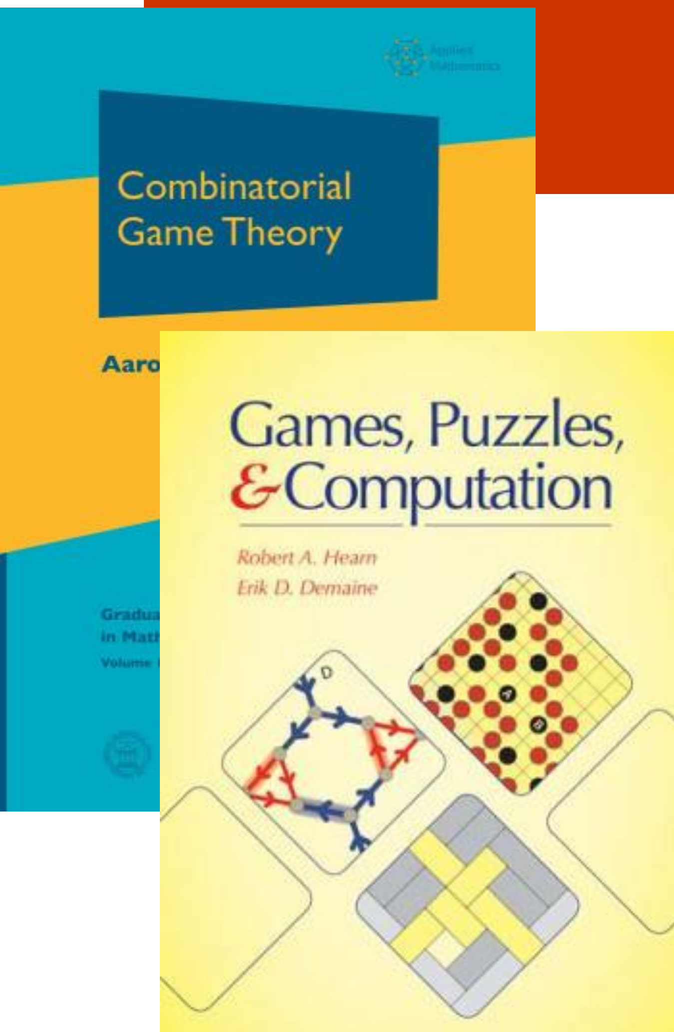


Game Complexity

IPA Advanced Course on Algorithmics and Complexity

Eindhoven, 25 Jan 2019

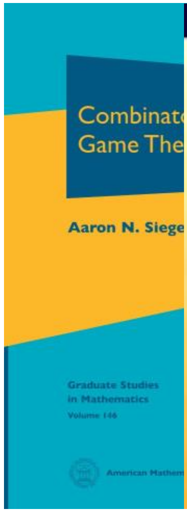
Walter Kosters
Hendrik Jan Hoogeboom

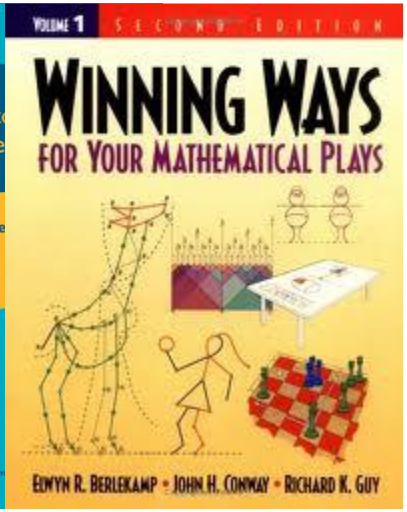
LIACS, Universiteit Leiden

schedule

10:00-11:00	HJ	Constraint Logic, classes
11:15-12:15	W	Gadgets, planarity, exercises
		Rush Hour, *Plank puzzle*
		PSPACE-complete
12:30-13:30		*lunch*
14:00-14:45	HJ	*Tip-Over* is NP-complete
15:00-16:00	W	Combinatorial Game Theory

‘game theory’ fields



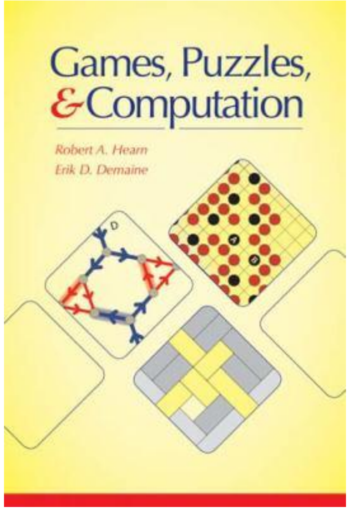


combinatorial game theory

algorithms
mathematical theory

economic game theory

von Neumann, Nash
strategy, optimization expected profit



computational complexity

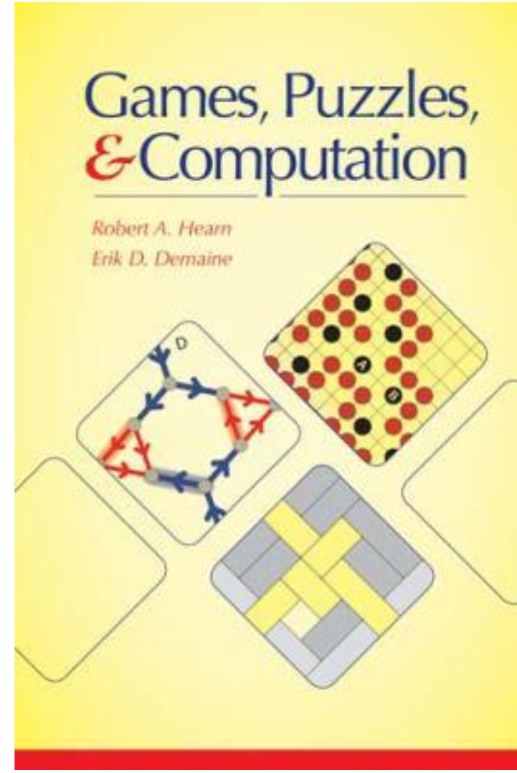
models of computation: *games*
turing machine

reference

Games, Puzzles,
& Computation

Robert A. Hearn
Erik D. Demaine

(2009, AKPeters)



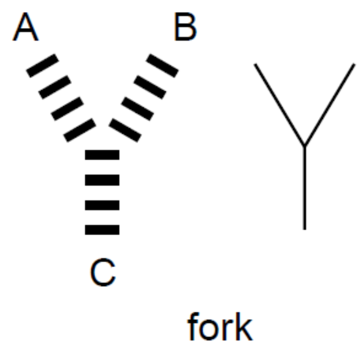
E. Demaine and R.A. Hearn. Constraint Logic: A Uniform Framework for Modeling Computation as Games. In: Proceedings of the 23rd Annual IEEE Conference on Computational Complexity, June 2008.

R.A. Hearn. Games, Puzzles, and Computation PhD thesis, MIT, 2006.

games & complexity classes

Computing with Planar Toppling Domino Arrangements

William M. Stevens



challenge:
(no) timing & (no) bridges

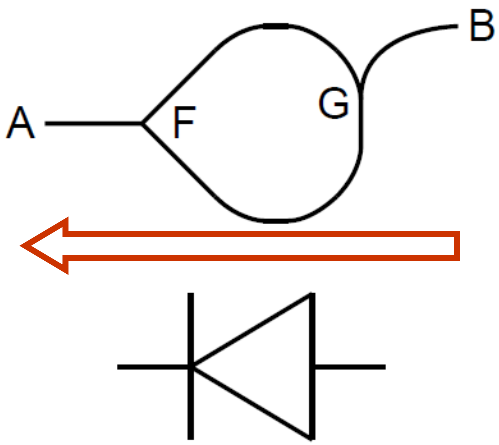


Fig. 3. A one way line

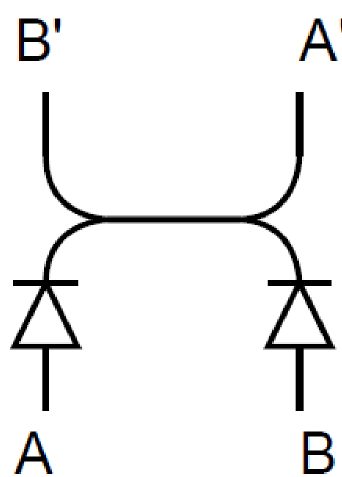


Fig. 4. A single line crossover

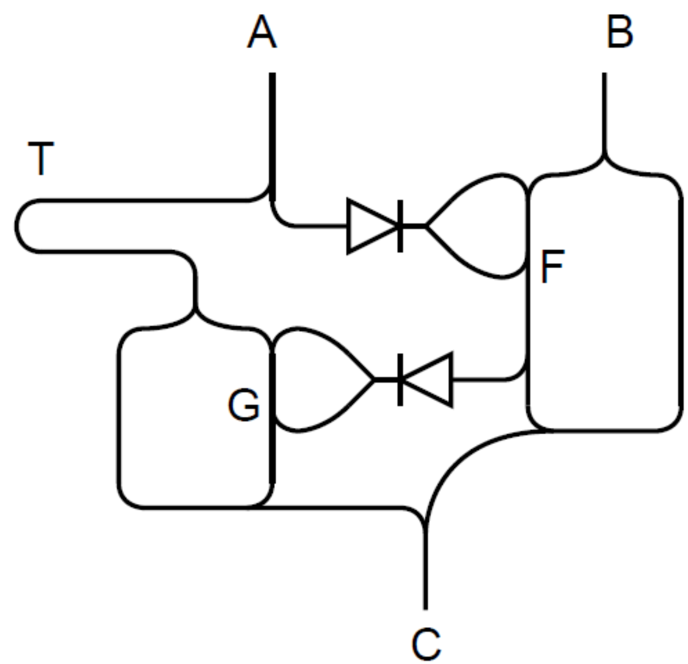


Fig. 5. A both mechanism

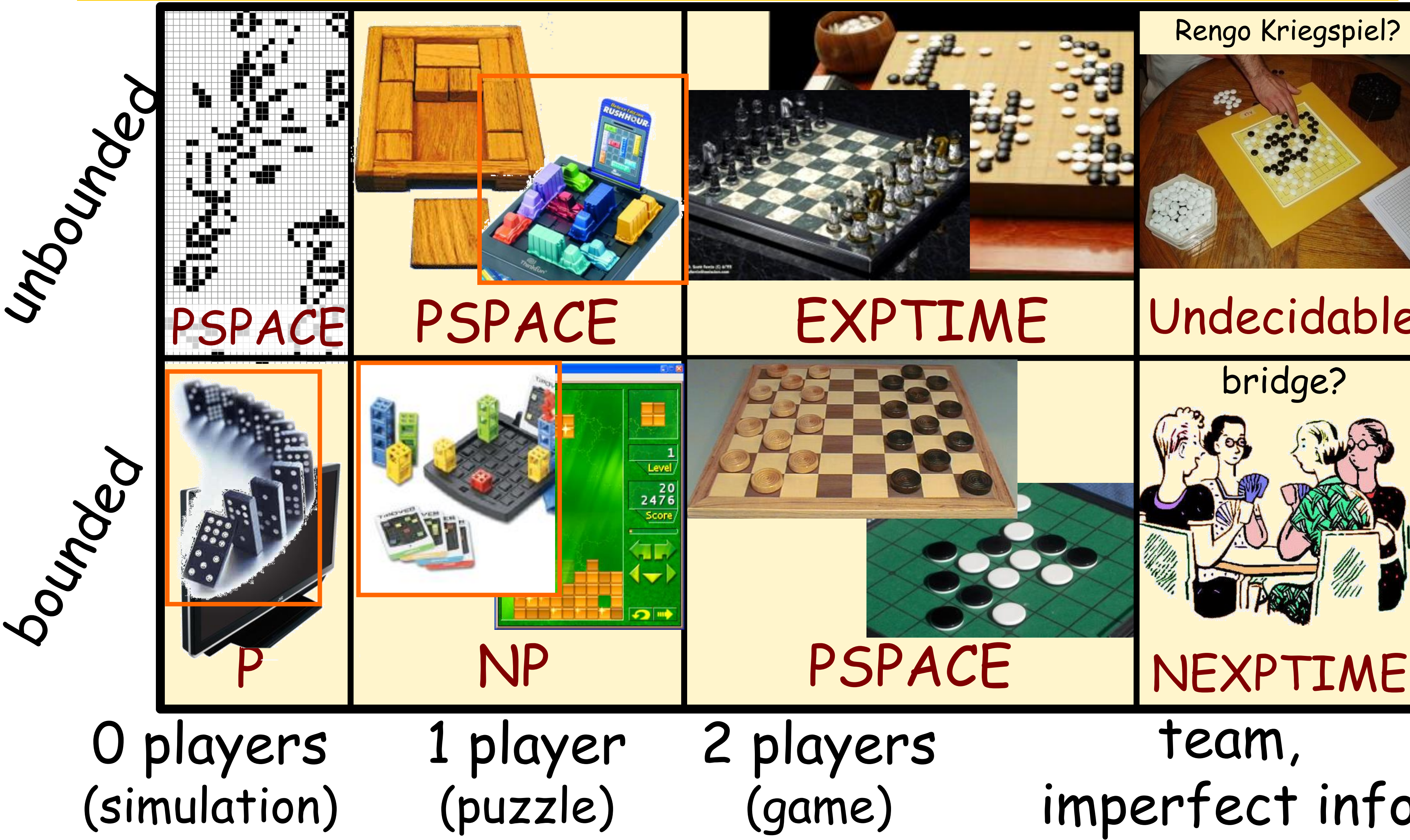
what is a game?

characteristics
- bounded state
- moves, repetition
- players, goal

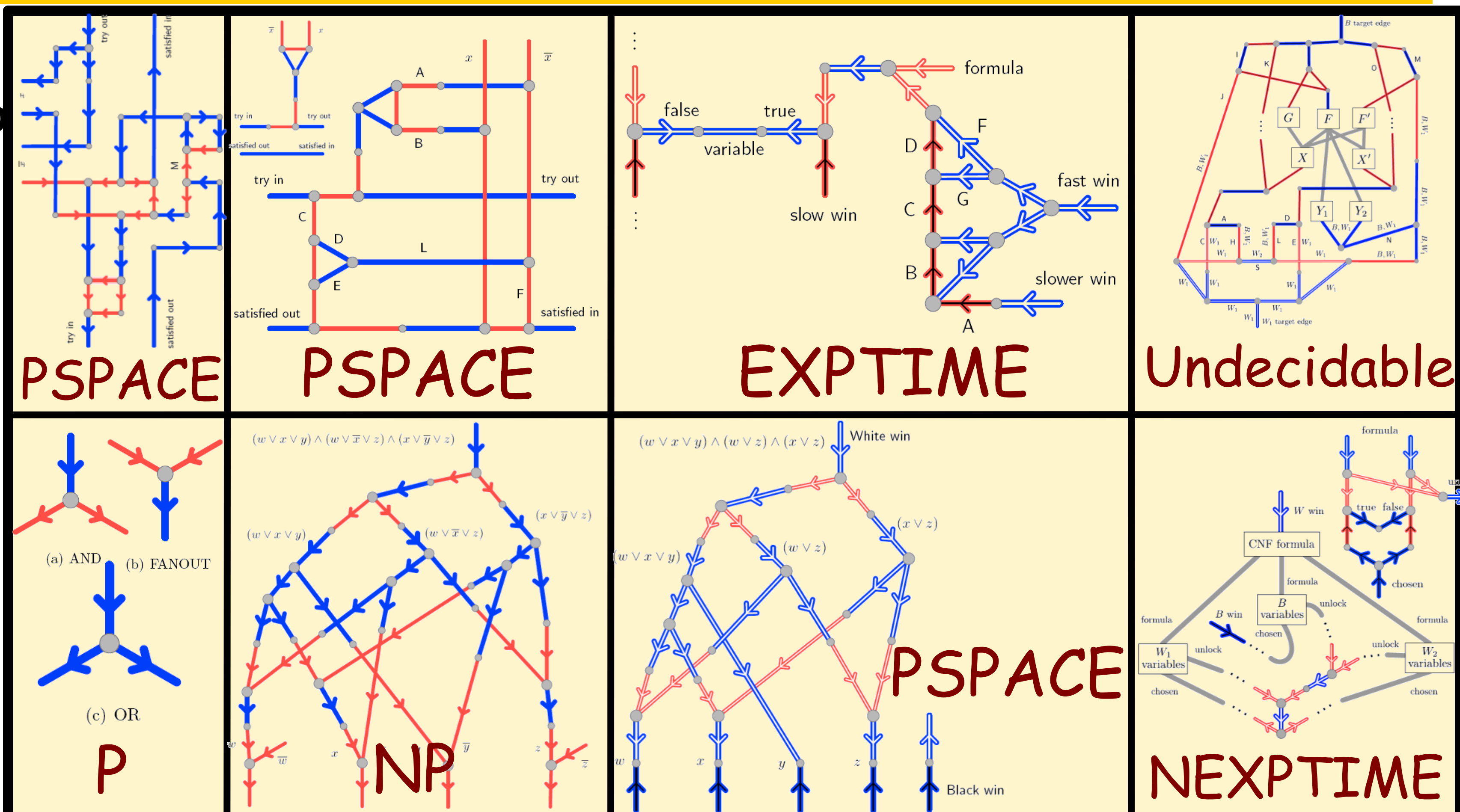
study the complexity of

- simulation (0p) *game of life*
- puzzles (1p) *rush hour*
- board games (2p) ‘generalized’ *chess*
- teams

Complexity of Games & Puzzles
[Demaine, Hearn & many others]



Constraint Logic
[Hearn & Demaine 2009]



unbounded

bounded

0 players (simulation)

