{\circ} \boxed{\bullet} \boxed{\circ} \boxed{\bullet} \boxed{\circ} \boxed{\bullet} \boxed{\circ} = \pm(\uparrow, \uparrow^{[2]} *, \{0 \mid \uparrow, \pm(*, \uparrow)\}, \{\uparrow * \mid \downarrow, \pm(*, \uparrow)\})$$



From the Siegel book:

$$0 < \{ 0 \mid * \} = \uparrow < 1/2 < \{ 1 \mid 1 \} = 1* < 2$$

		Right Options					
		-1	0,*	0	*	1	—
Left Options	1	$\pm 1$	$\{1 \mid 0, *\}$	$\{1 \mid 0\}$	$\{1 \mid *\}$	$1*$	2
	0,*	$\{0, * \mid -1\}$	$*2$	$\uparrow*$	$\uparrow$	$\frac{1}{2}$	1
	0	$\{0 \mid -1\}$	$\downarrow*$	*			
	*	$\{* \mid -1\}$	$\downarrow$		0		
	-1	$-1*$	$-\frac{1}{2}$				
	—	-2	-1				



Three main references:

## LessonsInPlay:

M.H. Albert, R.J. Nowakowski and D. Wolfe, Lessons in Play, **second edition**, CRC Press, 2019.

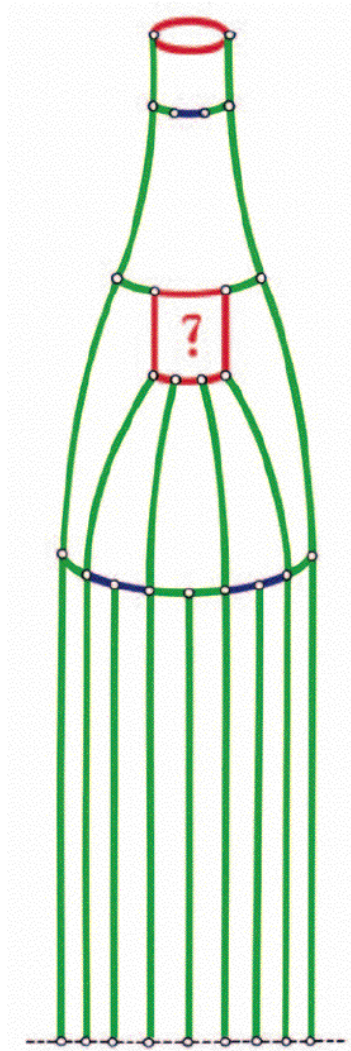
## Siegel:

A.N. Siegel, Combinatorial Game Theory, AMS, 2013.

## WinningWays:

E.R. Berlekamp, J.H. Conway and R.K. Guy, Winning Ways for your Mathematical Plays, 1982/2001.

(Note that there are two editions: the first has two volumes, the second has four volumes.)



from WinningWays

How is the seminar organized? Do the following twice:

Present a (chosen) “paper” during a 45 minutes **lecture**. Make slides, and use the “blackboard”.

Produce a 7–10 page **paper/report in L<sup>A</sup>T<sub>E</sub>X/PDF**. Use your own words, no copy-paste; English.

Grading is based on the four **P**s: **p**resentation (2×), **p**aper (2×), **p**articipation (including presence: discussions, questions) and maybe **p**eer review OR **p**rogramming.

Apply for participation: send e-mail<sup>†</sup> with proof of (\*) from slide 6 before Friday afternoon February 11, 2022. At most  $\approx 10$  participants.

<sup>†</sup> [w.a.kosters@liacs.leidenuniv.nl](mailto:w.a.kosters@liacs.leidenuniv.nl)

