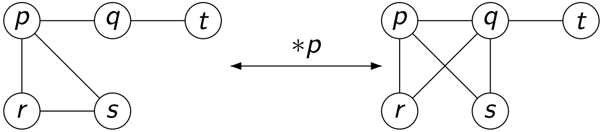
# Pivot and Loop Complementation on Graphs and Set Systems

Robert Brijder    Hendrik Jan Hoogeboom

Leiden Institute for Advanced Computer Science

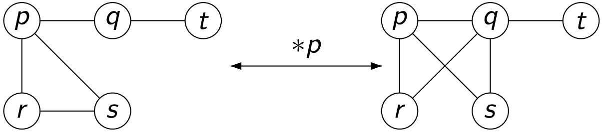
June 7, 2010

# Local Complementation



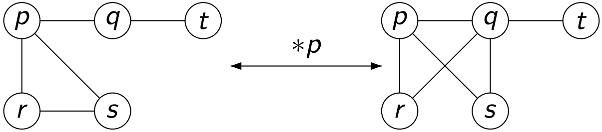

- Complement neighborhood of a vertex $p$ in a graph.
- Here: complement subgraph induced by $\{q, r, s\}$.

# Local Complementation




- Complement neighborhood of a vertex $p$ in a graph.
- Here: complement subgraph induced by $\{q, r, s\}$.
- Many applications: Transforming Euler circuits in 4-regular graphs (Kotzig, 1968), Quantum Computing, Interlace Polynomial.

# Local Complementation




- Complement neighborhood of a vertex $p$ in a graph.
- Here: complement subgraph induced by $\{q, r, s\}$.
- Many applications: Transforming Euler circuits in 4-regular graphs (Kotzig, 1968), Quantum Computing, Interlace Polynomial.
- Simple graphs considered.

# Curious Result for Simple Graphs

**Theorem (Bouchet,1988)**

*Let $G$ be a simple graph with edge $\{u, v\}$. We have $G * u * v * u = G * v * u * v$.*

- Define, in this case, $*u * v * u$ to be *edge complementation* (involution),

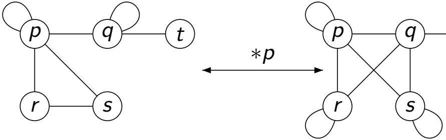
# Curious Result for Simple Graphs

**Theorem (Bouchet,1988)**

*Let $G$ be a simple graph with edge $\{u, v\}$. We have $G * u * v * u = G * v * u * v$.*

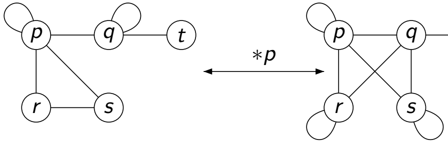
- Define, in this case, $*u * v * u$ to be *edge complementation* (involution),
- A goal: Understand nature of this equality (and obtain others like it).

# Local Complementation for Graphs with Loops



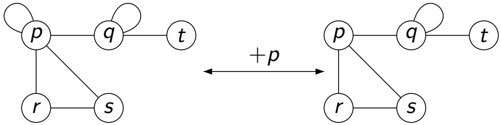

- Graphs where loops are allowed, called *graphs*,
- Local complementation on $p$ only applicable when loop is present for $p$.

# Local Complementation for Graphs with Loops



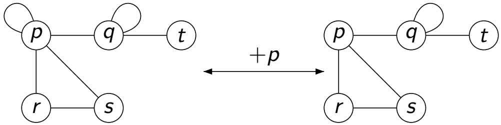

- Graphs where loops are allowed, called *graphs*,
- Local complementation on $p$ only applicable when loop is present for $p$.
- Original motivation: Gene Assembly in Ciliates (Computational Biology)

# Loop Complementation



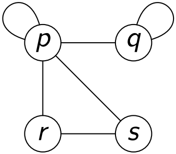

- Loop complementation on vertex $p$:
  if $p$ has a loop, then remove the loop, and
  if $p$ has no loop, then add a loop.

# Loop Complementation




- Loop complementation on vertex $p$:
  if $p$ has a loop, then remove the loop, and
  if $p$ has no loop, then add a loop.
- A main function: Bridge gap between
  1) local complementation on simple graphs, and
  2) local complementation on graphs.

