Pushdown Automata

transparencies made for a course at the

International PhD School
in Formal Languages and Applications

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`http://www.liacs.nl/~hoogeboo/praatjes/tarragona/`

* very unfinished

Handbooks

Autebert, Berstel & Boasson. Context-Free Languages and Pushdown Automata. *Handbook of Formal Languages* (Rozenberg & Salomaa, eds.) 1997

Berstel & Boasson. Context-Free Languages. *Handbook of Theoretical Computer Science* (van Leeuwen, ed.) 1990.

Hendrik Jan Hoogeboom & Joost Engelfriet, Pushdown Automata. *Formal Languages and Applications* (Martín-Vide, Mitrana & Păun, eds.), 2004. [this course]

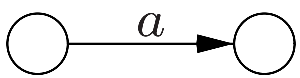
Textbooks

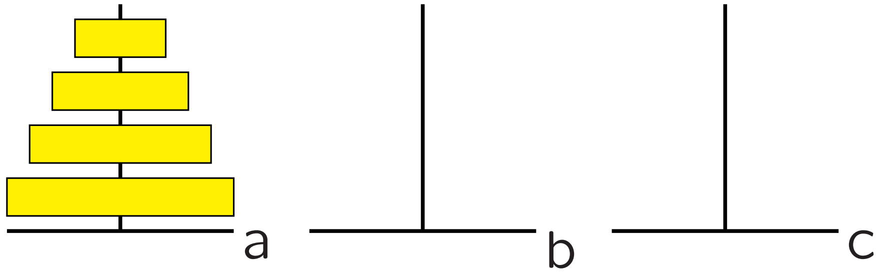
Harrison. *Introduction to Formal Language Theory*. Addison-Wesley, 1978.

Hopcroft & Ullman. *Introduction to Automata Theory, Languages, and Computation*, 2nd edition Addison-Wesley, 1979.

The Model

Introduction & Motivation

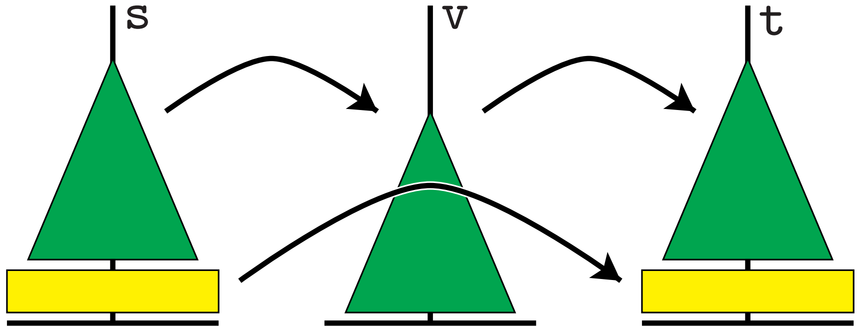
	grammar	**automaton**
3	regular	
	right-linear $A \longrightarrow aB$	finite state 
2	context-free	
	$A \longrightarrow \alpha$	pushdown (+lifo stack)
1	context-sensitive	
	$(\beta_\ell, A, \beta_r) \longrightarrow \alpha$ $\alpha \longrightarrow \beta \quad \lvert\beta\rvert \geq \lvert\alpha\rvert$ monotone	linear bounded
0	recursively enumerable	
	$\alpha \longrightarrow \beta$	turing machine