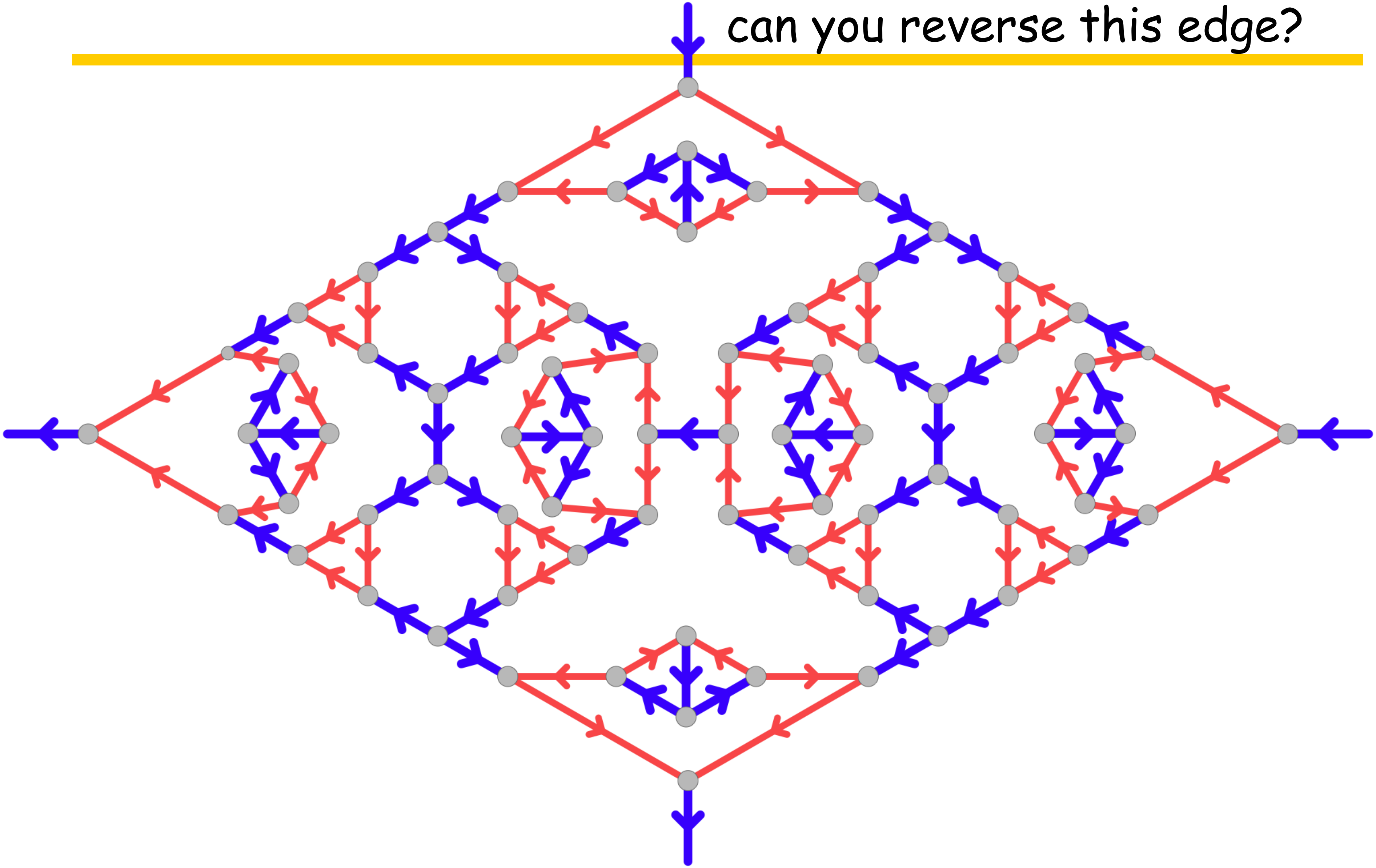
1 player (puzzle)

2 players (game)

team, imperfect info

Decision Problem

can you reverse this edge?



(details to follow)

constraint logic

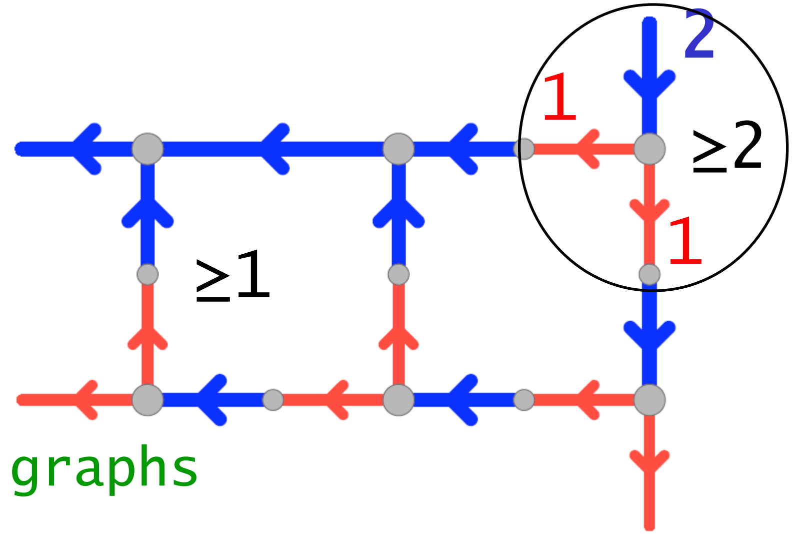
constraint graphs

directed 'oriented'

edge weight 1,2

inflow constraints

legal configuration



game/computation on constraint graphs

move: legal edge reversal

goal: reverse specific edge

NCL - nondet constraint logic

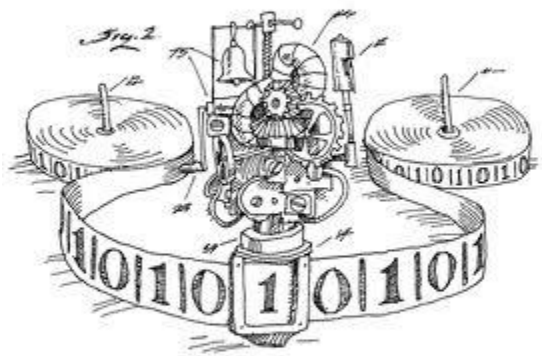
instance: constraint graph G, edge e

question: sequence which reverses e

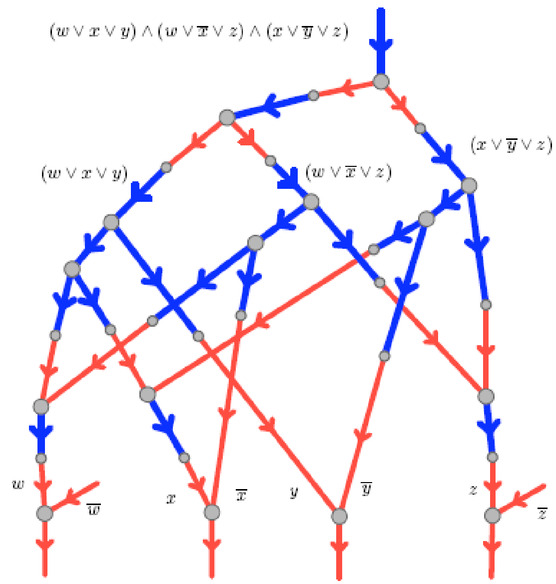
BOUNDED NCL

… reverses each edge *at most once*

NP & TipOver



$$(w \vee x \vee y) \wedge (w \vee \overline{x} \vee z) \wedge (x \vee \overline{y} \vee z)$$



$\Longleftrightarrow$ $\Longleftrightarrow$

NP

3SAT

part I

constraint logic

'graph games'

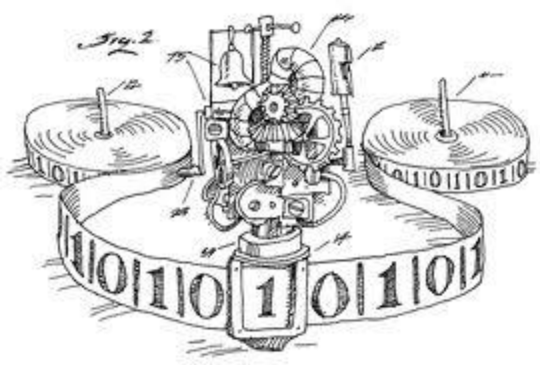
Bounded NCL

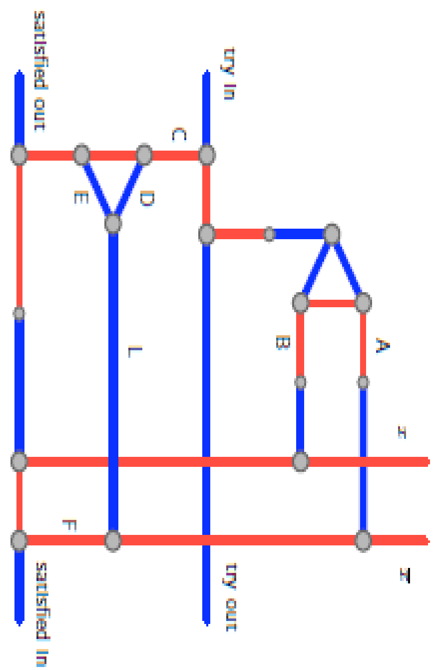
part II

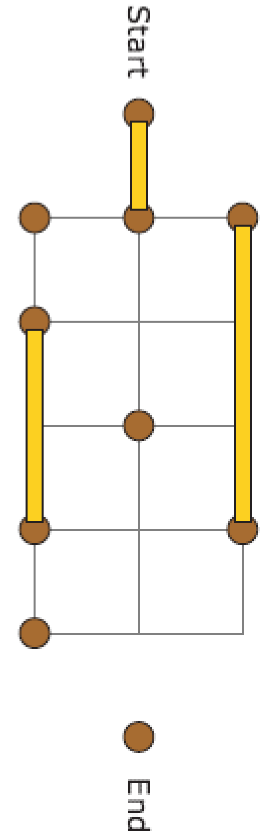
games in particular

TipOver

PSPACE & Plank Puzzle







PSPACE	part I constraint logic 'graph games'	part II games in particular
QBF	NCL	plank puzzle (river crossing)

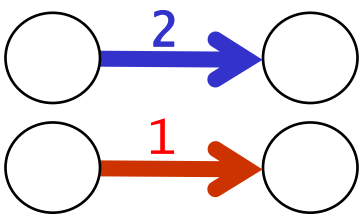
‘formal’ definition

constraint logic – a graph game

issues

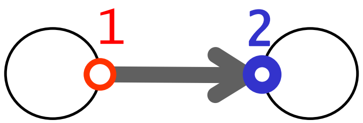
goal: generic ‘graph game’

- several instantiations for specific complexity classes / game types
- reduction to games & puzzles



Hearn & Demaine ‘constraint logic’

coloured edges / connectors



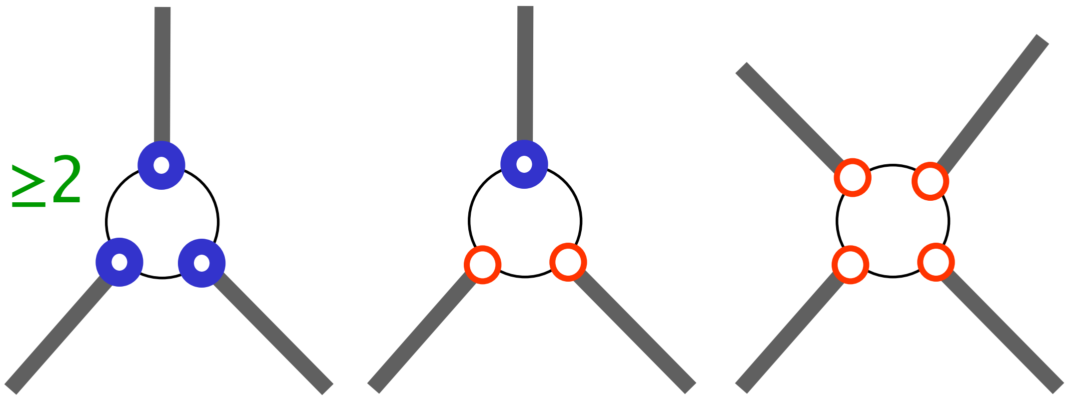
bounded vs. unbounded

(natural direction of computation)

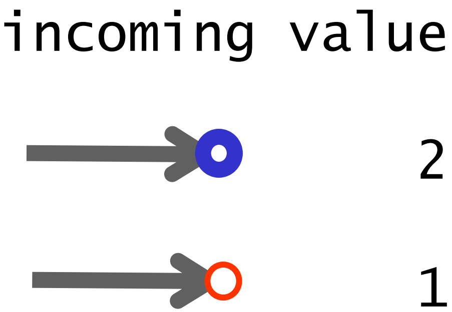
planarity

basic constraint logic

examples



edge connectors



constraint graph

oriented/directed edges + connectors

vertex constraint

inflow value ≥ 2

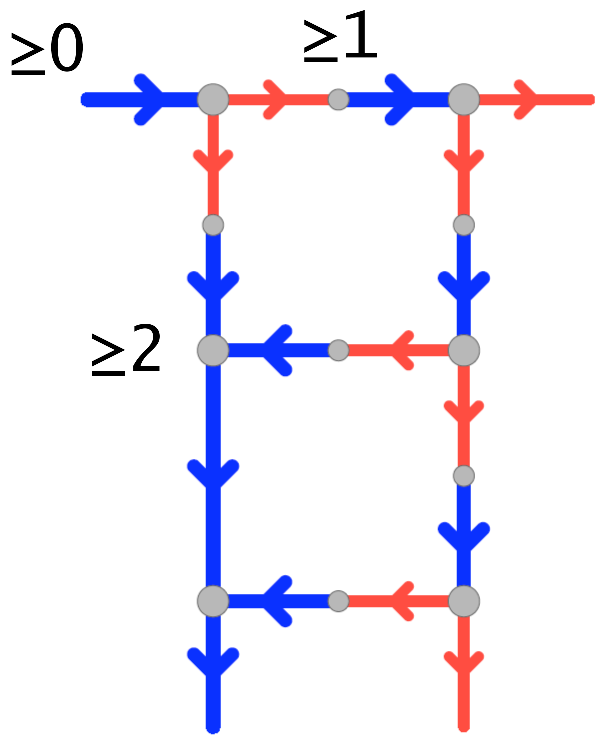
game: legal move

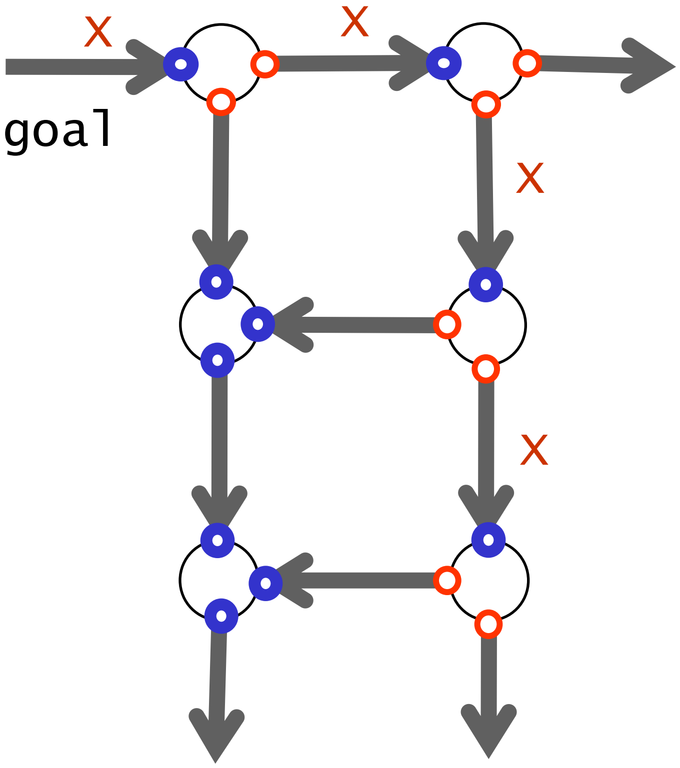
edge reversal satisfying constraint

goal

reversal given edge

‘special’ vertex constraints?



