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CHALLENGING MATHEMATICAL PROBLEMS WITH ELEMENTARY SOLUTIONS

Volume I: Combinatorial Analysis and Probability Theory

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SURVEY OF RECENT
EAST EUROPEAN MATHEMATICAL LITERATURE

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40. What is the smallest number of kings which can be arranged in such a way that every unoccupied square is controlled by at least one of them:

a. On an 8×8 chessboard?

b. On an $n \times n$ chessboard?

41. What is the greatest number of queens which can be arranged in such a way that no queen lies on a square controlled by another:

a. On an 8×8 chessboard?

b.*** On an $n \times n$ chessboard?

42a. What is the greatest number of knights which can be arranged on an 8×8 chessboard in such a way that none of them lies on a square controlled by another?

b.** Determine the number of different arrangements of knights on an 8×8 chessboard such that no knight controls the square on which another lies, and the greatest possible number of knights is used.

Some other combinatorial problems connected with arrangements of chess pieces can be found in L. Y. Okunev's booklet, *Combinatorial Problems on the Chessboard* (ONTI, Moscow and Leningrad, 1935).

IV. GEOMETRIC PROBLEMS INVOLVING COMBINATORIAL ANALYSIS

Some of the problems in this group are concerned with convex sets. A set in the plane or in three-dimensional space is called *convex* if the line segment joining any two of its points is contained in the set. For example, the interior of a circle or of a cube is convex. The set S in fig. 3 is not convex, since the line segment joining A and B is not entirely contained in S.

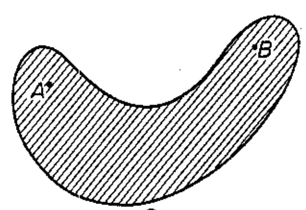

Fig. 3

43a. Each of the vertices of the base of a triangle is connected by straight lines to n points on the side opposite it. Into how many parts do these $2n$ lines divide the interior of the triangle?

b. Each of the three vertices of a triangle is joined by straight lines to n points on the opposite side of the triangle. Into how many parts do these $3n$ lines divide the interior of the triangle if no three of them pass through the same point?

44.* What is the greatest number of parts into which a plane can be divided by:

a. n straight lines?

b. n circles?

45.** What is the greatest number of parts into which three-dimensional space can be divided by:

a. n planes?

b. n spheres?

46.* In how many points do the diagonals of a convex n-gon meet if no three diagonals intersect inside the n-gon?

47.* Into how many parts do the diagonals of a convex n-gon divide the interior of the n-gon if no three diagonals intersect?

48. Two rectangles are considered different if they have either different dimensions or a different location. How many different rectangles consisting of an integral number of squares can be drawn

a. On an 8×8 chessboard?

b. On an $n \times n$ chessboard?

49. How many of the rectangles in problem 48 are squares

a. On an 8×8 chessboard?

b. On an $n \times n$ chessboard?

50.* Let K be a convex n-gon no three of whose diagonals intersect. How many different triangles are there whose sides lie on either the sides or the diagonals of K?

51.** *Cayley's problem.*[3] How many convex k-gons can be drawn, all of whose vertices are vertices of a given convex n-gon and all of whose sides are diagonals of the n-gon?

52. There are many ways in which a convex n-gon can be decomposed into triangles by diagonals which do not intersect inside the n-gon (see fig. 4, where two different ways of decomposing an octagon into triangles are illustrated).

a. Prove that the number of triangles obtained in such a decomposition does not depend on the way the n-gon is divided, and find this number.

[3] Arthur Cayley (1821–1895), an English mathematician.

arranged on an $n \times n$ chessboard in such a way as to control all squares of the board is $[(n+2)/3]^2$.

Remark. It is not hard to see from a consideration of fig. 38b that for n divisible by 3 there is *exactly one* way in which $(n/3)^2$ kings can be arranged on a board of n^2 squares so as to control the entire board. For values of n which leave a remainder of 1 or 2 upon division by 3, $[(n+2)/3]^2$ kings can be arranged on the board in such a way as to control all squares of the board in many different ways; we leave it to the reader to compute the number of such arrangements.

41a. There cannot be more than one queen in any column of the chessboard; hence it is impossible to arrange more than eight queens on an 8×8 chessboard in such a way that none of them lies on a square controlled by another.