move from a to c

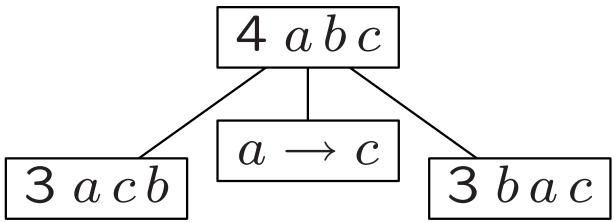
– respecting sizes

– one disk at a time

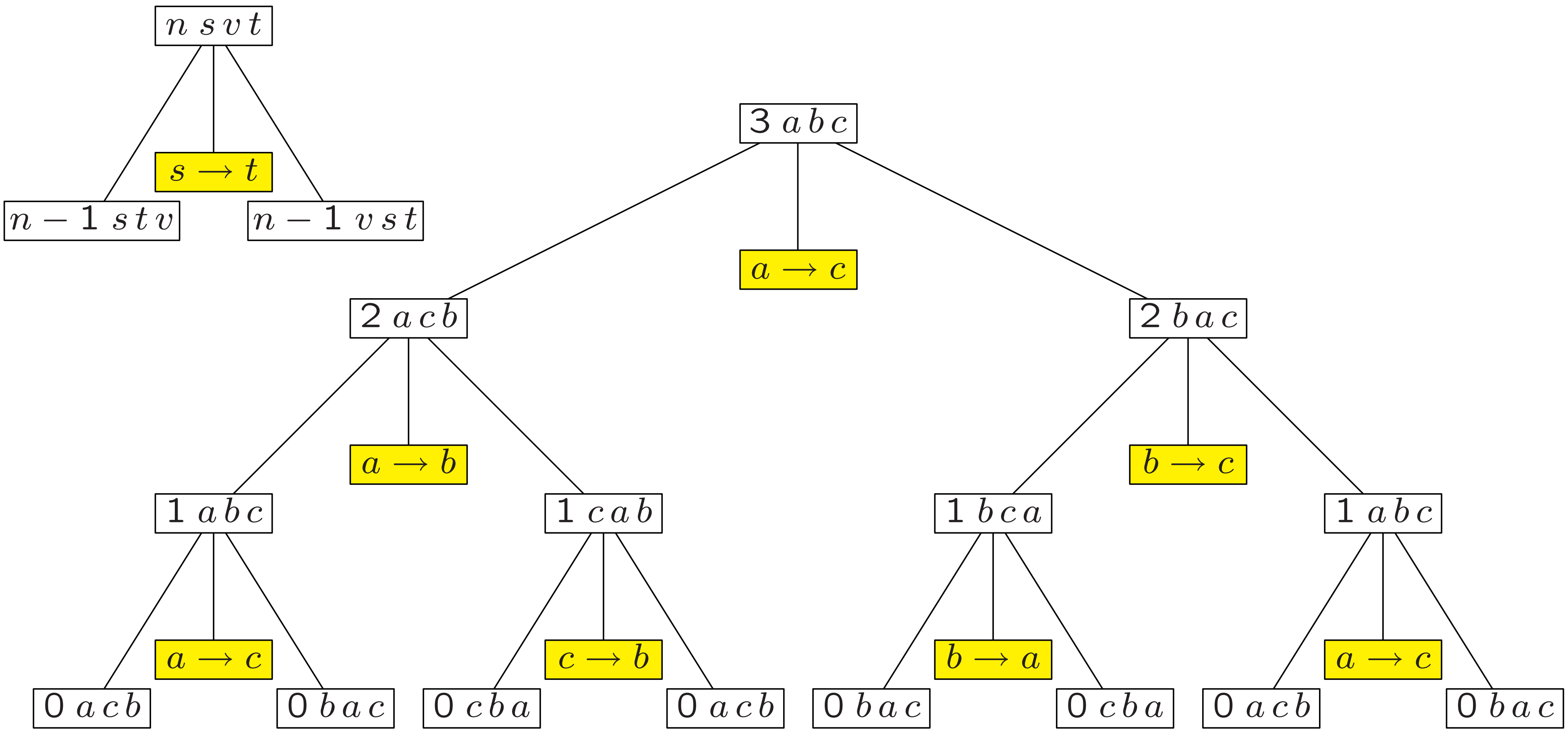


move(n, source, via, target) :- $\quad n \geq 1$

$$\left\{ \begin{array}{l} \text{move(} n-1\text{, source, target, via)} \\ \text{source } \xrightarrow{n} \text{ target} \\ \text{move(} n-1\text{, via, source, target)} \end{array} \right.$$