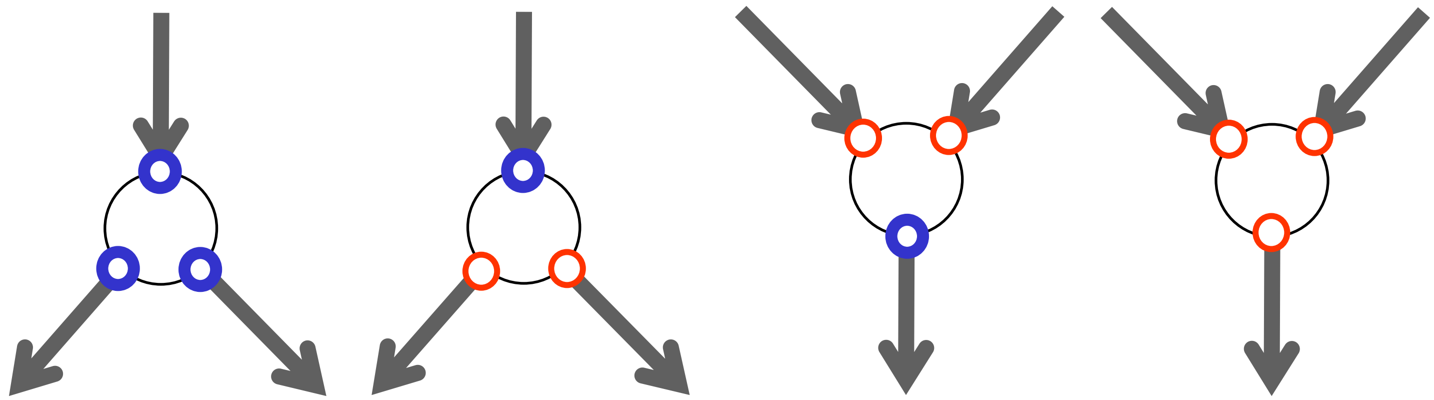


colour conversion

dangling edges   *edge terminators*

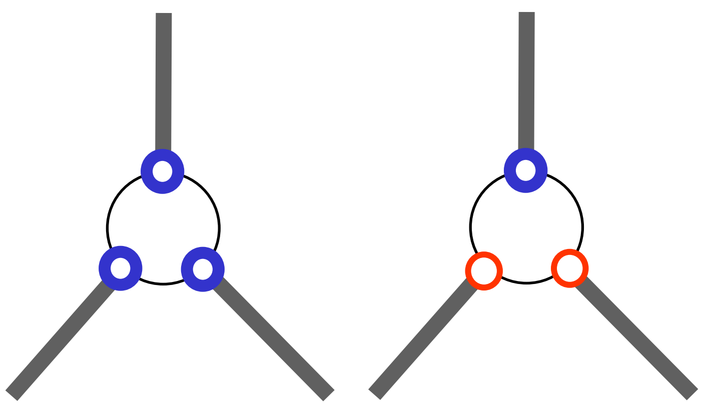
normal form vertices

bounded NCL (reverse only once)



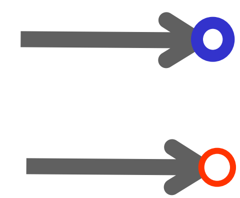
OR AND FANOUT CHOICE

NCL



OR AND

incoming value

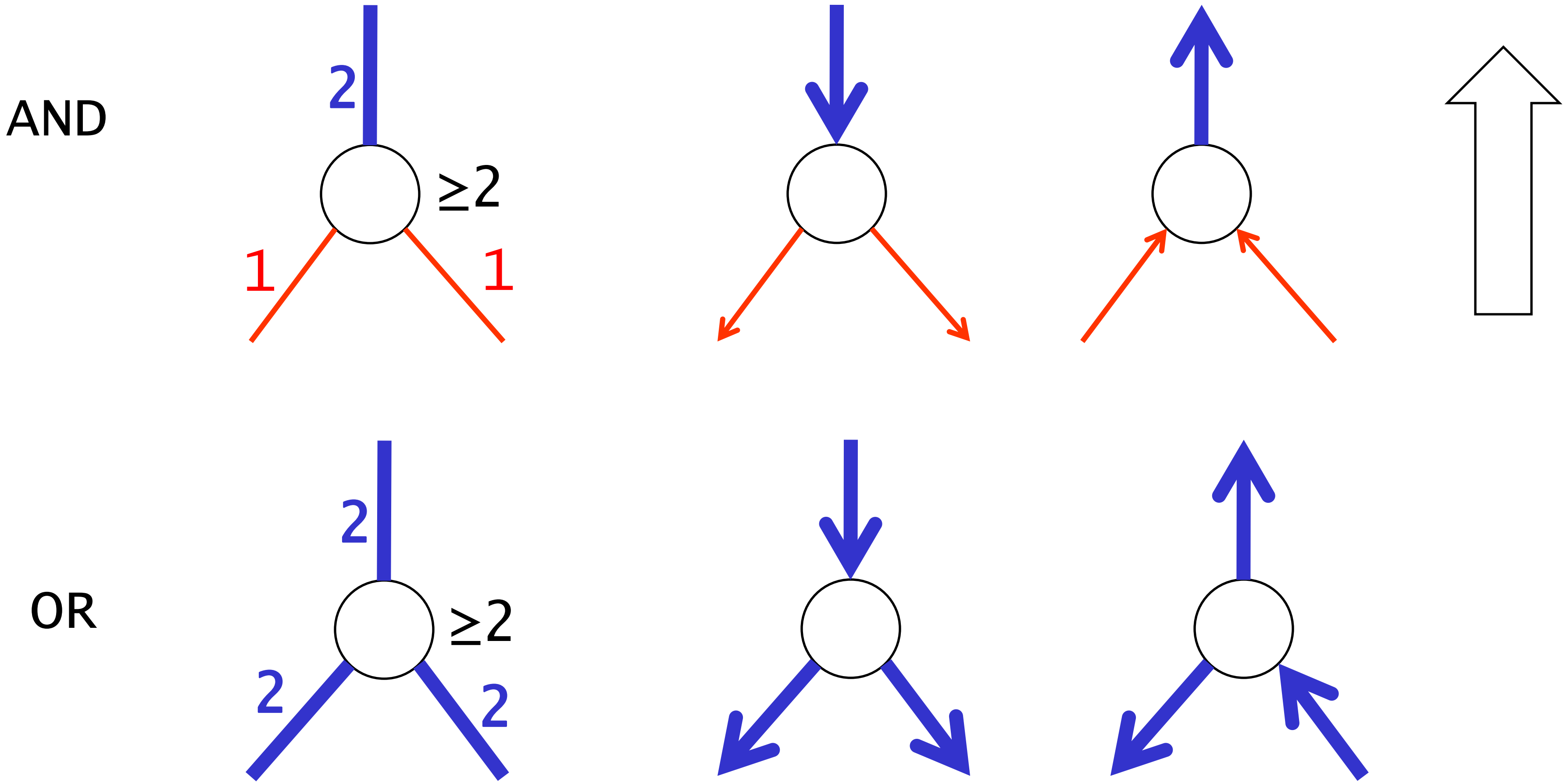


2

1

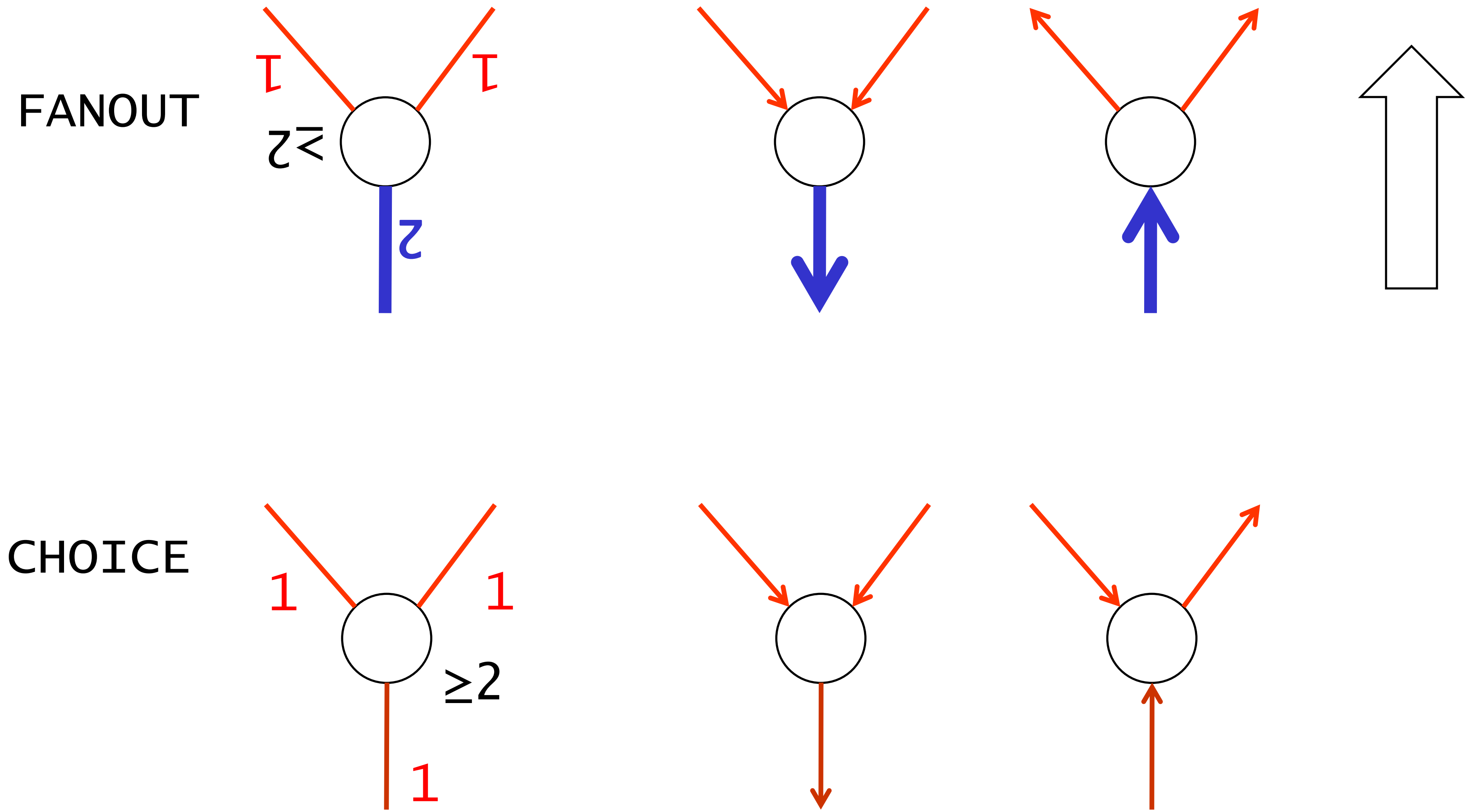
implementing gates

intuitive meaning of vertices



implementing gates

intuitive meaning of vertices



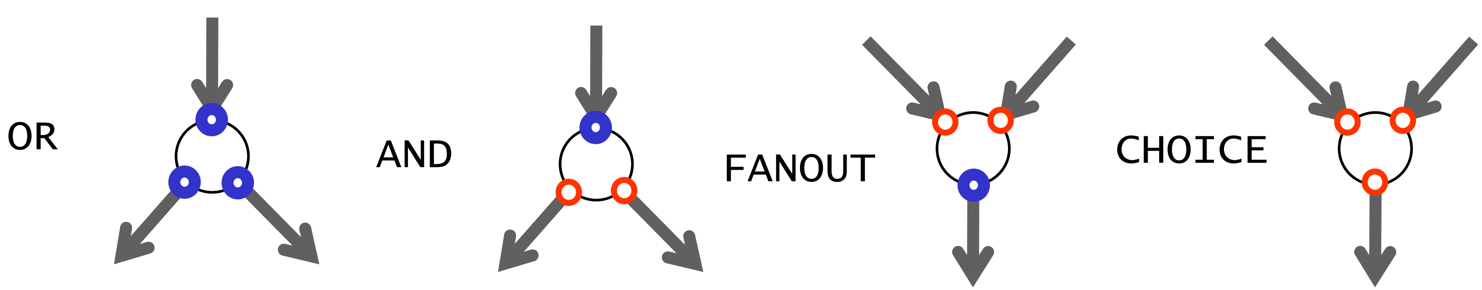
basic observation

“emulate” a logical formula as graph game

goal:
flip a given edge *iff* formula satisfiable

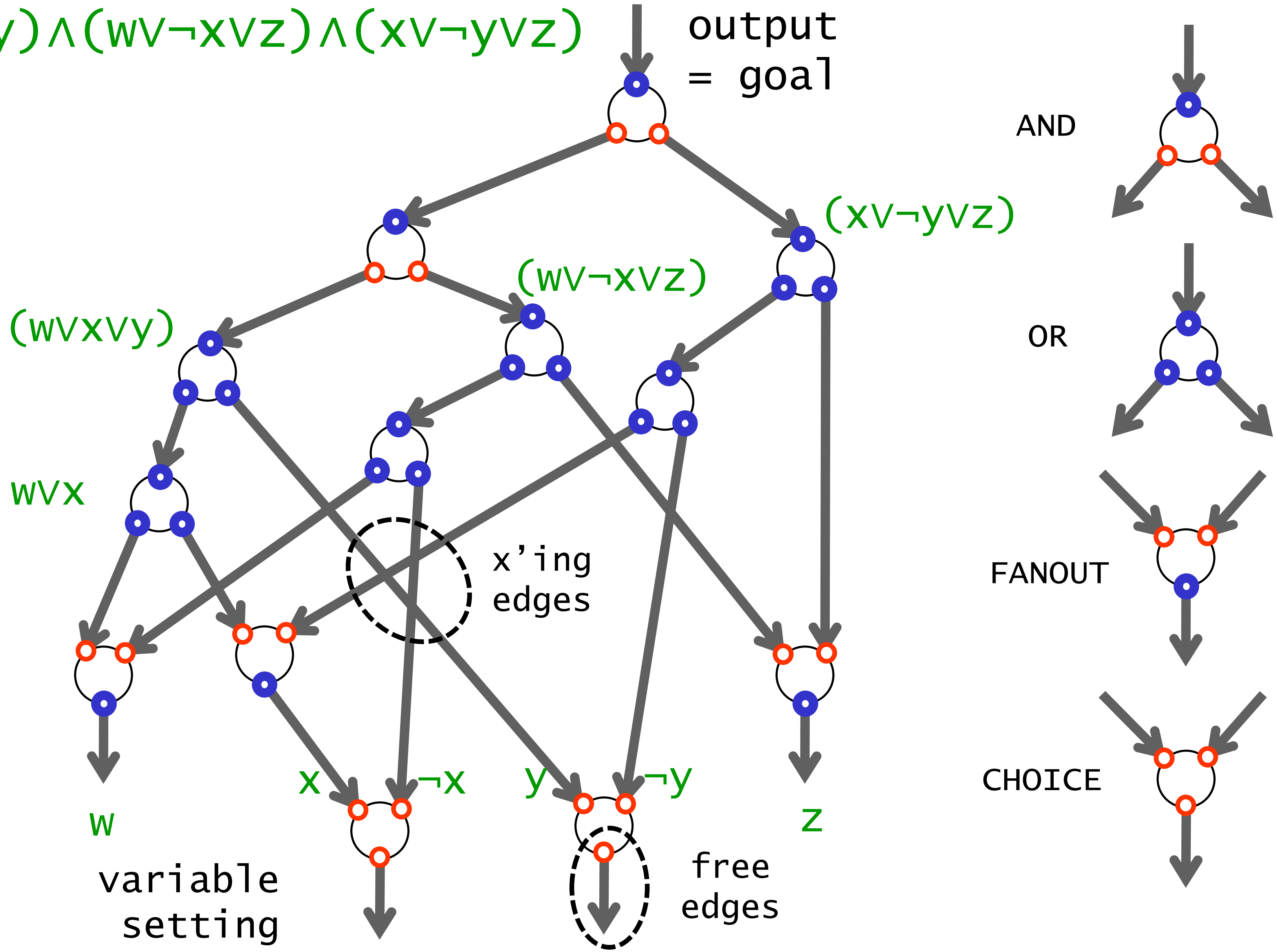
$$(w \lor x \lor y) \land (w \lor \neg x \lor z) \land (x \lor \neg y \lor z)$$

components



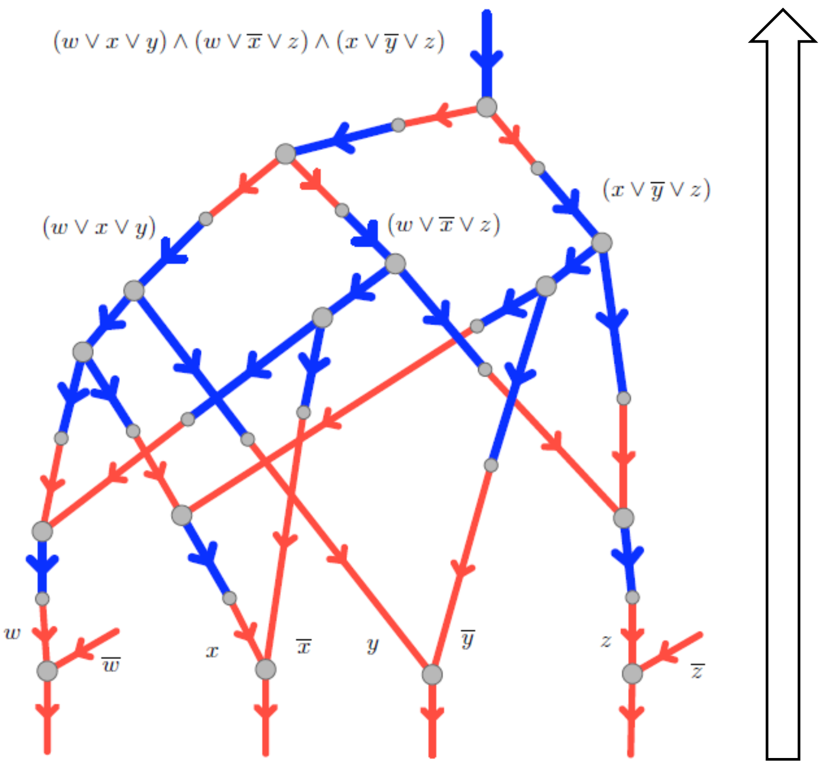
formula constraint graph

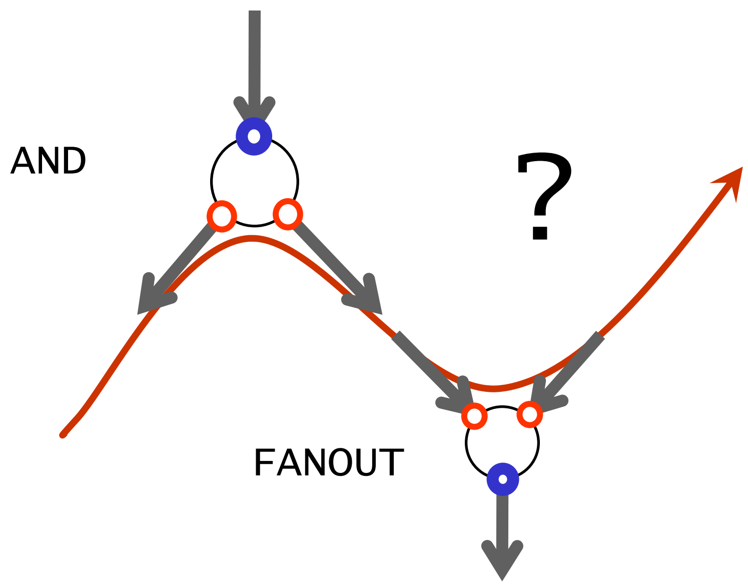
$(w \lor x \lor y) \land (w \lor \neg x \lor z) \land (x \lor \neg y \lor z)$



questions

- ‘can’ : not obliged to reverse edges upwards
  ie, we do not always set variable
- can we reverse the ‘wrong way’?
- do we need restriction to reverse edge once?