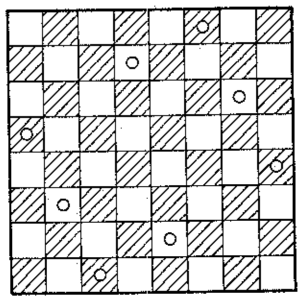

Fig. 39

On the other hand, we can actually put 8 queens on the board so as to satisfy this condition; one such arrangement is shown in fig. 39.

It can be shown that on an ordinary chessboard there are 92 different arrangements of eight queens which satisfy the condition imposed. (See, for example, M. Kraitchik, *Mathematical Recreations*, New York, 1942, p. 251.)

41b. There cannot be more than one queen in any column of the chessboard (since otherwise two queens would each control the square occupied by the other); hence it is impossible to arrange more than n queens on an $n \times n$ chessboard so as to satisfy the hypothesis of the problem.

If a single queen is put on a 2×2 chessboard, then it will control all squares of the board and thus no second queen can be put on the board (fig. 40a). On a 3×3 chessboard, one can arrange two queens so as to satisfy the hypothesis (fig. 40b), but it is impossible to do so with three queens. On a 4×4 or 5×5 chessboard it is possible to arrange four or five queens respectively, none of which lies on a square controlled by another (fig. 40c and d).

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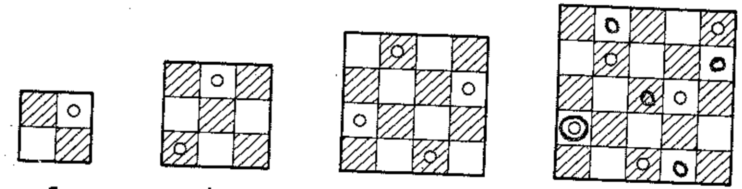

Fig. 40

We will now show that for $n \geq 4$ it is possible to arrange n queens on an $n \times n$ chessboard so that none lies on a square controlled by another. Consider first the case of *even* $n = 2k$. In adjacent columns one cannot put two queens either in the same row or in adjacent rows (otherwise these queens would control each other horizontally or diagonally). We will therefore try putting each queen in a row two away from that in which we put the preceding one. Let us start by putting a queen on the second (that is, next to bottom) square of the first column, then on the fourth square of the second column, on the sixth square of the third column, etc., until we hit the top row of the board; then start again on the bottom square of the next column, then the third square of the next column, etc. (fig. 41). Since no two queens are in the same row or column, it remains only to prove that no two queens are on the same diagonal.

Let us treat separately the cases of positive and negative diagonals. The first $n/2 = k$ queens are arranged in such a way that the positive diagonal on which any of them lies is the one immediately above that on

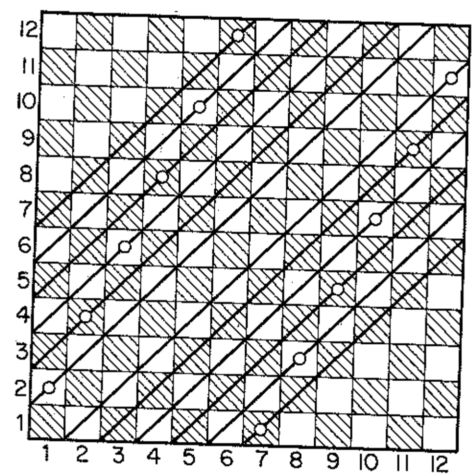

Fig. 41

which the previous one lies; similarly with the remaining k queens. Thus, the only way in which two of them could lie on the same positive diagonal would be for one of the first k queens to lie on the same positive diagonal as one of the second k queens. But this is impossible since the first k queens lie above the diagonal which joins the lower left-hand corner of the board to the upper right-hand corner, and the other k queens lie below this diagonal. Hence no two of the queens lie on the same positive diagonal.