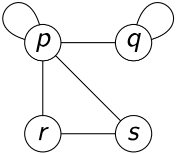
# Adjacency Matrix




$$\begin{array}{c} \\ p \\ q \\ r \\ s \end{array} \begin{array}{c} \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

- Identify a graph $G = (V, E)$ with its adjacency matrix,

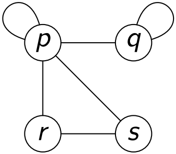
# Adjacency Matrix




| | p | q | r | s |
|---|---|---|---|---|
| p | 1 | 1 | 1 | 1 |
| q | 1 | 1 | 0 | 0 |
| r | 1 | 0 | 0 | 1 |
| s | 1 | 0 | 1 | 0 |

- Identify a graph $G = (V, E)$ with its adjacency matrix,
- A symmetric $V \times V$-matrix (columns and rows are indexed by $V$) over $\mathbb{F}_2$ corresponds precisely to a graph.

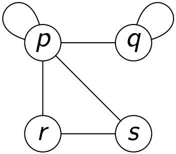
# Adjacency Matrix




| | $p$ | $q$ | $r$ | $s$ |
|---|---|---|---|---|
| $p$ | 1 | 1 | 1 | 1 |
| $q$ | 1 | 1 | 0 | 0 |
| $r$ | 1 | 0 | 0 | 1 |
| $s$ | 1 | 0 | 1 | 0 |

- Identify a graph $G = (V, E)$ with its adjacency matrix,
- A symmetric $V \times V$-matrix (columns and rows are indexed by $V$) over $\mathbb{F}_2$ corresponds precisely to a graph.
- Choice for $\mathbb{F}_2$ is important:
  addition is logical exclusive-or $\oplus$, and
  multiplication is logical conjugation $\wedge$.

# Adjacency Matrix




| | p | q | r | s |
|---|---|---|---|---|
| p | 1 | 1 | 1 | 1 |
| q | 1 | 1 | 0 | 0 |
| r | 1 | 0 | 0 | 1 |
| s | 1 | 0 | 1 | 0 |

- Identify a graph $G = (V, E)$ with its adjacency matrix,
- A symmetric $V \times V$-matrix (columns and rows are indexed by $V$) over $\mathbb{F}_2$ corresponds precisely to a graph.
- Choice for $\mathbb{F}_2$ is important:
  addition is logical exclusive-or $\oplus$, and
  multiplication is logical conjugation $\wedge$.
- Now: consider local complementation as a special case of a general matrix operation.

# The Bigger Picture: Principal Pivot Transform

**Definition**

Let $A$ be a $V \times V$-matrix (over an arbitrary field), and let $X \subseteq V$ with $A[X]$ is nonsingular. If $A = \left(\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right)$ with $P = A[X]$, then the *pivot* of $A$ on $X$ is

$$A * X = \left(\begin{array}{c|c} P^{-1} & -P^{-1}Q \\ \hline RP^{-1} & S - RP^{-1}Q \end{array}\right).$$

The pivot is the partial (component-wise) inverse:

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \text{ iff } A * X \begin{pmatrix} y_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix}, \qquad (1)$$

where the vectors $x_1$ and $y_1$ correspond to the elements of $X$. Relation (1) forms alternative definition of pivot.

# Properties of Pivot

- If $A$ is skew-symmetric, then $A * X$ is too. Hence if $G$ is a graph, then $G * X$ is too.

# Properties of Pivot

- If $A$ is skew-symmetric, then $A * X$ is too. Hence if $G$ is a graph, then $G * X$ is too.
- For graph $G$, $G * \{u\}$ is local complementation, and $G * \{u, v\}$ is edge complementation!! Although observed by Geelen, 1997, (and by Bouchet for edge complementation) this observation is almost unknown.

# Properties of Pivot

- If $A$ is skew-symmetric, then $A * X$ is too. Hence if $G$ is a graph, then $G * X$ is too.
- For graph $G$, $G * \{u\}$ is local complementation, and $G * \{u, v\}$ is edge complementation!! Although observed by Geelen, 1997, (and by Bouchet for edge complementation) this observation is almost unknown.
- Local and edge complementation together define pivot for graphs (they form the elementary pivots).