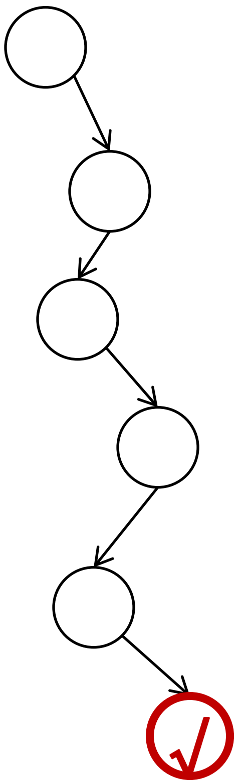
basic complexity classes

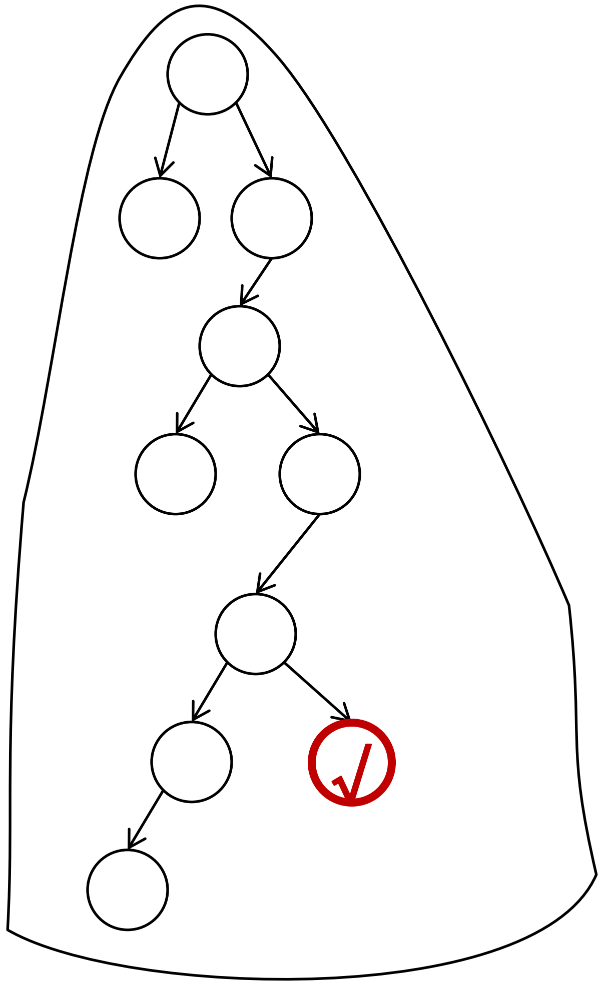
game complexity classes
vs.
TM resources: *space* & *time*

Cook/Levin NP completeness SAT

Savitch PSPACE = NPSPACE

computation tree

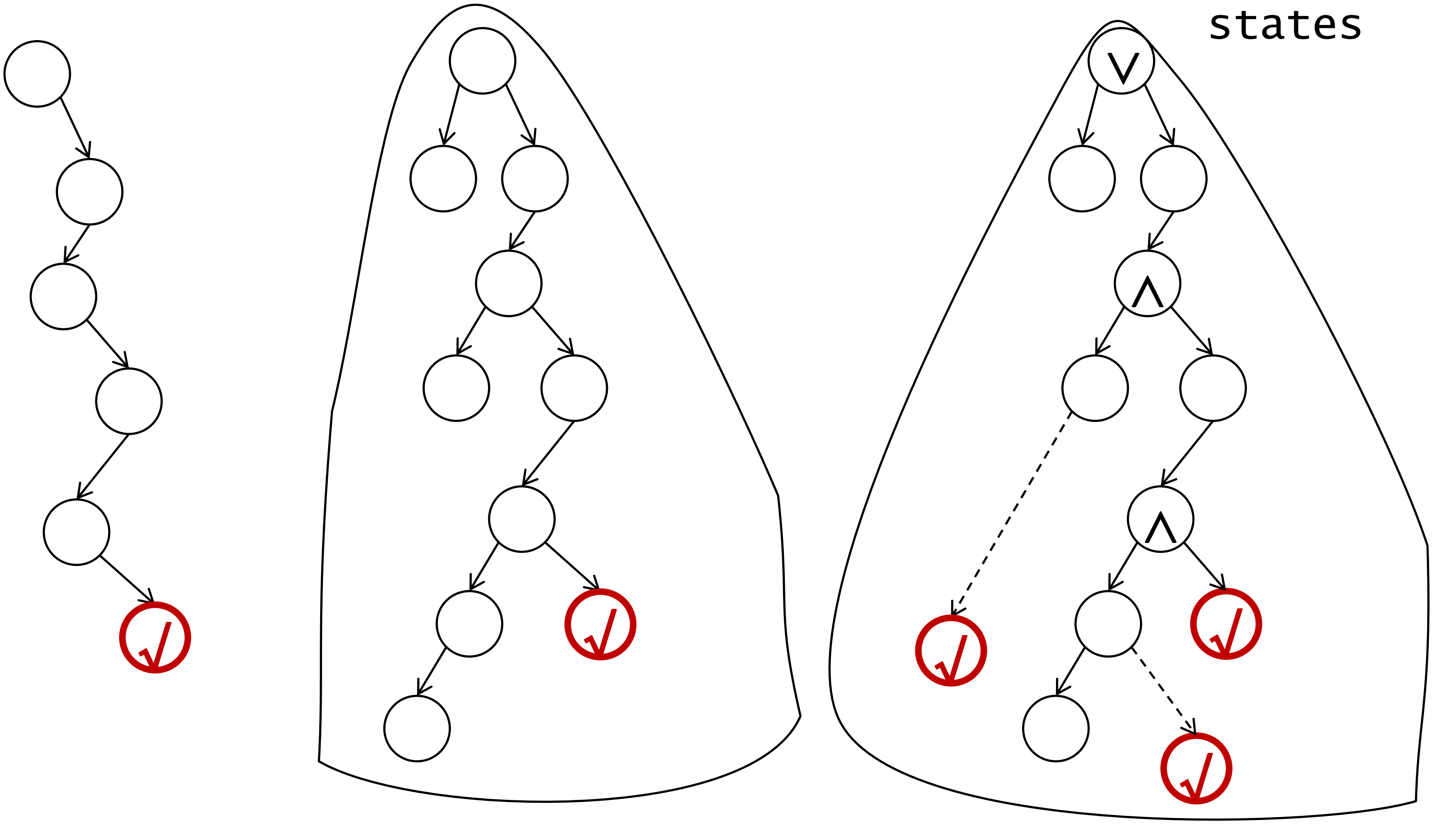




determinism nondeterminism

computation tree

existential and *universal* states



determinism nondeterminism alternation

Savitch

$\boxed{\text{NSPACE}(\ s(n)\ ) \subseteq \text{SPACE}(\ s^2(n)\ )}$

can we reach a halting configuration?
at most exponentially many steps $s(n)|\Sigma|^{s(n)}$

solve recursively "re-use space"

reach(*ini*,*fin*,1) = step(*ini*,*fin*)
reach(*ini*,*fin*,2k)
  foreach configuration mid
    test reach(*ini*,*mid*,k) ∧ reach(*mid*,*fin*,k)

stack depth s(n) of configs, each size s(n)

$\boxed{\text{NPSPACE} = \text{PSPACE}}$

NSPACE(s(n)) ⊆ ATIME(s^2(n)) "parallel in time"

dimensions

existential and *universal* states
computation = tree

	log.	*polynomial*		*exp.*
	space	time	space	time
determinism	L	P	PSPACE	EXPTIME
nondeterminism	NL	NP	NPSPACE	NEXPTIME
alternation	AL	AP	APSPACE	AEXPTIME

AL AP APSPACE AEXPTIME

$\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} \subseteq \mathrm{EXPTIME} \subseteq \mathrm{NEXPTIME} \subseteq \mathrm{EXPSPACE}$

NPSPACE NEXPSPACE

A.K. Chandra, D.C. Kozen, and L.J. Stockmeyer. 'Alternation', Journal of the ACM, Volume 28, Issue 1, pp. 114–133, 1981.

game categories

game categories and their natural complexities

Rush Hour
River Crossing

unbounded	**PSPACE**	**PSPACE**	**EXPTIME**	**undecid**
bounded	**P**	**NP**	**PSPACE**	**NEXPTIME**
#	**zero** *simulation*	**one** *puzzle*	**two** *game*	**team** *imperfect informat.*

TipOver

NL ⊆ P ⊆ NP ⊆ PSPACE ⊆ EXPTIME ⊆ NEXPTIME

game categories

game categories and their natural complexities

(polynomial)
TM resources

Rush Hour
River Crossing

unbounded **SPACE**	**PSPACE**	**PSPACE** NPSPACE	**EXPTIME** APSPACE	undecid
bounded **TIME**	**P**	**NP**	**PSPACE** AP	NEXPTIME
#	**zero** *simulation* ***determ.***	**one** *puzzle* ***nondeterm.***	**two** *game* ***alternat.***	team *imperfect informat.*

TipOver

NL $\subseteq$ P $\subseteq$ NP $\subseteq$ PSPACE $\subseteq$ EXPTIME $\subseteq$ NEXPTIME
= NPSPACE = AP

3SAT

literal
(negated) variable

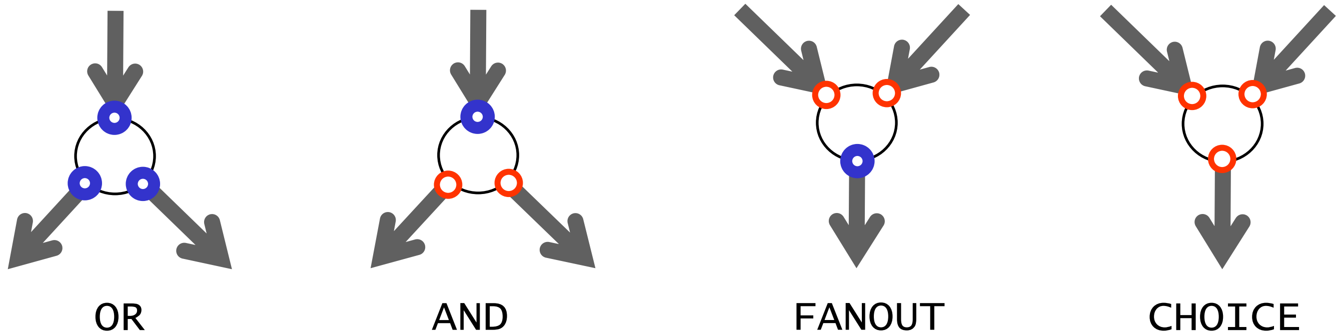
$$\underbrace{(w \vee x \vee y)}^{\text{clause}} \wedge (w \vee \neg x \vee z) \wedge (x \vee \neg y \vee z) \wedge \ldots$$

3 conjunctive normalform

3SAT
given: given formula ϕ in 3CNF
question: is ϕ satisfiable?
(can we find a variable assignment making formula true)

Cook/Levin   3SAT is NP-complete

conclusion (B-NCL)



BOUNDED NCL - nondet constraint logic

instance: constraint graph G, edge e

question: sequence which reverses *each edge at most once*, ending with e

- reduction from 3SAT into Bounded NCL
- Bounded NCL is in NP

thm. Bounded NCL is NP-complete

however: topling domino's cannot cross

formula games – complete problems

NL **2SAT**

$(x_1 \vee x_3) \wedge (\neg x_5 \vee \neg x_3) \wedge (x_5 \vee x_1)$

P **HORN-SAT**

$(\neg x_3 \vee \neg x_2 \vee \neg x_5 \vee x_1)$ i.e. $(x_3 \wedge x_2 \wedge x_5 \rightarrow x_1)$

NP **SAT** satisfiablity

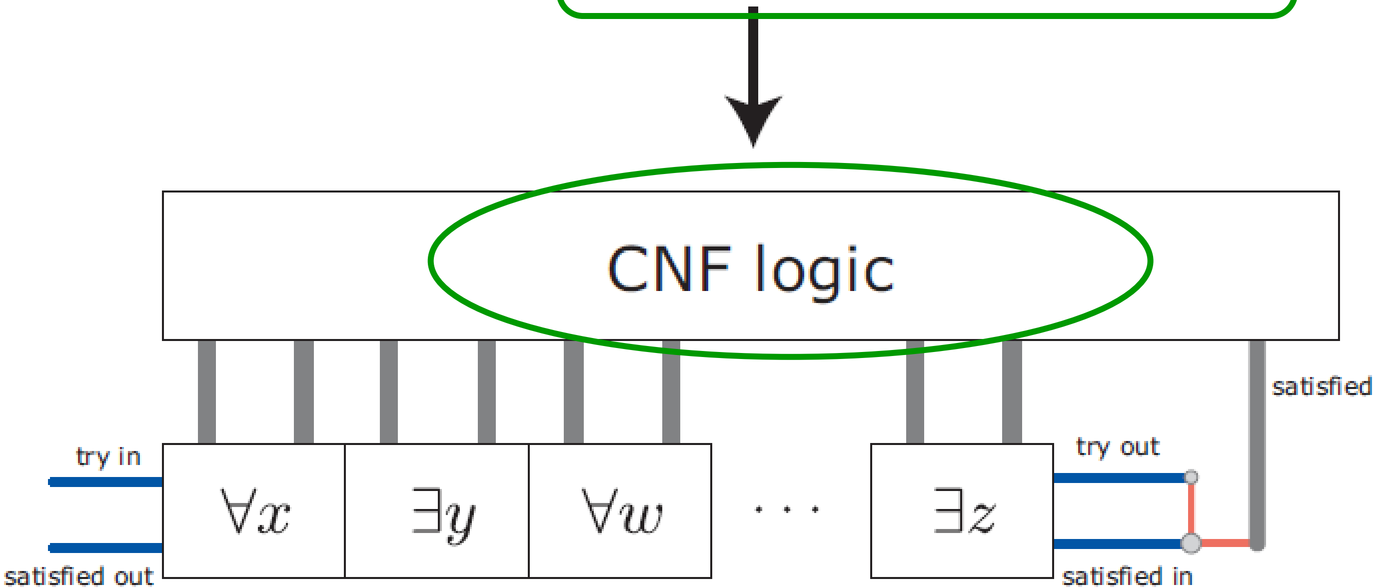
$\exists x_1\ \exists x_3\ \exists x_5\ (x_1 \vee x_3 \vee \neg x_5) \wedge (\neg x_1 \vee \neg x_3) \wedge (x_5 \vee x_1)$

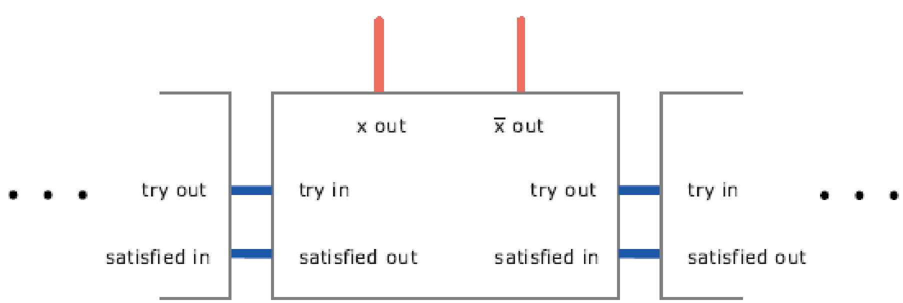
(N)PSPACE **QBF** aka QSAT

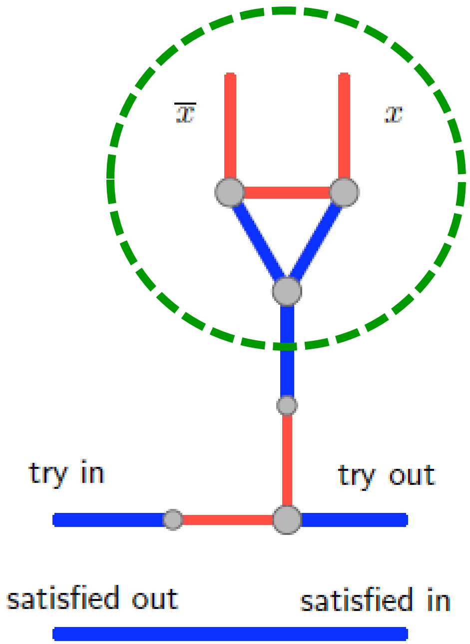
$\exists x_1\ \forall x_3\ \exists x_5\ (x_1 \vee x_3 \vee \neg x_5) \wedge (\neg x_1 \vee \neg x_3) \wedge (x_5 \vee x_1)$

quantification

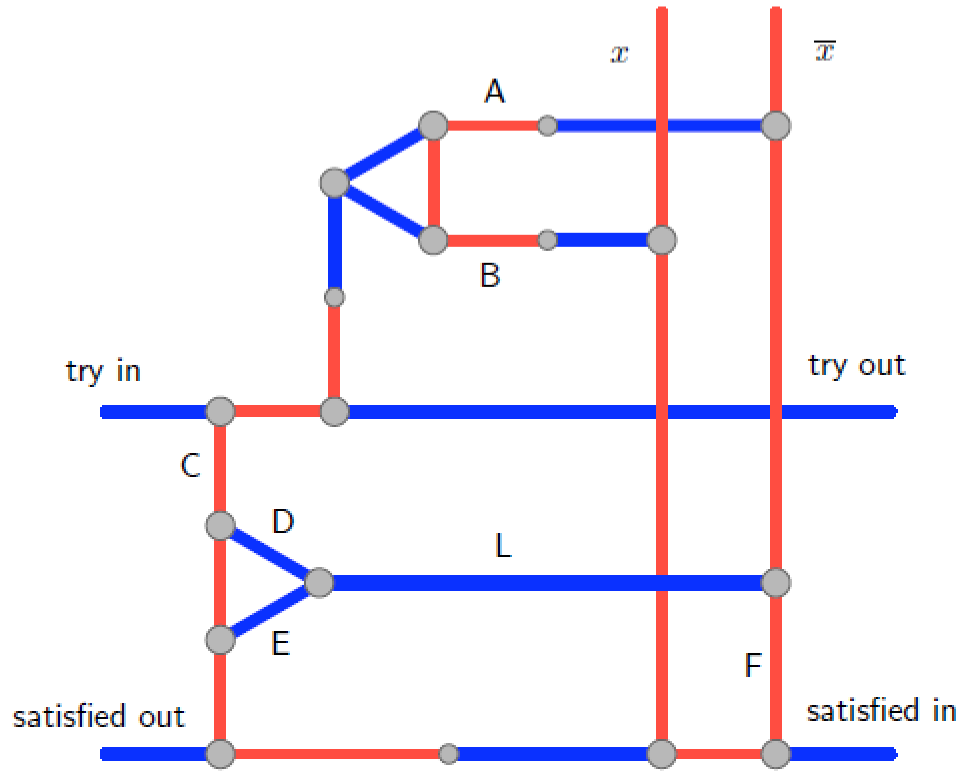
$$\forall x \exists y \forall w \cdots \exists z \, \boxed{[(x \vee y) \wedge \cdots \wedge (\overline{z} \vee x \vee \overline{w})]}$$