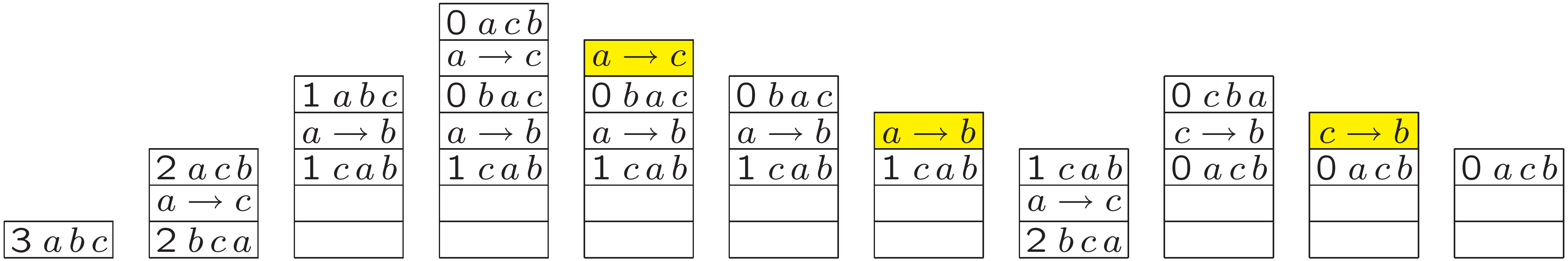


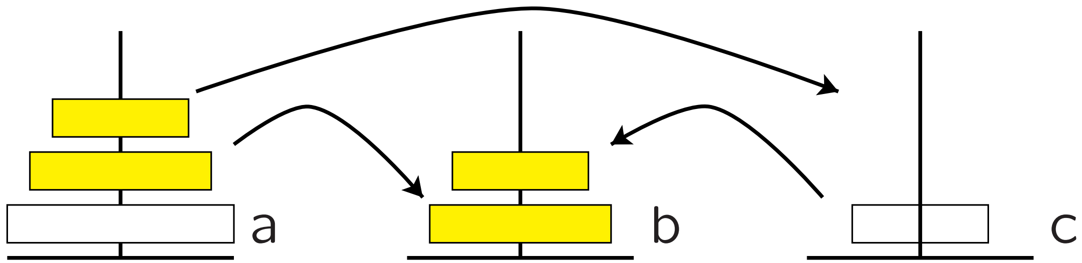
recursion, context-free grammar (-like)


n s v t
s ⟶ t
n − 1 s t v
n − 1 v s t
3 a b c
a ⟶ c
2 a c b
2 b a c
a ⟶ b
b ⟶ c
1 a b c
1 c a b
1 b c a
1 a b c
a ⟶ c
c ⟶ b
b ⟶ a
a ⟶ c
0 a c b
0 b a c
0 c b a
0 a c b
0 b a c
0 c b a
0 a c b
0 b a c

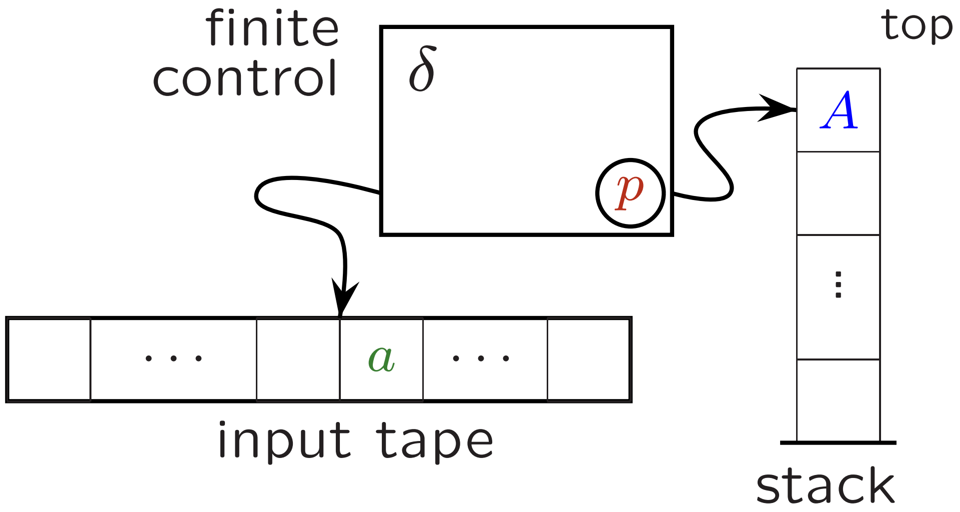
move(n, source, via, target) :- $n \geq 1$