# Properties of Pivot

- If $A$ is skew-symmetric, then $A * X$ is too. Hence if $G$ is a graph, then $G * X$ is too.
- For graph $G$, $G * \{u\}$ is local complementation, and $G * \{u, v\}$ is edge complementation!! Although observed by Geelen, 1997, (and by Bouchet for edge complementation) this observation is almost unknown.
- Local and edge complementation together define pivot for graphs (they form the elementary pivots).

**Theorem (Tucker, 1960)**

*Let $A$ be a $V \times V$-matrix, and let $X \subseteq V$ be such that $A[X]$ is nonsingular. Then, for $Y \subseteq V$,*
$\det(A * X)[Y] = \det A[X \oplus Y] / \det A[X]$.

- $(A * X)[Y]$ is nonsingular iff $A[X \oplus Y]$ is nonsingular.

# Set Systems

- A *set system* (over $V$) is a tuple $M = (V, D)$ with $V$ a finite set and $D \subseteq \mathcal{P}(V)$ a family of subsets of $V$.

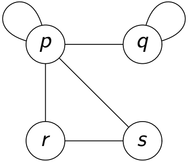
# Set Systems

- A *set system* (over $V$) is a tuple $M = (V, D)$ with $V$ a finite set and $D \subseteq \mathcal{P}(V)$ a family of subsets of $V$.
- Let, for graph $G$, $\mathcal{M}_G = (V, D_G)$ be the set system with $D_G = \{X \subseteq V \mid \det G[X] = 1\}$ (computed over $\mathbb{F}_2$).

# Set Systems

- A *set system* (over $V$) is a tuple $M = (V, D)$ with $V$ a finite set and $D \subseteq \mathcal{P}(V)$ a family of subsets of $V$.
- Let, for graph $G$, $\mathcal{M}_G = (V, D_G)$ be the set system with $D_G = \{X \subseteq V \mid \det G[X] = 1\}$ (computed over $\mathbb{F}_2$).
- $\mathcal{M}_G$ is known to be a $\Delta$-matroid. (We will not use this property here.)

# Set Systems Example

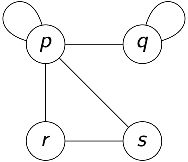



$$\begin{array}{c} \\ p \\ q \\ r \\ s \end{array} \begin{array}{c} \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$(V, \{\emptyset, \{p\}, \{q\}, \{p,r\}, \{p,s\}, \{r,s\}, \{p,q,r\}, \{p,q,s\}, \{p,r,s\}, \{q,r,s\}\})$

- $V = \{p,q,r,s\}$. For example, $\{p,r\} \in \mathcal{M}_G$ as $G[\{p,r\}] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is nonsingular over $\mathbb{F}_2$.

# Set Systems Example

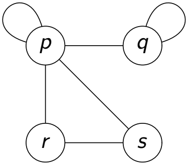



$$\begin{array}{c} \\ p \\ q \\ r \\ s \end{array} \begin{array}{c} \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$(V, \{\varnothing, \{p\}, \{q\}, \{p, r\}, \{p, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}\})$

- $V = \{p, q, r, s\}$. For example, $\{p, r\} \in \mathcal{M}_G$ as $G[\{p, r\}] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is nonsingular over $\mathbb{F}_2$.
- Define, for $X \subseteq V$, the *pivot* $M * X = (V, D * X)$, where $D * X = \{Y \oplus X \mid Y \in D\}$.