If two squares of the board lie on the same negative diagonal, then the sum of the row number and the column number is the same for both of them. Conversely, if the sum of the row number and column number is the same for two squares, then they lie on the same negative diagonal. The row number of each of the squares on which the first k queens lie is twice the column number. The remaining k queens lie in the $(k+1)$st through $2k$-th columns; the column number of the square in which one of these queens lies is thus of the form $k+s$, where s is a positive integer at most equal to k; it is not hard to see that the corresponding row number is $2s-1$. For $r = 1, 2, \ldots, k$, the sum of the row and the column numbers of the square containing the r-th queen is $2r + r = 3r$; consequently, for each of the first k queens this sum has a different value, which means that no two of them lie on the same negative diagonal. Similarly, the sum of the row and column numbers for the $(k+s)$th queen $(s = 1, 2, \ldots, k)$ is $(2s-1) + (k+s) = 3s + k - 1$, which takes a different value for each value of s; consequently, no two of the last k queens lie on the same negative diagonal. The only remaining possibility is that of one of the first k queens (say, the r-th) lies on the same negative diagonal as one of the last k queens (say, the $(k+s)$th). This will happen if and only if

$$3r = 3s + k - 1, \quad \text{that is,} \quad 3(r-s) + 1 = k = \tfrac{1}{2}n, \text{ or}$$
$$6(r-s) + 2 = n.$$

This is possible only when n leaves a remainder of 2 upon division by 6. Thus, for even n of the form $6m$ or $6m+4$, fig. 41 gives an arrangement of n queens on the chessboard for which none of the queens lies on a square controlled by another.

For $n = 6m + 2$, fig. 41 leads to an arrangement in which two queens control each other. But even in this case we can find an arrangement of n queens, none of which lies on a square controlled by another, although this arrangement is more complicated than the preceding one. One such arrangement is shown in fig. 42 for the case of $n = 14$ (compare also with fig. 39). Here, in the $n/2 - 3$ columns starting with the 2nd and ending with the $(n/2-2)$nd, a queen is put in every other row starting with the 3rd (that is, the queen in the 2nd column lies in the 3rd row, that in the 3rd

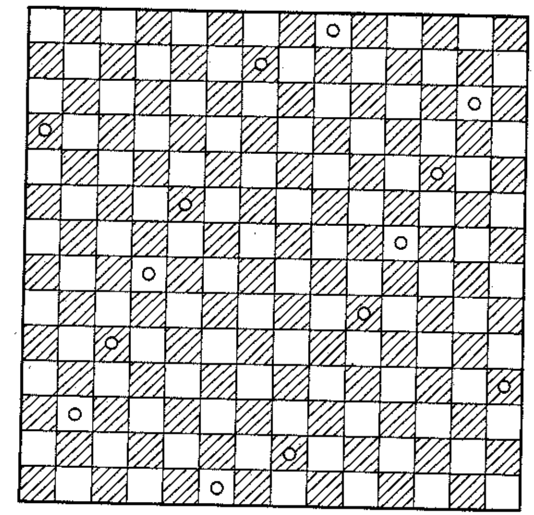

Fig. 42

column lies in the 5th row, that in the 4th column lies in the 7th row, etc.). In the $n/2 - 3$ columns starting with the $(n/2+3)$rd and ending with the $(n-1)$st, queens are put in every other row, starting with the 6th (that is, these queens are put respectively into the 6th, 8th, ..., and $(n-2)$nd rows). This leaves us with columns $1, n/2 - 1, n/2, n/2 + 1, n/2 + 2$, and n and rows $1, 2, 4, n-3, n-1$, and n unoccupied. In the 1st, $(n/2-1)$st, $(n/2)$nd, $(n/2+1)$st, $(n/2+2)$nd, and nth columns, the queens are placed respectively in rows $n-3, 1, n-1, 2, n$, and 4. It is clear that no two queens are in the same row or column; we thus have only to verify that no more than one queen lies on any diagonal.

Let us label the positive diagonals by assigning the numbers 1 to $2n-1$ to the squares of the bottom row and the leftmost column as indicated in fig. 43a and giving each positive diagonal the number of the numbered square which belongs to it. We label the negative diagonals in a way similar to this by assigning numbers to the squares of the bottom row and the rightmost column as indicated in fig. 43b. If by the first queen we mean that lying in the first column, by the second queen that lying in the second column, etc. then the 1st, 2nd, 3rd, ..., and n-th queen will lie respectively on the positive diagonals whose numbers are

$$2n-4,\ n+1,\ n+2,\ n+3, \ldots, 3n/2-3,\ n/2+2,\ 3n/2-1,$$
$$n/2+1,\ 3n/2-2,\ n/2+3,\ n/2+4,\ n/2+5, \ldots, n-1,\ 4;$$