$$\begin{cases} \text{move(} n-1\text{, source, target, via)} \\ \text{source} \xrightarrow{n} \text{target} \\ \text{move(} n-1\text{, via, source, target)} \end{cases}$$





move(2, a, c, b) = a $\rightarrow$ c; a $\rightarrow$ b; c $\rightarrow$ b;



formally 7-tuple

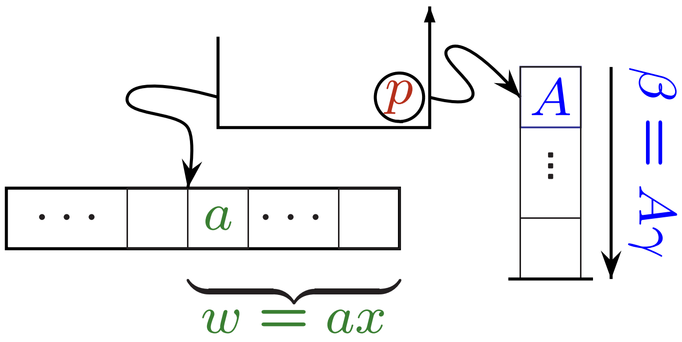
$$\mathcal{A} = (Q, \Delta, \Gamma, \delta, q_{in}, Z_{in}, F)$$

Q	*states*	p, q	
$q_{in} \in Q$	*initial state*		
$F \subseteq Q$	*final states*		
Δ	*input alphabet*	a, b	w, x
Γ	*stack alphabet*	A, B	α
$Z_{in} \in \Gamma$	*initial stack symbol*		

transition relation (finite)

$$\delta \subseteq Q \times (\Delta \cup \{\lambda\}) \times \Gamma \times Q \times \Gamma^*$$

	from			to		
(	p	a	A	q	α	)
		read	pop		push	
	before			after		

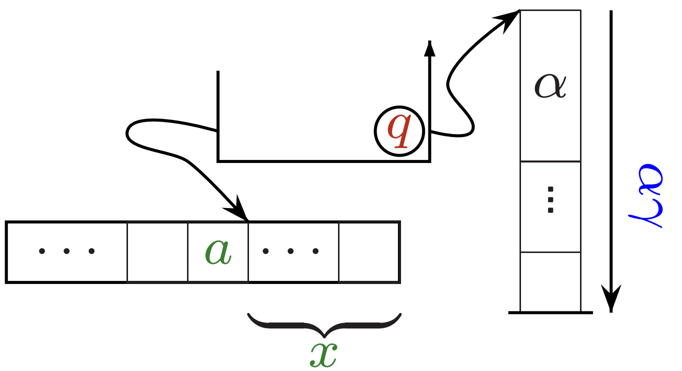


$Q \times \Delta^* \times \Gamma^*$ instantaneous descriptions

(p, w, β)
- p state
- w input, unread part
- β stack, top-to-bottom

move (step) $\vdash_{\mathcal{A}}$

$(p, ax, A\gamma) \vdash_{\mathcal{A}} (q, x, \alpha\gamma)$ iff
$(p, a, A, q, \alpha) \in \delta$, $x \in \Delta^*$ and $\gamma \in \Gamma^*$



computation $\vdash_{\mathcal{A}}^*$

$L(\mathcal{A})$ *final state language*

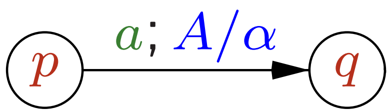
$\{ x \in \Delta^* \mid (q_{in}, x, A_{in}) \vdash_{\mathcal{A}}^* (q, \lambda, \gamma)$
for some $q \in F$ and $\gamma \in \Gamma^* \}$

$N(\mathcal{A})$ *empty stack language*

$\{ x \in \Delta^* \mid (q_{in}, x, A_{in}) \vdash_{\mathcal{A}}^* (q, \lambda, \lambda)$
for some $q \in Q \}$