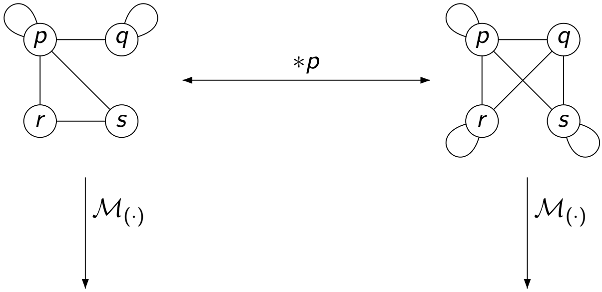
# Set Systems Example




$$\begin{array}{c} \\ p \\ q \\ r \\ s \end{array} \begin{array}{c} \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$(V, \{\varnothing, \{p\}, \{q\}, \{p,r\}, \{p,s\}, \{r,s\}, \{p,q,r\}, \{p,q,s\}, \{p,r,s\}, \{q,r,s\}\})$

- $V = \{p,q,r,s\}$. For example, $\{p,r\} \in \mathcal{M}_G$ as $G[\{p,r\}] = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is nonsingular over $\mathbb{F}_2$.
- Define, for $X \subseteq V$, the *pivot* $M * X = (V, D * X)$, where $D * X = \{Y \oplus X \mid Y \in D\}$.
- By determinant formula: $\mathcal{M}_{G*X} = \mathcal{M}_G * X$ (if $X \in \mathcal{M}_G$). Explicit: Exclusive-or $\oplus$ "simulates" pivot $*$.




$\{\varnothing, \{p\}, \{q\}, \{p, r\}, \{p, s\}, \{r, s\}, \{p, q, r\}, \{p, q, s\}, \{p, r, s\}, \{q, r, s\}\}$

$\xleftrightarrow{*p}$

$\{\varnothing, \{p\}, \{r\}, \{s\}, \{p, q\}, \{q, r\}, \{q, s\}, \{r, s\}, \{p, r, s\}, \{p, q, r, s\}\}$

- $V = \{p, q, r, s\}$. Indeed $\mathcal{M}_{G*p} = \mathcal{M}_G * p$.

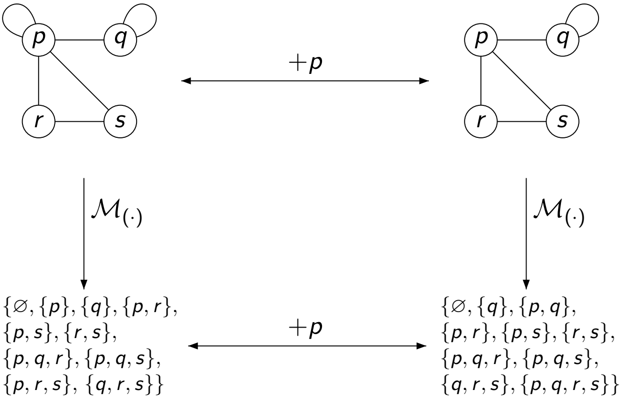
# Loop Complementation on Set Systems

- Let $M = (V, D)$ be a set system.
- Define, for $u \in V$, *loop complementation* of $M$ on $u$, as $M + u = (V, D')$, where $D' = D \oplus \{X \cup \{u\} \mid X \in D, u \notin X\}$.

**Theorem**

*Let $G$ be a graph and $u \in V$. Then $\mathcal{M}_{G+u} = \mathcal{M}_G + u$.*

# Loop Complementation on Set Systems Example




- $V = \{p, q, r, s\}$.
  $\mathcal{M}_G + p = \mathcal{M}_G \oplus \{\{p\}, \{p, q\}, \{p, r, s\}, \{p, q, r, s\}\}$.
  Indeed, $\mathcal{M}_{G+p} = \mathcal{M}_G + p$.

# Interplay Loop Complementation and Pivot

**Theorem (Commutation on different elements)**

*Let $M$ be a set system and $u, v \in V$ with $u \neq v$. Then $M * u * v = M * v * u$, $M + u + v = M + v + u$, and $M + u * v = M * v + u$.*

Proof is by considering both pivot and loop complementation as special cases of a more general operation (called *vertex flip*), and proving that vertex flips commute on different elements.

**Theorem ($S_3$ on single elements)**

*Let $M$ be a set system and $u \in V$. Then $M * u + u * u = M + u * u + u$.*

Proof is by showing that $+u$ and $*u$ generate the group $S_3$ of permutations on three elements.

- Define for $X = \{u_1, \ldots, u_n\}$, $M + X = M + u_1 \cdots + u_n$ (in any order). Similarly for $M * X$.
- We have: 1) [$S_3$] $M + X * X + X = M * X + X * X$, and 2) [commutative] for $Y \cap X = \emptyset$, $M + X * Y = M * Y + X$.