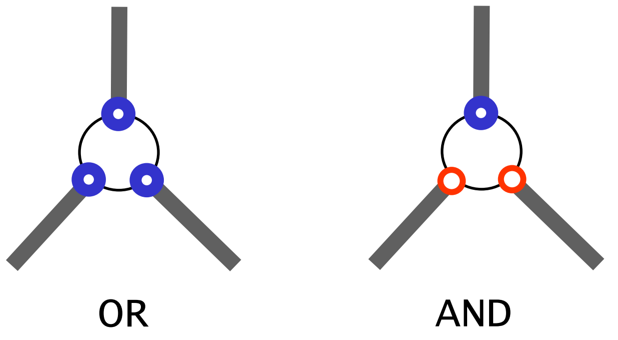


(a) Existential quantifier



(b) Universal quantifier

conclusion (NCL)



initial orientation
arrows not specified

NCL - nondet constraint logic
instance: constraint graph G, edge e
question: sequence which reverses e

thm. NCL is PSPACE-complete

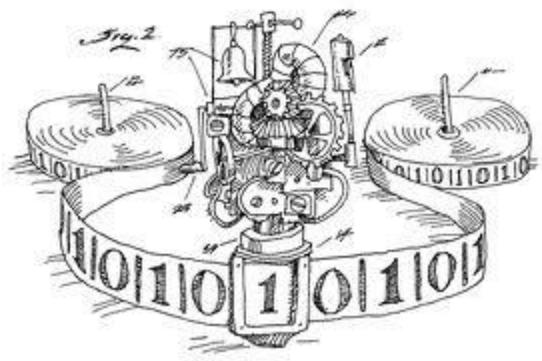
next: concrete games

bounded: NP

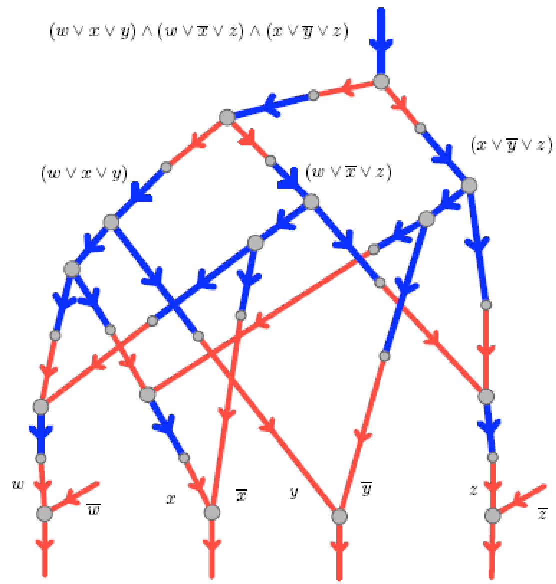
unbounded: PSPACE

TipOver is NP-Complete

NP & TipOver



$$(w \vee x \vee y) \wedge (w \vee \overline{x} \vee z) \wedge (x \vee \overline{y} \vee z)$$



NP

3SAT

part I

constraint logic

'graph games'

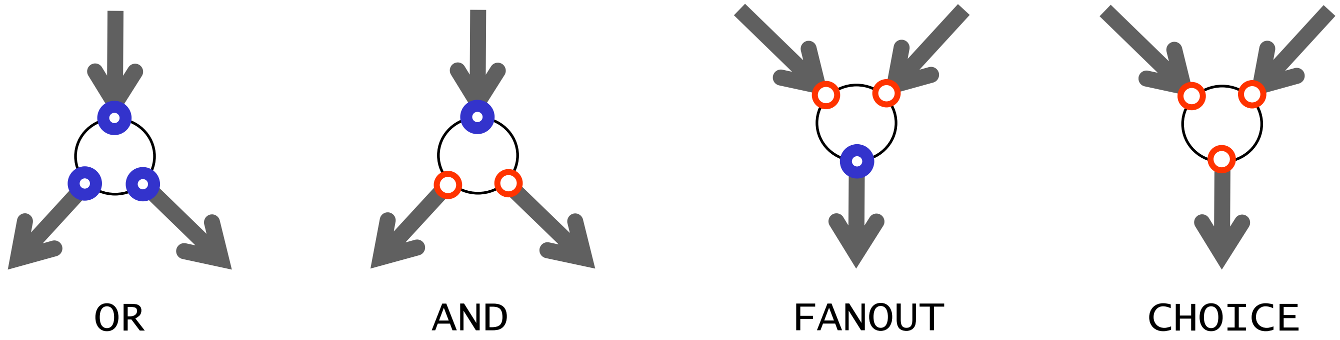
Bounded NCL

part II

games in particular

TipOver

conclusion



BOUNDED NCL - nondet constraint logic

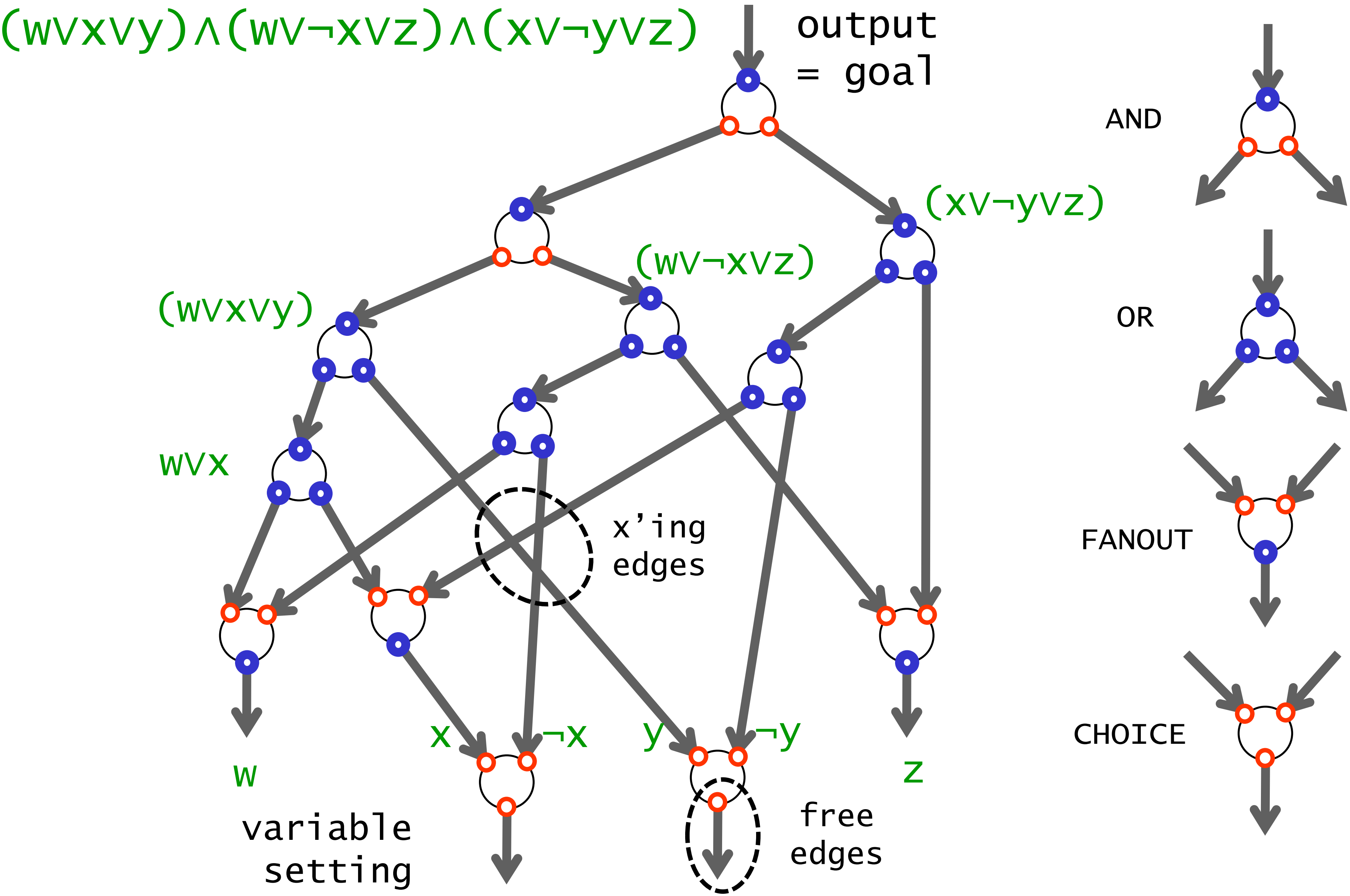
instance: constraint graph G, edge e

question: sequence which reverses *each edge at most once*, ending with e

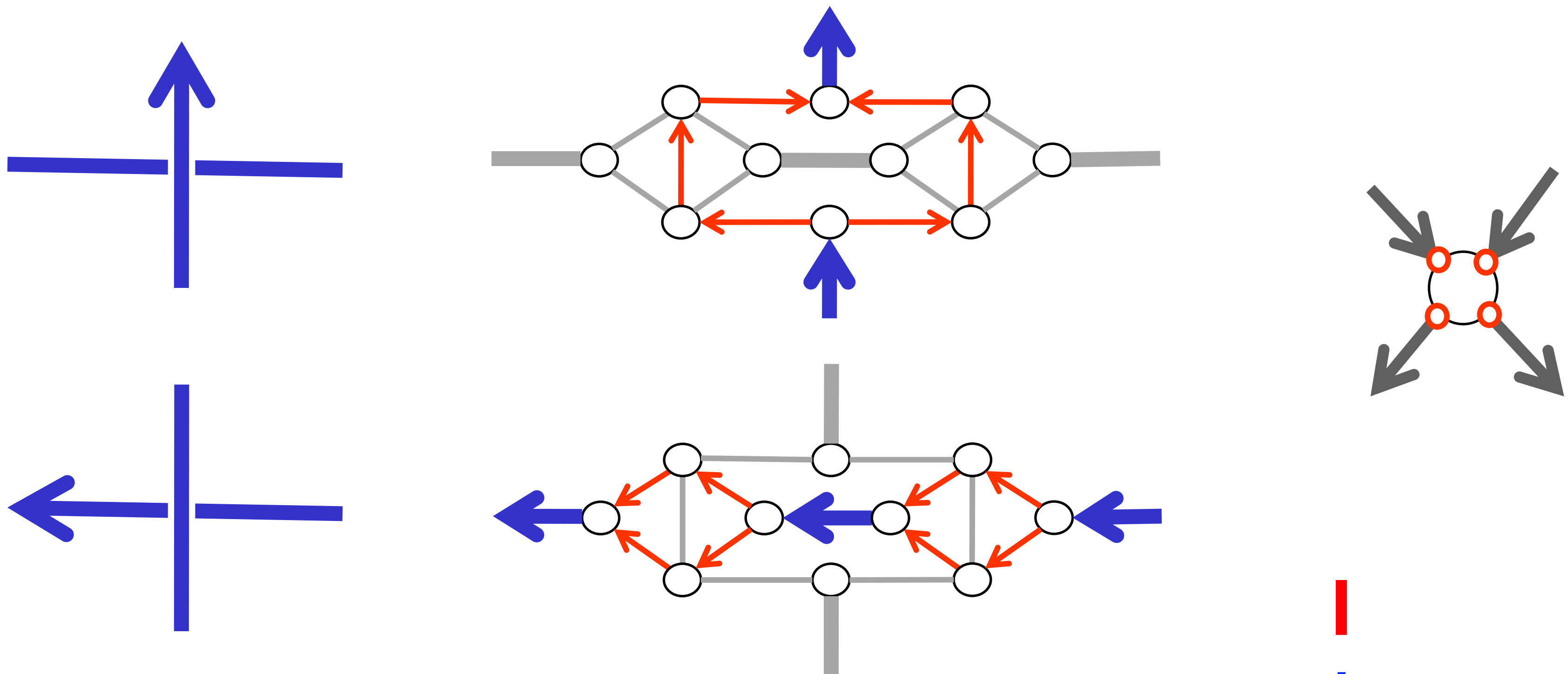
Bounded NCL is NP-complete

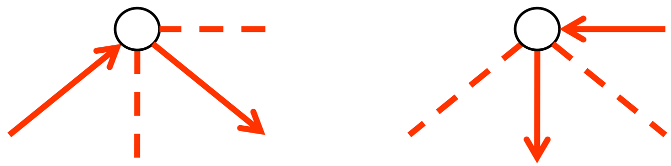
however: topling domino's and crates cannot cross

formula constraint graph



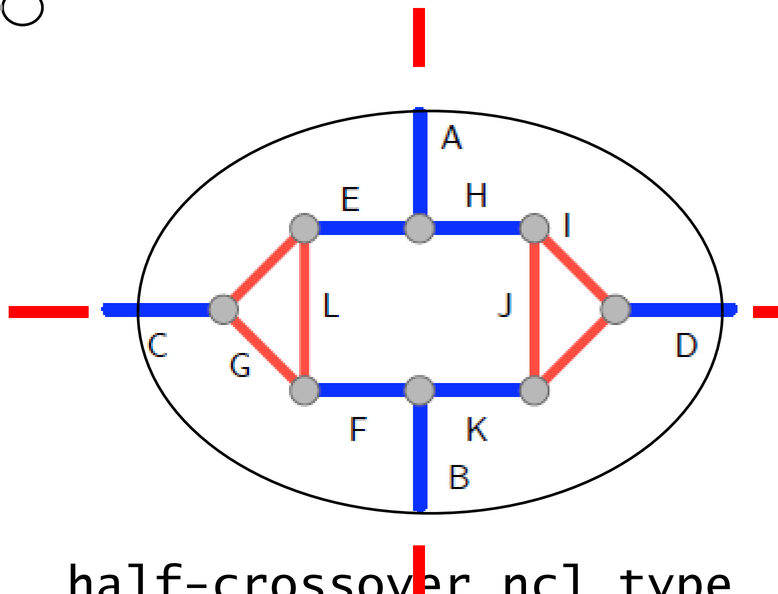
planar crossover gadget





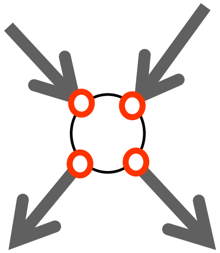
necessary behaviour

either of them or both!

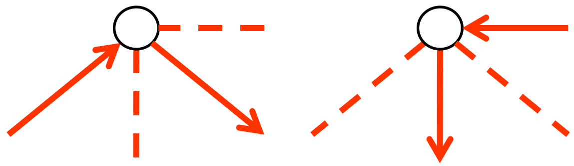


half-crossover ncl type

bounded NCL half-crossover

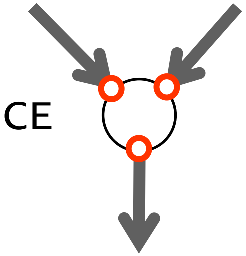


to be replaced

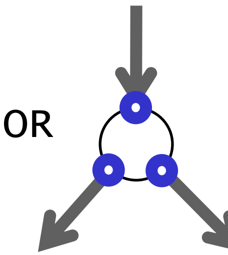


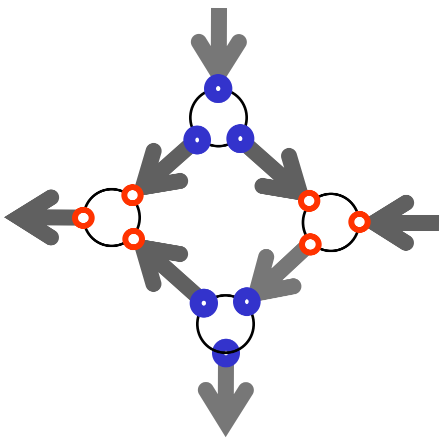
restricted behaviour

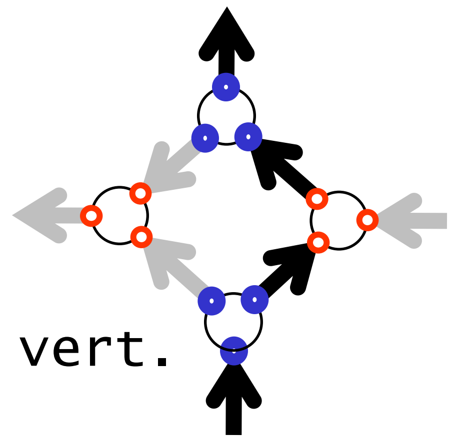
CHOICE



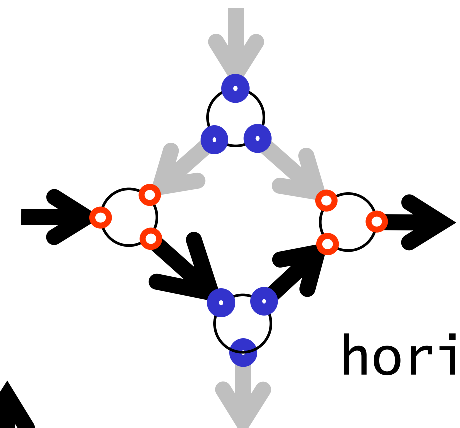
OR



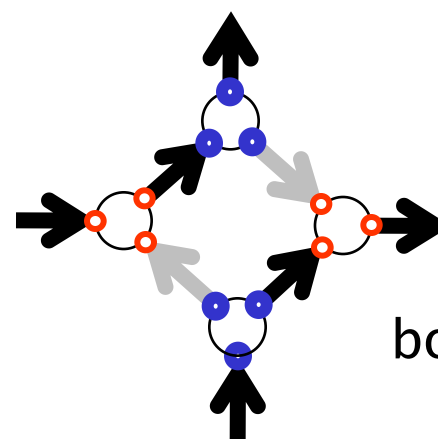




vert.