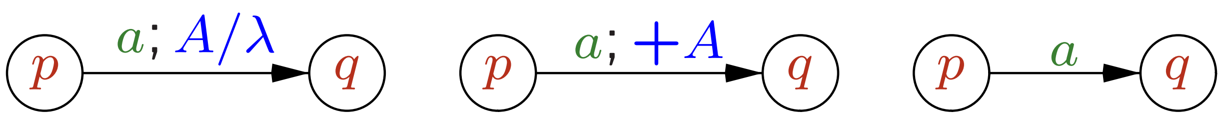
general form $(\,p, a, A, q, \alpha\,)$

$$(p, a, A) \mapsto (q, \alpha)$$



intuitive	*formalized as*	
pop A	$(\,p, a, A, q, \lambda\,)$	$\alpha = \lambda$
push A	$(\,p, a, X, q, AX\,)$	for all $X \in \Gamma$
read a	$(\,p, a, X, q, X\,)$	for all $X \in \Gamma$

our convention



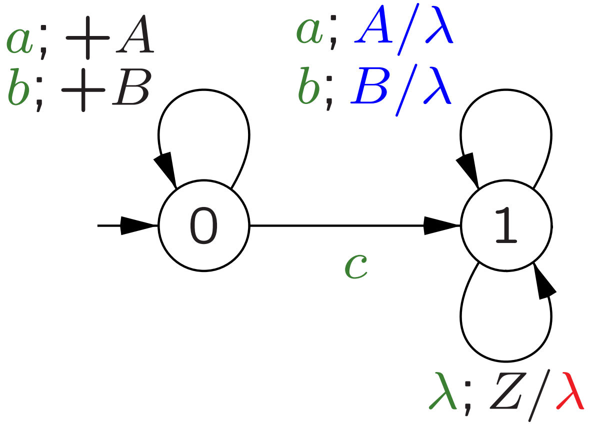
empty stack

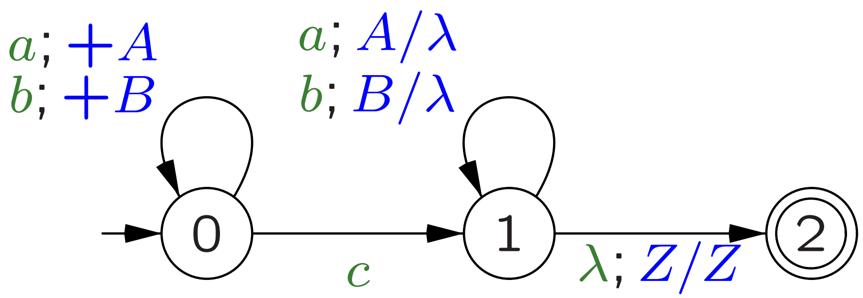
$N(\mathcal{A}_2) = L$

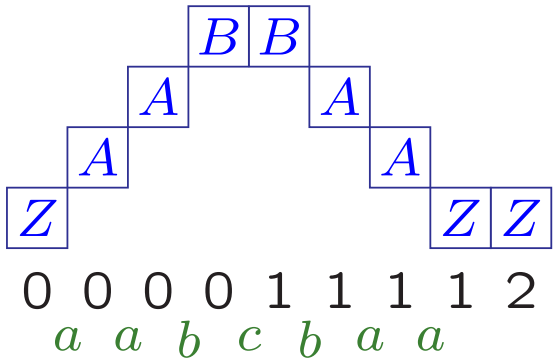
$$L = \{\, w\,c\,w^R \mid w \in \{a,b\}^* \,\}$$