no two of these numbers will be equal provided that

$$4 < n/2 + 1, \qquad 2n - 4 > 3n/2 - 1,$$

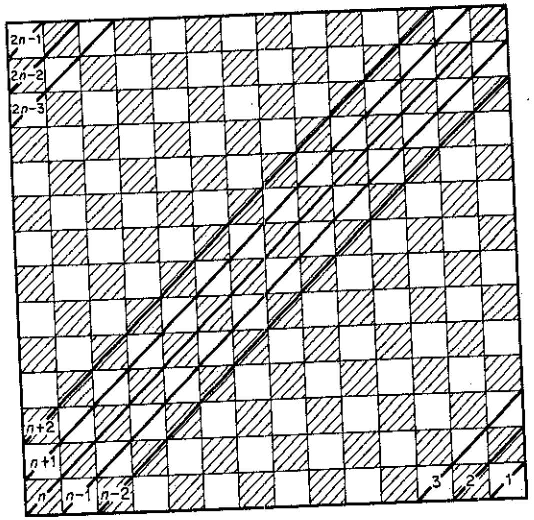

Fig. 43a

that is, if $n > 6$. Similarly, the queens lie respectively on the negative diagonals whose numbers are

$$n-3,\ 4,\ 7,\ 10,\ 13,\ \ldots,\ 3n/2-8,\ n/2-1,\ 3n/2-2,$$
$$n/2+2,\ 3n/2+1,\ n/2+8,\ n/2+11,\ n/2+14,$$
$$n/2+17,\ \ldots,\ 2n-4,\ n+3,$$

where the dots denote terms of an arithmetic progression with difference 3. The numbers $4, 7, 10, 13, \ldots, 3n/2-8, 3n/2-2, 3n/2+1$ all give remainders of 1 on division by 3; $n/2-1, n/2+2, n/2+8, n/2+11, n/2+17, \ldots, 2n-4$ are all divisible by 3 (recall that we are dealing with an n of the form $6m+2$); $n-3$ and $n+3$ give remainders of 2 on division by 3. It is immediately clear from this that none of the numbers occurs more than once.

It now remains only to show that on an $n \times n$ board, where the number n is *odd* and ≥ 5, it is possible to arrange n queens in such a way that none of them lies on a square which another controls. But this becomes clear if one notes that in all the above arrangements constructed for even n, there are no queens on the diagonal joining the lower left-hand corner to the upper right-hand corner. Consequently, we can arrange n queens on an $n \times n$ board (n odd) in the following way: on the leftmost $n-1$ columns and bottom $n-1$ rows, $n-1$ queens are arranged in such a way that none of them controls another according to the above scheme (this is possible since $n-1$ is even), and the remaining queen is placed in the upper right-hand corner of the board. These n queens will

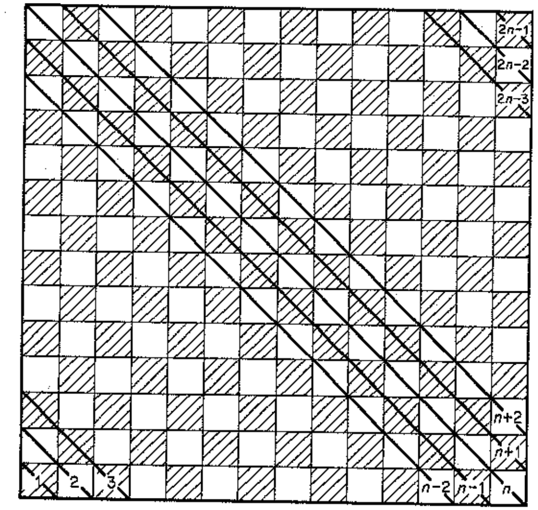

Fig. 43b

satisfy the required condition (see, for example, fig. 44 ,where an arrangement of 15 queens on a 15×15 board squares is illustrated).

To determine the *number* of different arrangements of n queens on an $n \times n$ board in which none of the queens lies on a square controlled by

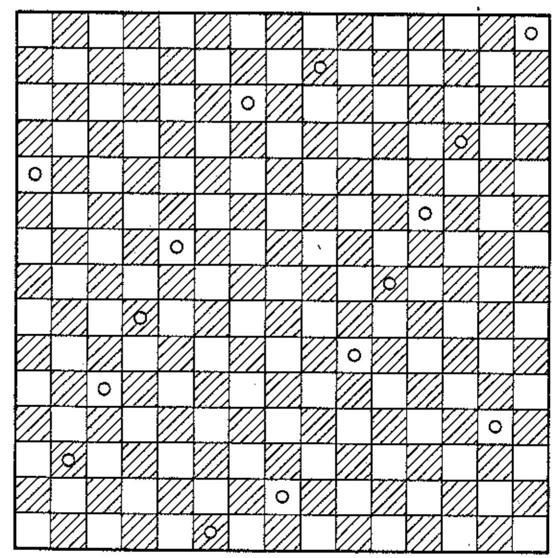

Fig. 44

another is extremely difficult, and so far no one has succeeded in doing so for the general case.