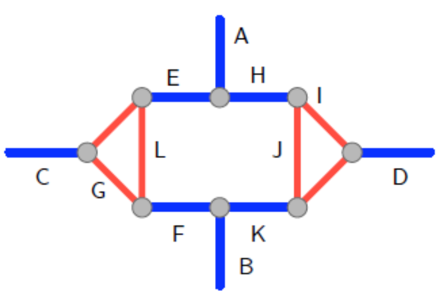


hori.



both

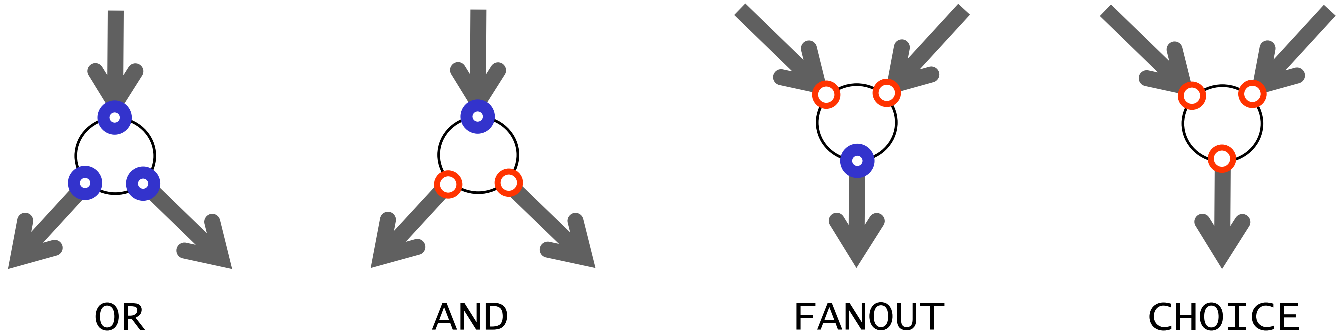
race condition



(b) Half-crossover

half-crossover *bounded* ncl type

conclusion (planer BNCL)



BOUNDED NCL - nondet constraint logic

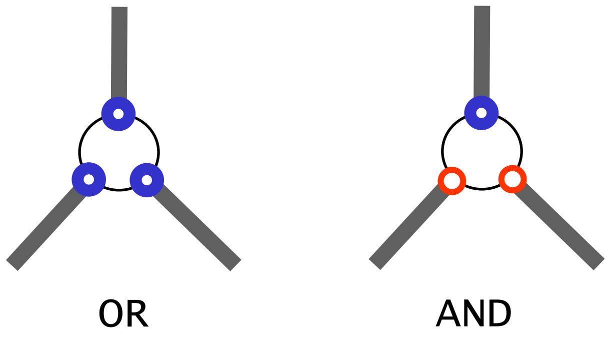
instance: constraint graph G, edge e

question: sequence which reverses *each edge at most once*, ending with e

Bounded NCL is NP-complete,

even for planar graphs,
with restricted vertices

conclusion (planar NCL)