final state

$L(\mathcal{A}_1) = L$







$(0, aabcbaa, Z) \vdash$

$(0, abcbaa, AZ) \vdash$

$(0, bcbaa, AAZ) \vdash$

$(0, cbaa, BAAZ) \vdash$

$(1, baa, BAAZ) \vdash$

$(1, aa, AAZ) \vdash$

$(1, a, AZ) \vdash$

$(1, \lambda, Z) \vdash$

$(2, \lambda, Z)$

δ consists of

$(0, a, Z, 0, AZ)$
$(0, a, A, 0, AA)$
$(0, a, B, 0, AB)$

$(0, b, Z, 0, BZ)$
$(0, b, A, 0, BA)$
$(0, b, B, 0, BB)$

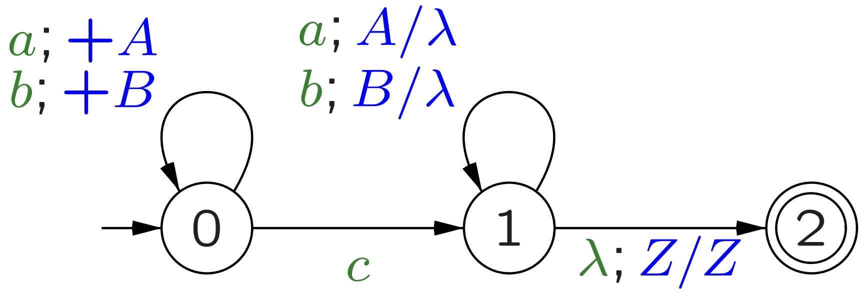
$(0, c, Z, 1, Z)$
$(0, c, A, 1, A)$
$(0, c, B, 1, B)$

$(1, a, A, 1, \lambda)$

$(1, b, B, 1, \lambda)$

$(1, \lambda, Z, 2, Z)$

$$L = \{ \, w \, c \, w^R \mid w \in \{a, b\}^* \, \}$$



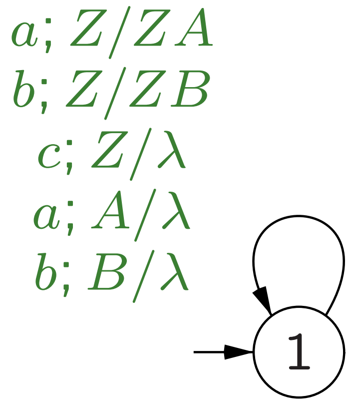
final state
$L_f(\mathcal{A}_1) = L$

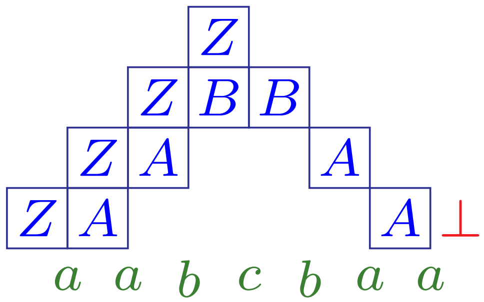
$\mathcal{A} = (Q, \Delta, \Gamma, \delta, q_{in}, Z_{in}, F)$, where

$Q = \{0, 1, 2\}$
$\Delta = \{a, b, c\}$
$\Gamma = \{A, B, Z\}$
$q_{in} = 0$
$Z_{in} = Z$
$F = \{2\}$