Another as yet unsolved problem is that of determining the minimum number of queens which can be arranged on an $n \times n$ chessboard so that

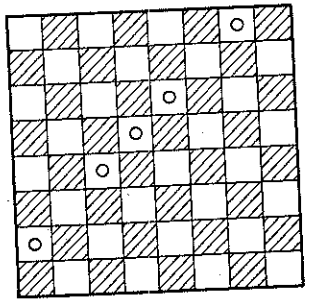

Fig. 45

they will control all squares of the board. For an ordinary 8×8 board, this number is 5 (see, for example, fig. 45); the number of different arrangements of 5 queens on an 8×8 board such that the queens control all squares of the board is 4860.

42a. Since a knight on a white square controls only black squares, it is obvious that 32 knights can be arranged in such a way that none lies on a

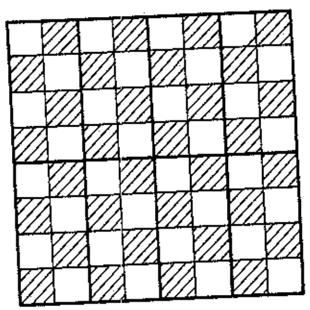

Fig. 46

square controlled by another; to do this, it suffices to put a knight on each white square (of which there are $64/2 = 32$). Let us show that such an arrangement using more than 32 knights is not possible. For this purpose, let us divide the board into eight rectangular sections, each two squares wide and four squares high (fig. 46). It is easy to see that a knight situated on a square of one of these rectangles R controls one and only one other square of R. Thus the squares of R can be divided into 4 pairs, and only

one square of each pair can be occupied by a knight. It follows from this that no more than four knights can be arranged in one of these rectangles in such a way that none of them lies on a square controlled by another. Therefore, the total number of knights which can be arranged in such a way on the chessboard is at most $4 \times 8 = 32$.

42b. We must determine how many arrangements of 32 knights on a chessboard are such that none of them lies on a square controlled by another. Two such arrangements present themselves immediately: we can put the 32 knights on all the white squares of the board, or on all the black squares of the board. Let us prove that there are no other arrangements.

Divide the board once more into eight rectangular sections as indicated in fig. 46. On each section we must arrange exactly four knights (since we have 32 knights to dispose of and by the argument of part a, no more than four can be in any one section). Consider now how four knights can be arranged on the lower left-hand rectangle (we will call this the first rectangle).

Let us first try putting knights in each of the bottom two squares of this rectangle (these squares are marked by circles in fig. 47a). In this case we must leave empty the squares of the first rectangle which are marked by crosses: the two squares in the third row are controlled by the two knights, and the square in the second row marked with a cross must be left free since otherwise the three knights would control five squares of the second rectangle (that is, the one to the right of the first rectangle), and consequently it would be impossible to arrange four knights in that rectangle without one of them lying on a square controlled by another knight. Since the 2 squares marked with asterisks in fig. 47a cannot both be occupied, we must have a knight in the upper left-hand corner of the first rectangle. This leaves only two possible arrangements: those indicated by the circles in fig. 47b and 47c. If we arrange the knights on the squares of the first rectangle marked by circles in fig. 47b, then the squares of the second rectangle marked with circles will have to be the ones with knights on them (since the other four squares of the second rectangle are controlled by the knights in the first rectangle); but then only two knights could be put in the third rectangle (namely, on the squares marked with circles), since the other six squares are controlled by the knights in the second rectangle. Consequently, this possibility must be discarded. Finally, if we arrange the knights as in fig. 47c, then the knights in the second rectangle can be placed only in the first and fourth rows; then in the upper left rectangle, the knights can be placed only in the top two rows (since the other four squares of this rectangle are controlled by the four knights in the fourth row). But then the knights on