NCL - nondet constraint logic

instance: constraint graph G, edge e

question: sequence which reverses e

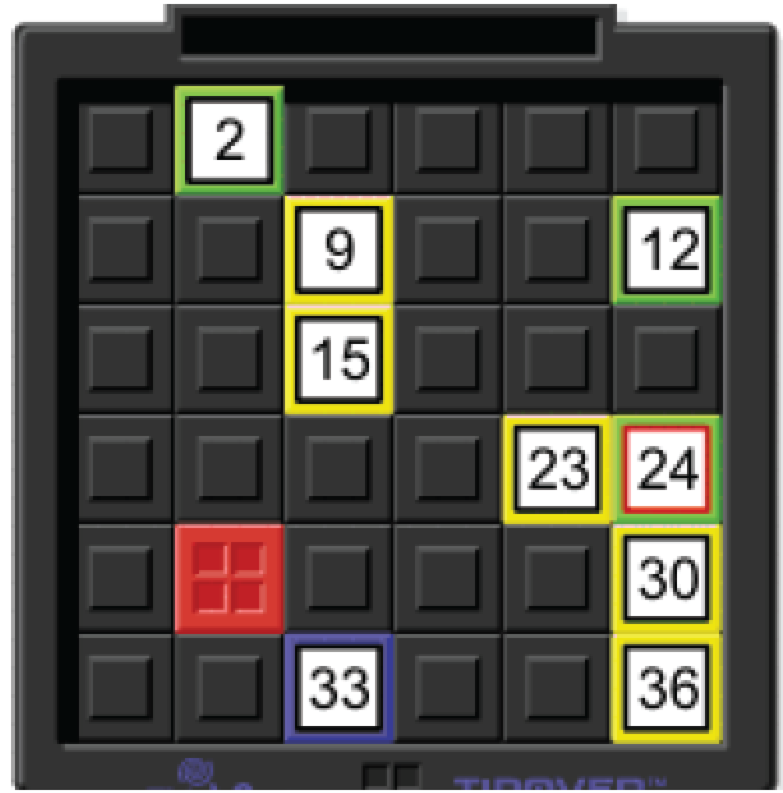
NCL is PSPACE-complete,

even for planar graphs,

with restricted vertices

application: TipOver

http://www.puzzles.com/products/tipover/PlayOnLine.htm



24 initial position

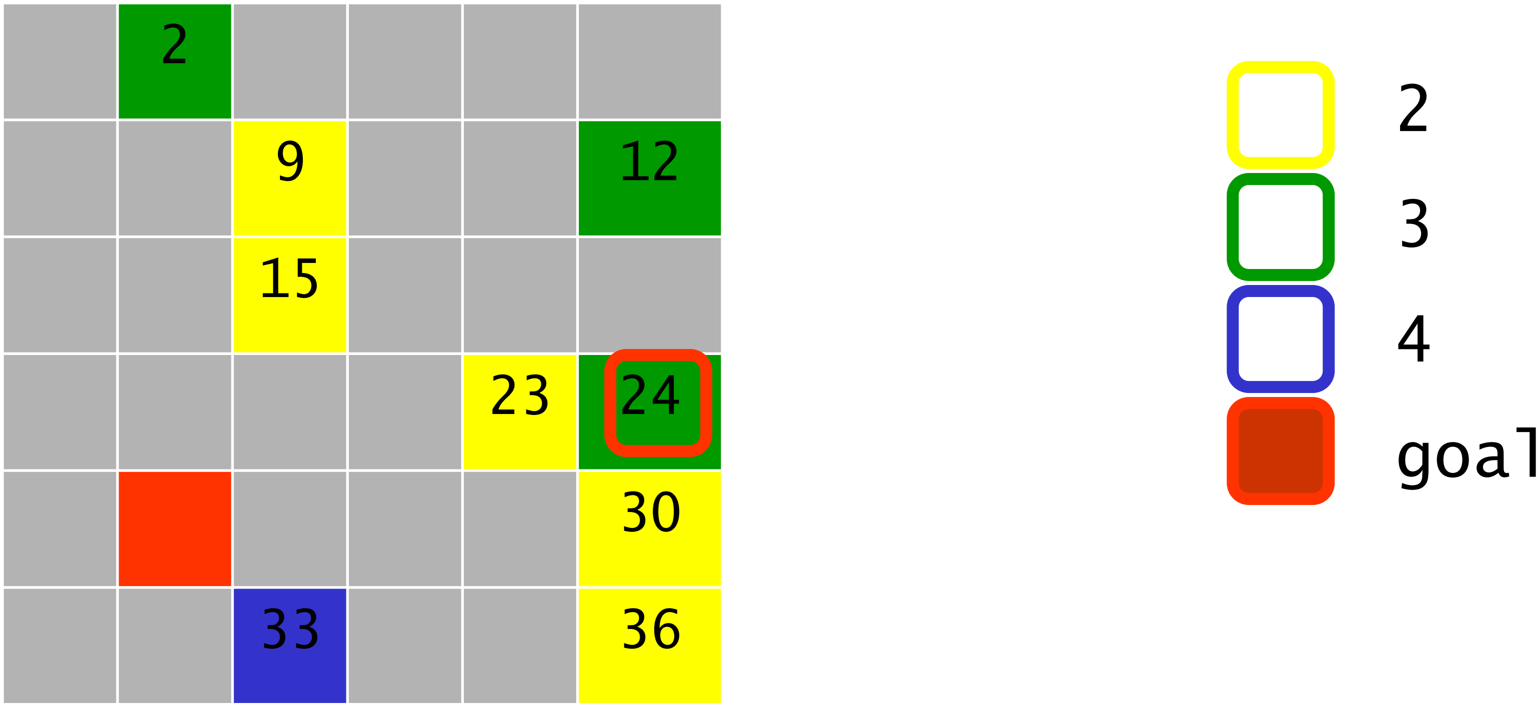
2

3

4

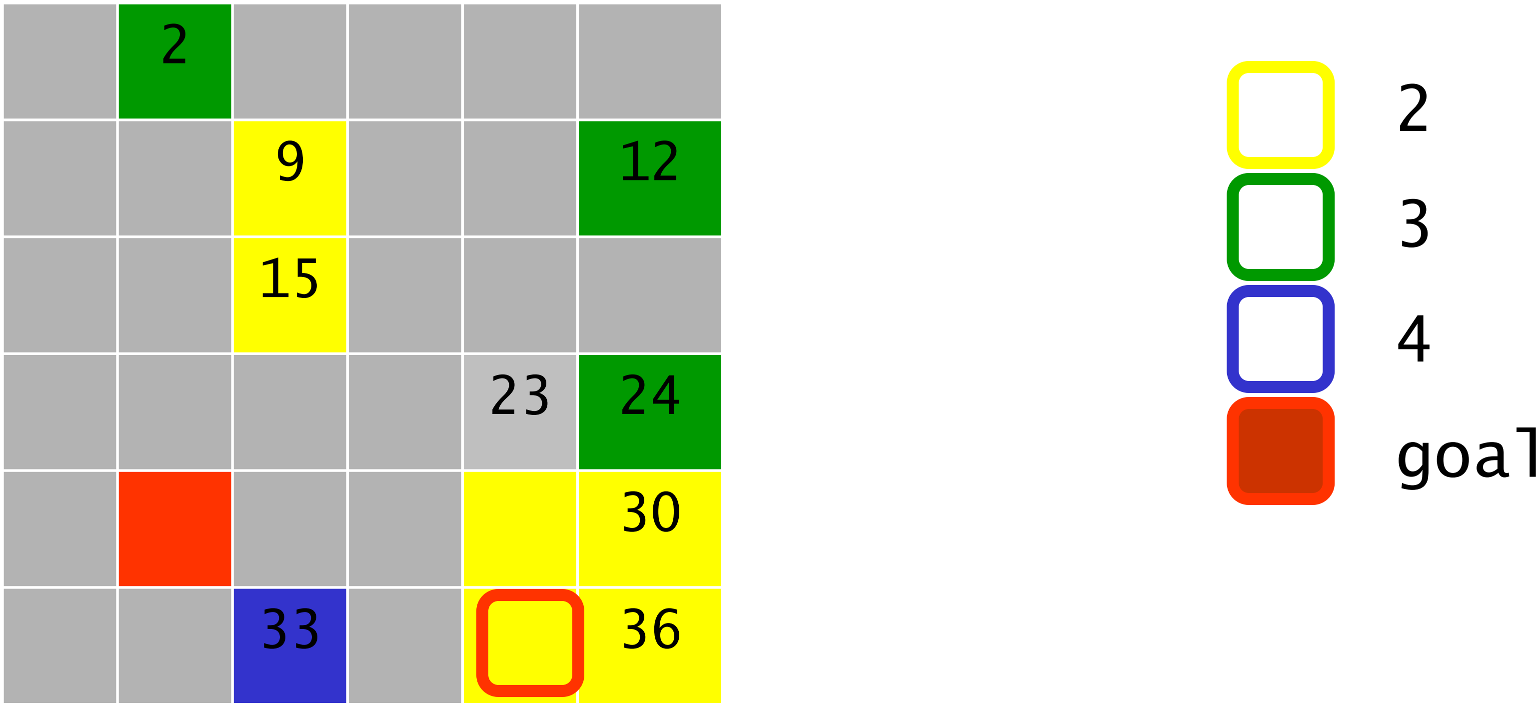
goal

solution advanced



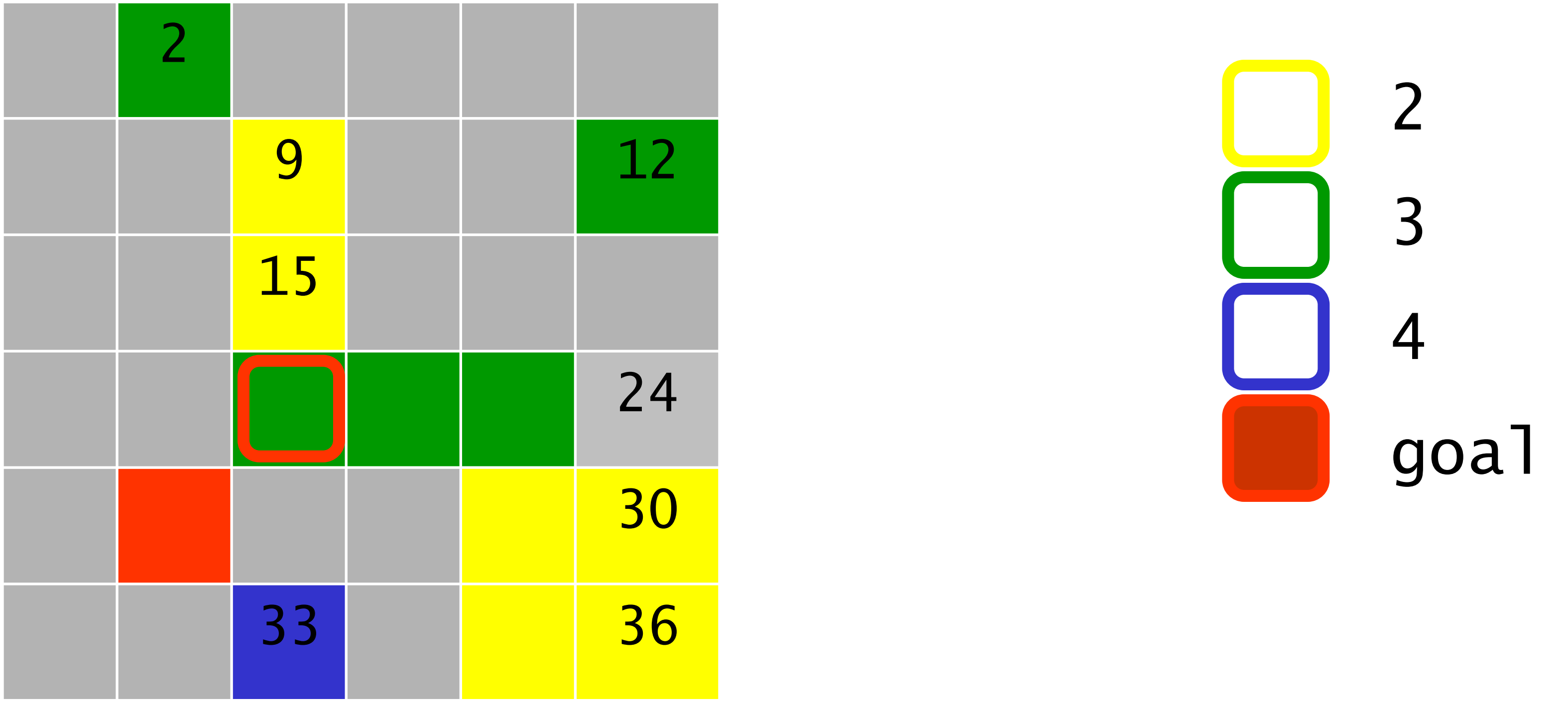
23D, 24L, 30U, 9R, 15U, 2D

solution advanced



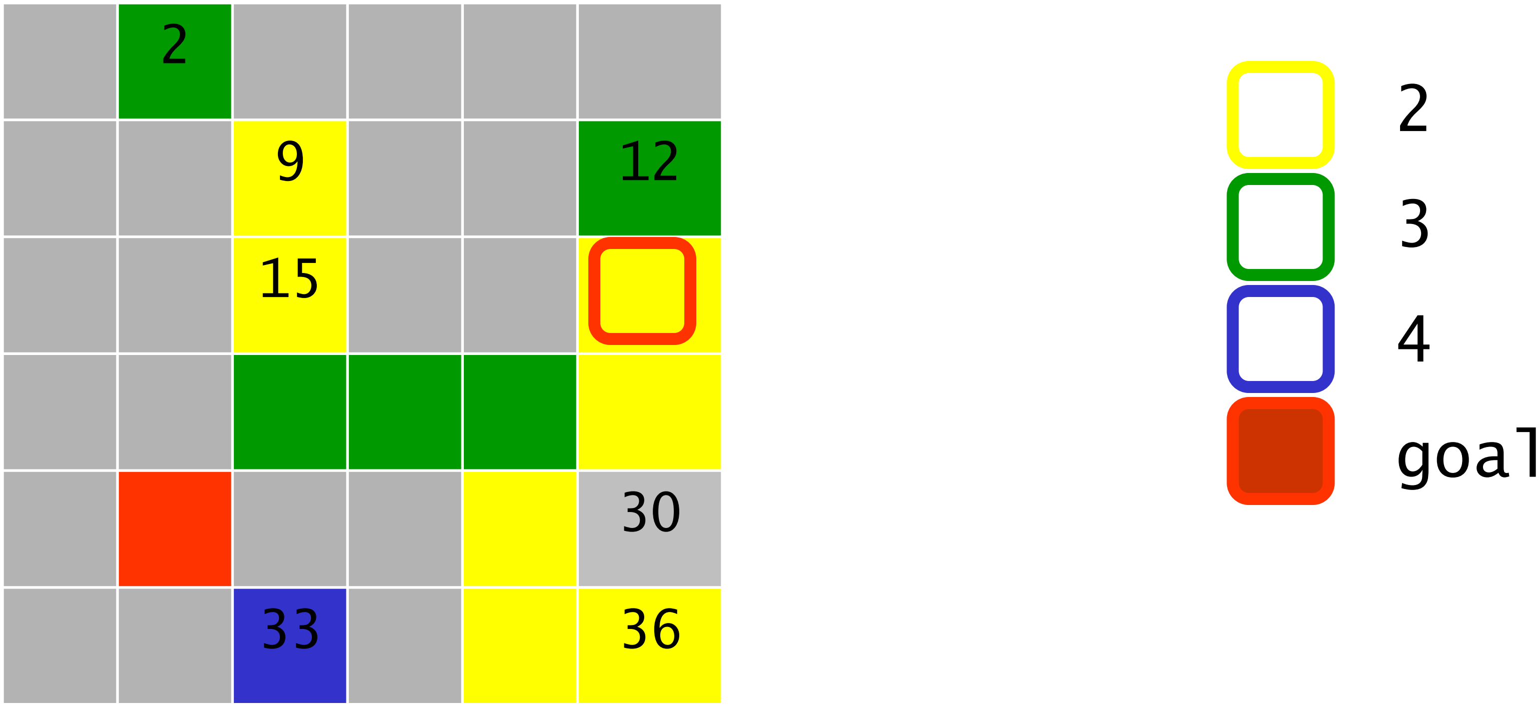
23D, 24L, 30U, 9R, 15U, 2D

solution advanced



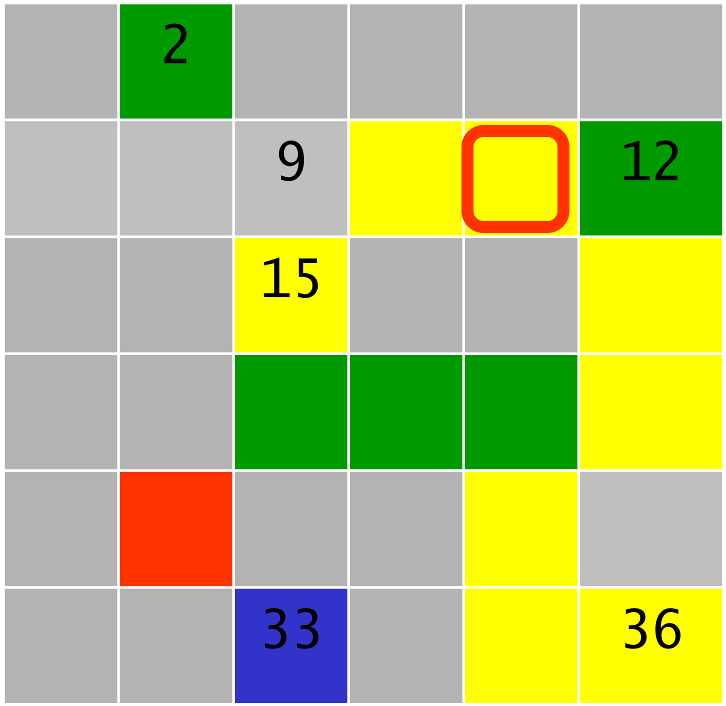
23D, 24L, 30U, 9R, 15U, 2D

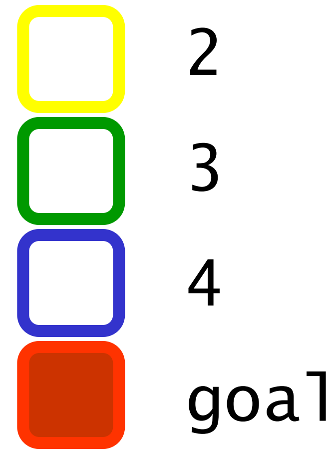
solution advanced



23D, 24L, 30U, 9R, 15U, 2D

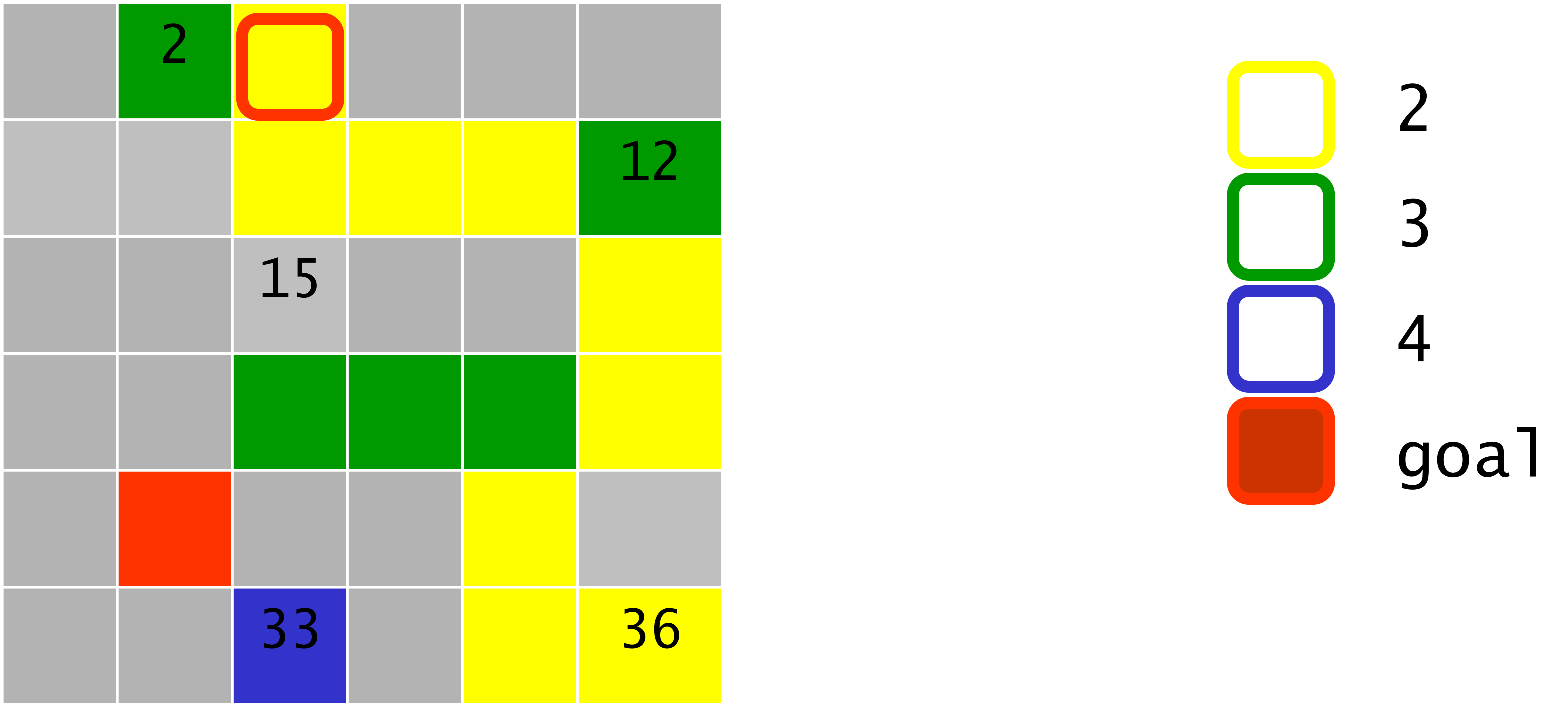
solution advanced





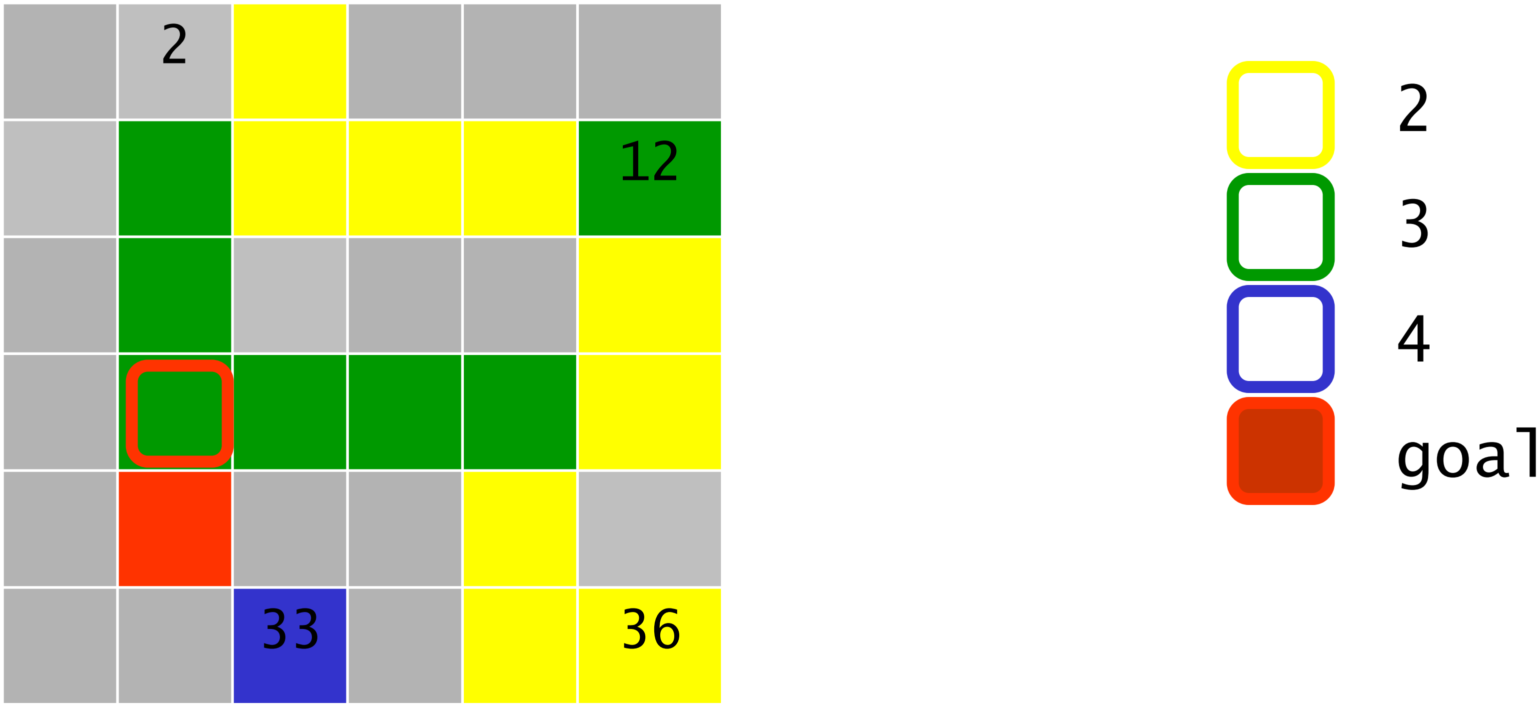
23D, 24L, 30U, 9R, 15U, 2D

solution advanced



23D, 24L, 30U, 9R, 15U, 2D

solution advanced



23D, 24L, 30U, 9R, 15U, 2D

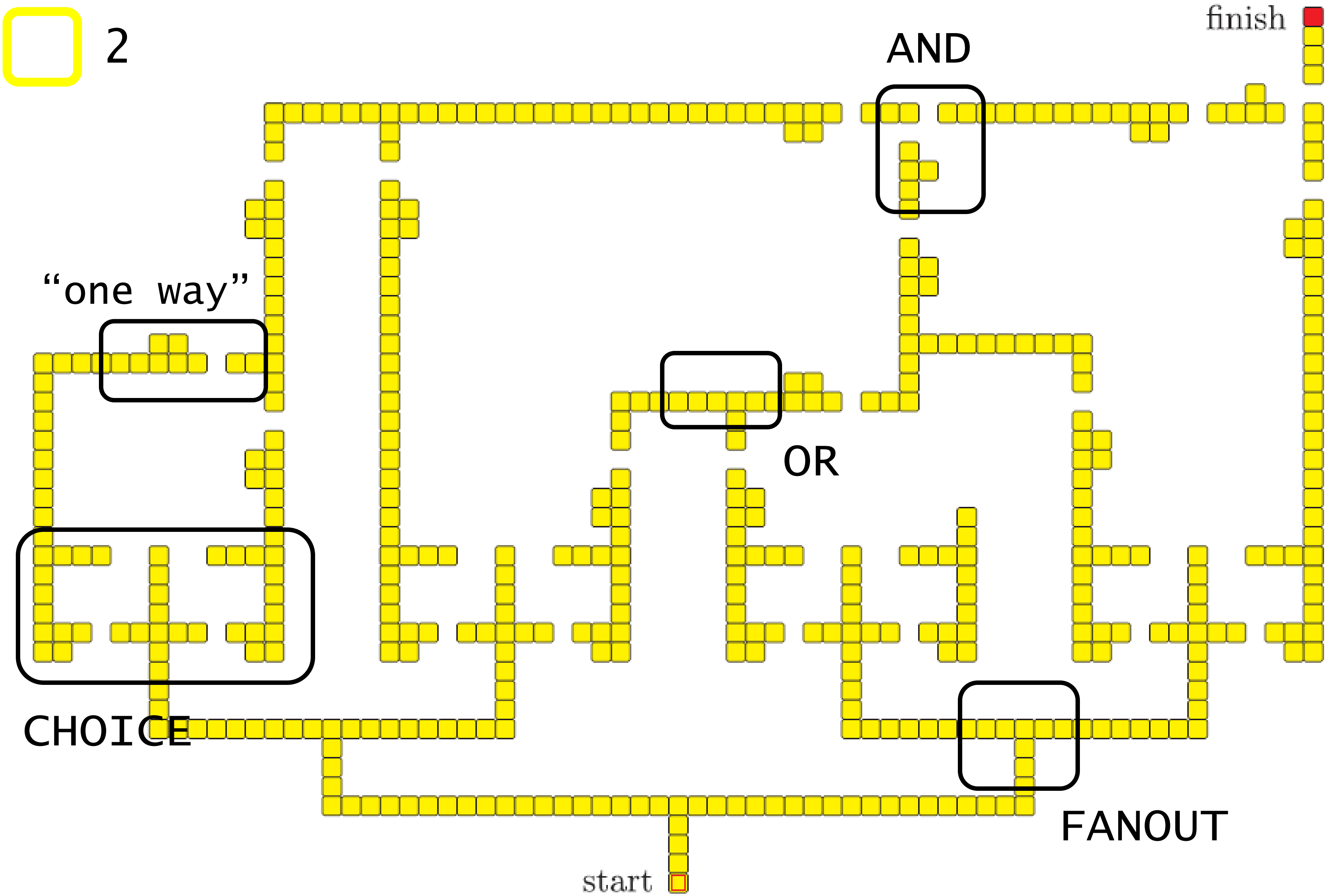


Figure 9-7: TipOver puzzle for a simple constraint graph.

gadgets: “one way”, OR

invariant:

- can be reached ⇔ can be inverted
- all visited positions remain connected

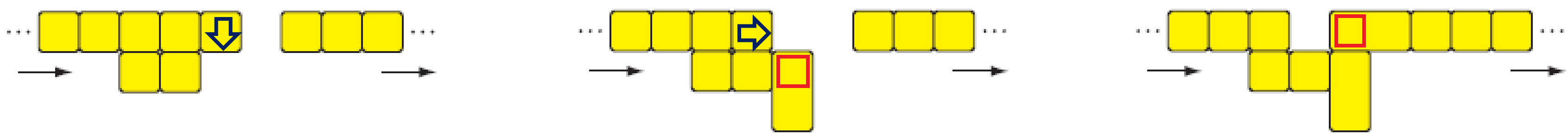
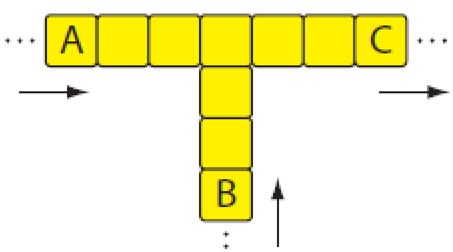
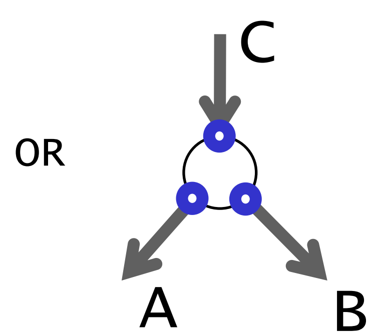


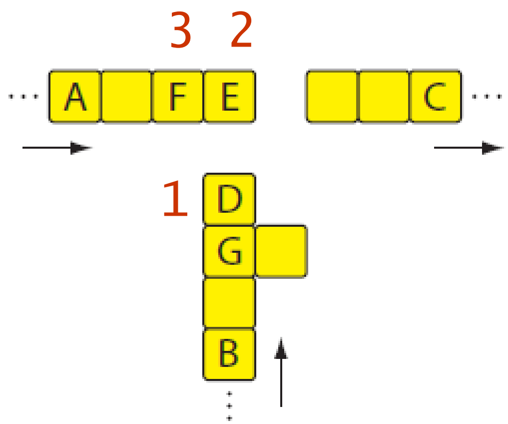
Figure 9-3: A wire that must be initially traversed from left to right. All crates are height two.



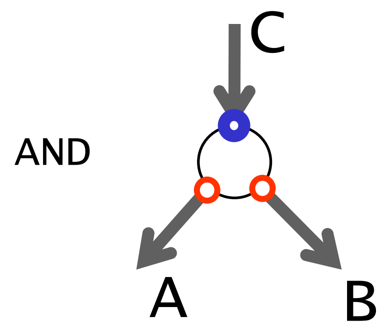
(a) OR gadget. If the tipper can reach either A or B, then it can reach C.



gadgets: AND



(b) AND gadget. If the tipper can reach both A and B, then it can reach C.