$$L = \{\, w\,c\,w^R \mid w \in \{a,b\}^* \,\}$$

single state :- stack codes state



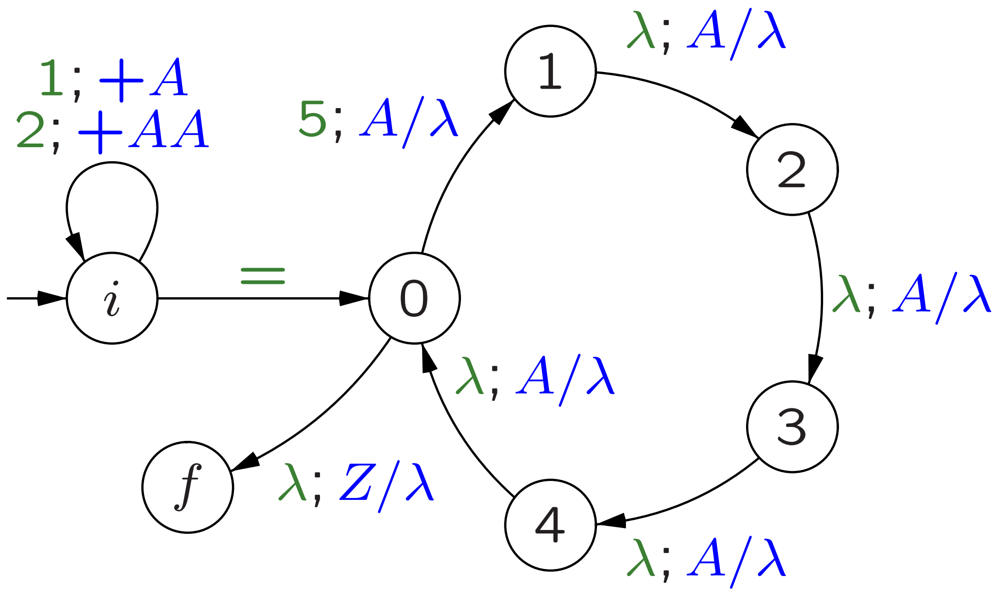


$$
\begin{aligned}
&(1, aabcbaa, Z) \vdash \\
&(1, abcbaa, ZA) \vdash \\
&(1, bcbaa, ZAA) \vdash \\
&(1, cbaa, ZBAA) \vdash \\
&(1, baa, BAA) \vdash \\
&(1, aa, AA) \vdash \\
&(1, a, A) \vdash \\
&(1, \lambda, \lambda)
\end{aligned}
$$

①②⑤ alphabet $\{1, 2, 5, =\}$

$$\{\ x = y \mid x \in \{1,2\}^*, y \in \{5\}^*,\ \#_1 x + 2\#_2 x = 5\#_5(y)\ \}$$

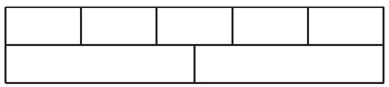
$\#_a x$ number of a occurrences in x



$212 = 5$

$22222 = 55$

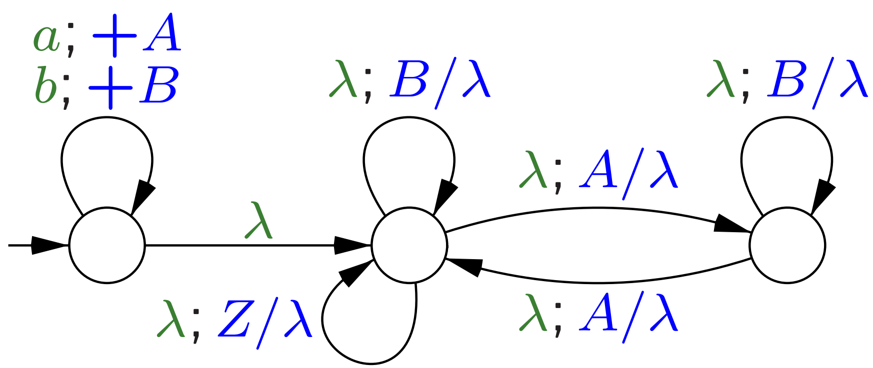
$12(122)^3 = 5^4$



empty stack / final state f

- ▷ context-free grammar?
- ▷ single state PDA?

- ⋆ completely read input
  - ∗ input+stack may block
  - ∗ infinite λ-computations!
- ⋆ no steps on empty stack
- ⋆ computations without reading
  - ∗ at the end to reach acceptance



$\{A, B, Z\}$, initial Z

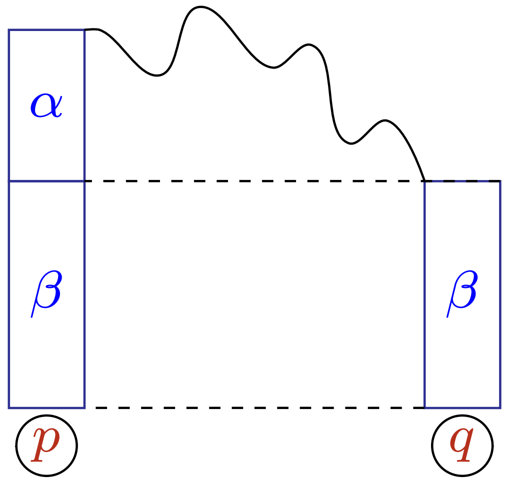
cutting and pasting PDA computations

- input

  $(p, w, \alpha) \vdash^* (q, \lambda, \lambda)$

  iff

  $(p, wz, \alpha) \vdash^* (q, z, \lambda)$



- stack

  $(p, w, \alpha) \vdash^* (q, \lambda, \lambda)$

  then

  $(p, w, \alpha\beta) \vdash^* (q, z, \beta)$

  $(p, w, \alpha) \vdash^* (q, \lambda, \lambda)$

  iff

  $(p, w, \alpha\beta) \vdash^* (q, z, \beta)$

  <u>and</u> every stack is longer than β.