# Interplay Loop Complementation and Pivot for Graphs

- Define for $X = \{u_1, \ldots, u_n\}$, $M + X = M + u_1 \cdots + u_n$ (in any order). Similarly for $M * X$.
- We have: 1) [$S_3$] $M + X * X + X = M * X + X * X$, and 2) [commutative] for $Y \cap X = \emptyset$, $M + X * Y = M * Y + X$.
- Identities must hold for graphs as well. However, $G * X$ is only defined when $X \in \mathcal{M}_G$.
- For graph $G$, $G + X * X + X = G * X + X * X$ when both sides are defined. Turns out: right-hand side defined, implies left-hand side defined.

# Consequences for Simple Graphs

Remember:

**Theorem (Bouchet,1988)**

*Let $G$ be a simple graph with edge $\{u, v\}$. We have* $G * u * v * u = G * v * u * v$.

- In this case, $*u * v * u$ is *edge complementation* (for simple graphs)

**Theorem**

*Let $F$ be a graph with edge $\{u, v\}$ with no loops for $u$ and $v$. We have* $F * \{u, v\} = F + u * u + u * v * u + u = F + v * v + v * u * v + v$.

- So "modulo loops", "$F * \{u, v\} = F * u * v * u = F * v * u * v$". Hence alternative proof of result for simple graphs.

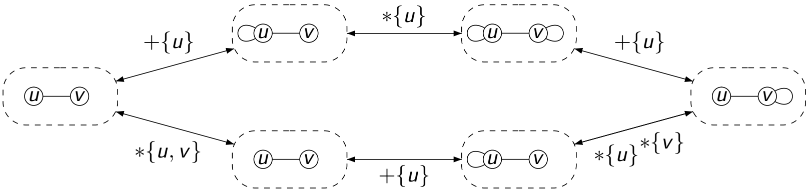
# Proof

## Theorem

*Let $F$ be a graph with edge $\{u, v\}$ with no loops for $u$ and $v$. We have*
$F * \{u, v\} = F + u * u + u * v * u + u = F + v * v + v * u * v + v$.

## Proof.

$\mathcal{M}_F * \{u, v\} + u * u * v + u * u + u = \mathcal{M}_F * u * v + u * u * v + u * u + u = \mathcal{M}_F * u + u * u + u * u + u * v * v = \mathcal{M}_F$. Both sides are applicable by the figure. □



# New Results for Simple Graphs

**Theorem**

*Let $G$ be a simple graph, and let $u, v, w \in V(G)$ be such that the subgraph of $G$ induced by $\{u, v, w\}$ is a complete graph. Then* $G(*\{u\} * \{v\} * \{w\})^2 = G * \{v\}$.

**Theorem**

*Let $G$ be a simple graph, and let $\varphi$ be a sequence of local complementation operations applicable to $G$. Then* $G\varphi \approx G + X * Y$ *for some* $X, Y \subseteq V$ *with* $X \subseteq Y$.

# Discussion

- Interplay pivot and loop complementation is $S_3$ on identical vertices.

# Discussion

- Interplay pivot and loop complementation is $S_3$ on identical vertices.
- Bridges gap between simple graphs and graphs with loops.

# Discussion

- Interplay pivot and loop complementation is $S_3$ on identical vertices.
- Bridges gap between simple graphs and graphs with loops.
- Nature of classic result $G * u * v * u = G * v * u * v$ for simple graphs explained.

# Discussion

- Interplay pivot and loop complementation is $S_3$ on identical vertices.
- Bridges gap between simple graphs and graphs with loops.
- Nature of classic result $G * u * v * u = G * v * u * v$ for simple graphs explained.
- Characterization of sequences of local complementation on simple graphs.

# Discussion

- Interplay pivot and loop complementation is $S_3$ on identical vertices.
- Bridges gap between simple graphs and graphs with loops.
- Nature of classic result $G * u * v * u = G * v * u * v$ for simple graphs explained.
- Characterization of sequences of local complementation on simple graphs.
- Framework setting is set systems in general.