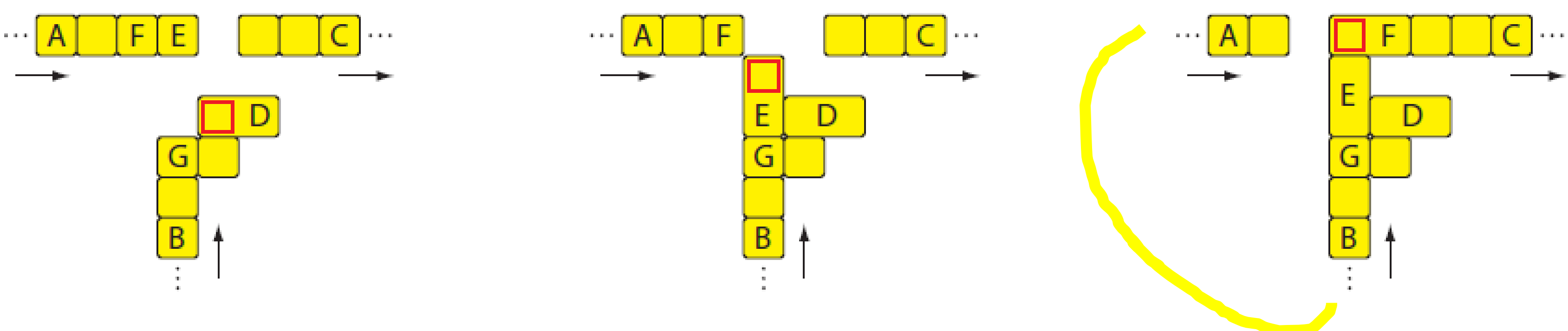


Figure 9-5: How to use the AND gadget.

remains connected

gadgets: CHOICE, FANOUT

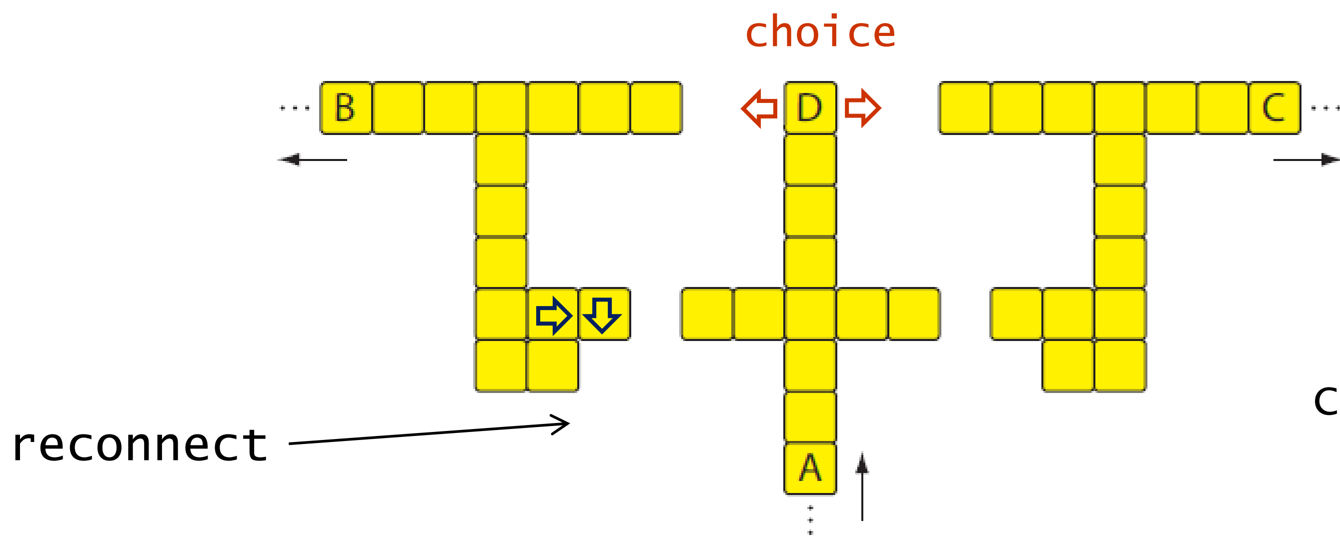
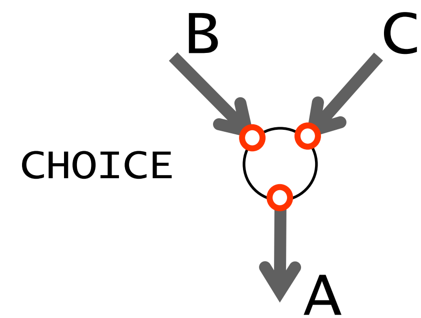
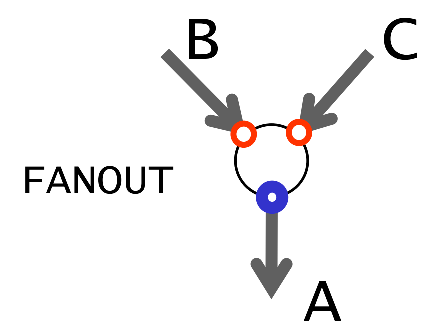


Figure 9-6: TipOver CHOICE gadget. If the tipper can reach A, then it can reach B or C, but not both.

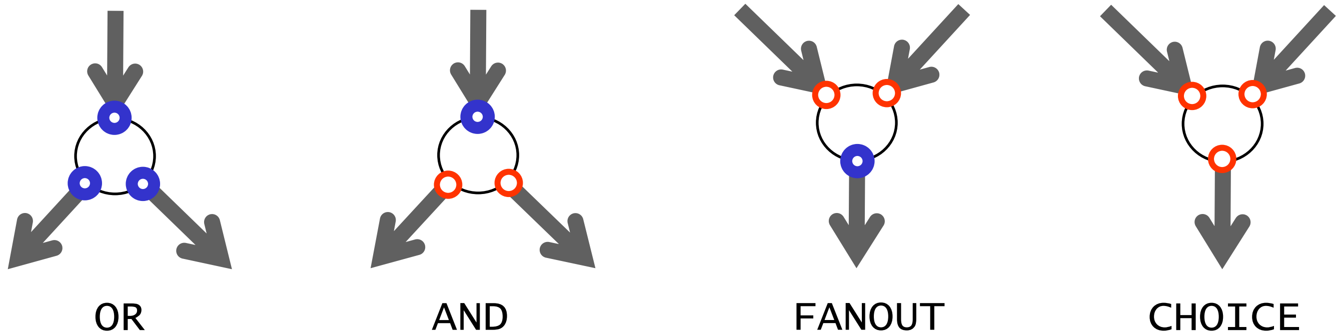




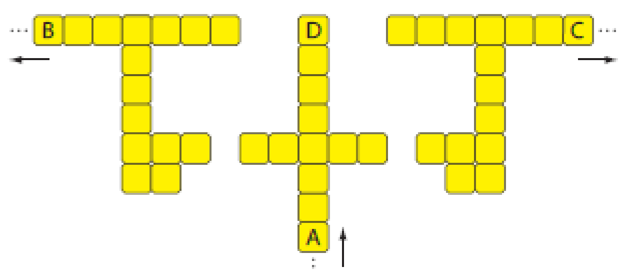
use one-way gadgets at B and C

(control information flow)

conclusion



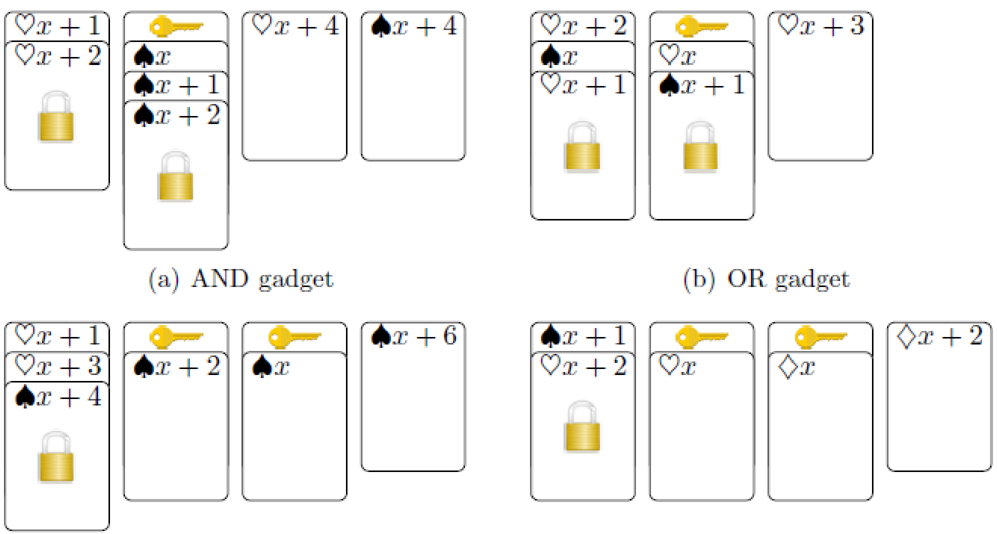
Bounded NCL is NP-complete,
even for planar graphs,
with above restricted vertices



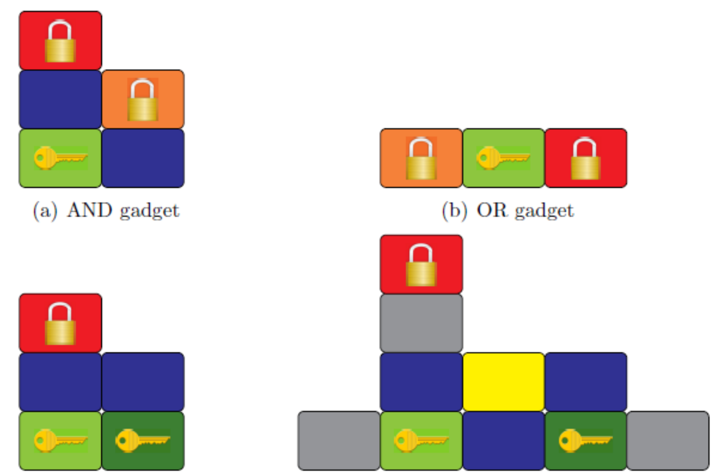
thm. TipOver is NP-complete

NP complete bounded games

Jan van Rijn: Playing Games:
The complexity of Klondike, Mahjong, Nonograms
and Animal Chess
(Master Thesis, 2013, Leiden)



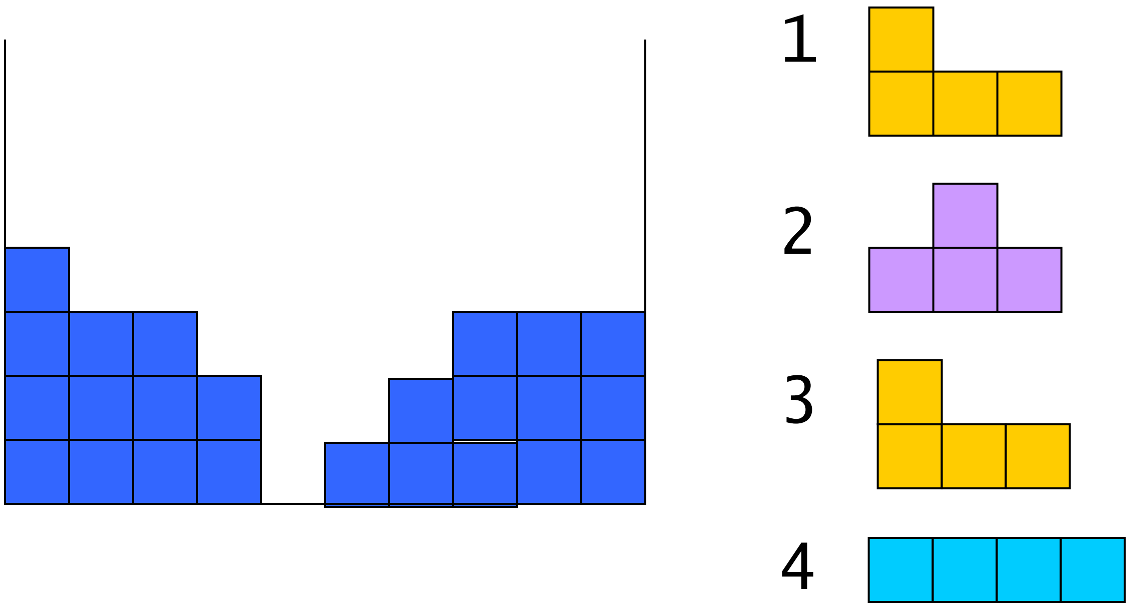
(a) AND gadget (b) OR gadget (c) FANOUT gadget (d) CHOICE gadget



(a) AND gadget (b) OR gadget (c) FANOUT gadget (d) CHOICE gadget

Tetris is NP complete

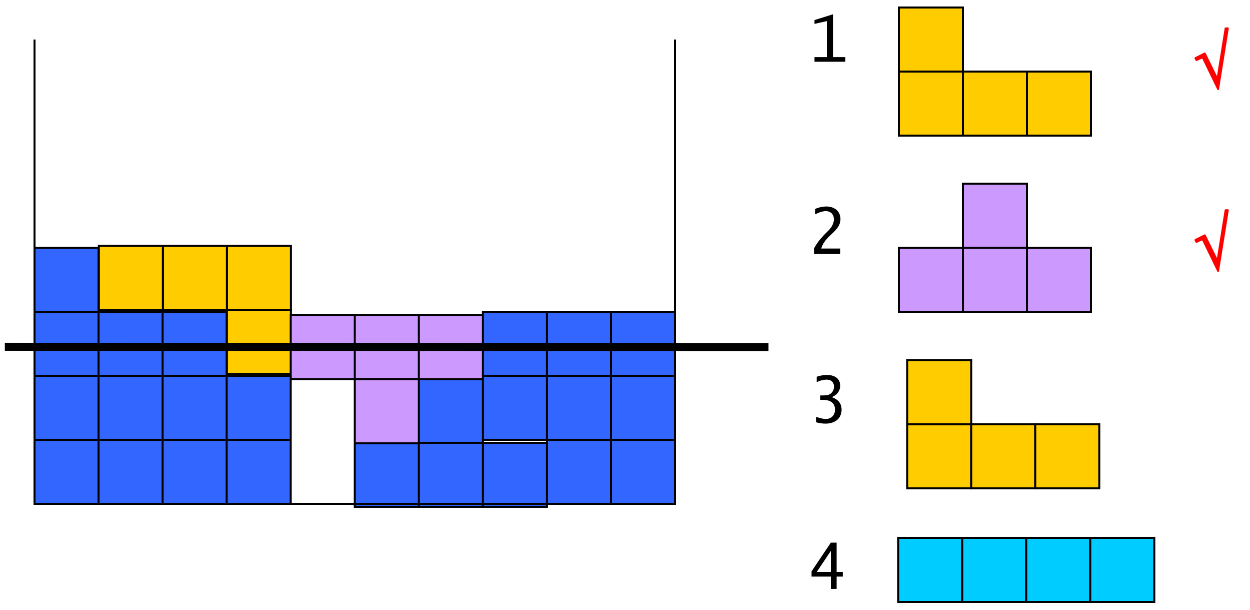
"Given an initial game board and a sequence of pieces, can the board be cleared?"



Breukelaar, Demaine, Hohenberger, Hoogeboom, Kosters, Liben-Nowell. Tetris is Hard, Even to Approximate. Selected Papers from the Ninth Int. Computing and Combinatorics Conf. (COCOON 2003). *Int. J. of Computational Geometry and Applications* 14 (2004) 41-68.

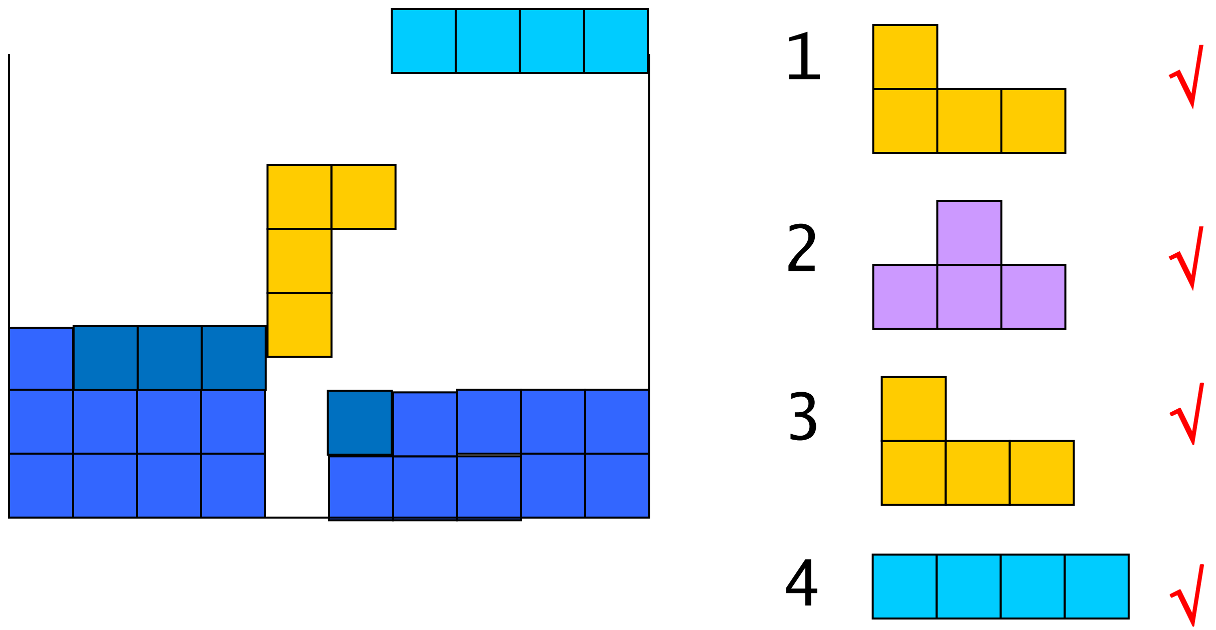
Tetris is NP complete

"Given an initial game board and a sequence of pieces, can the board be cleared?"



Tetris is NP complete

"Given an initial game board and a sequence of pieces, can the board be cleared?"



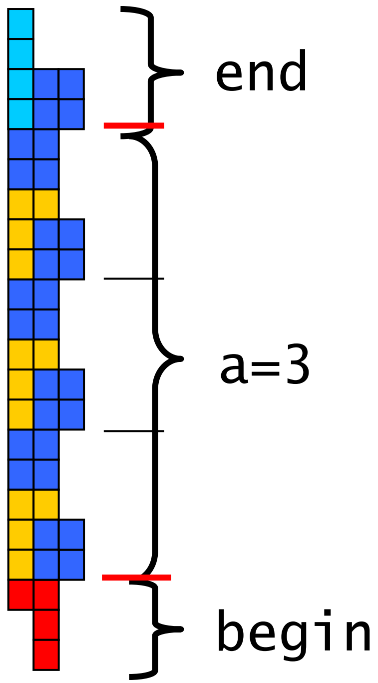
yes!

Tetris is NP complete

reduction from 3-partitioning problem
(can we divide set of numbers into triples?)

OPEN: directly with Bounded NCL ?

find OR, AND, FANOUT, CHOICE



Wrap Up

conclusion: nice uniform family of graph games, suitable for the various game classes

not in this presentation:

deterministic classes are hard to prove complete: timing constraints

bounded det. ncl has no known planar normal form

2pers. games need two types of edges (apart from colours), for each of the players

for teams one needs hidden info, otherwise equivalent to 2p games

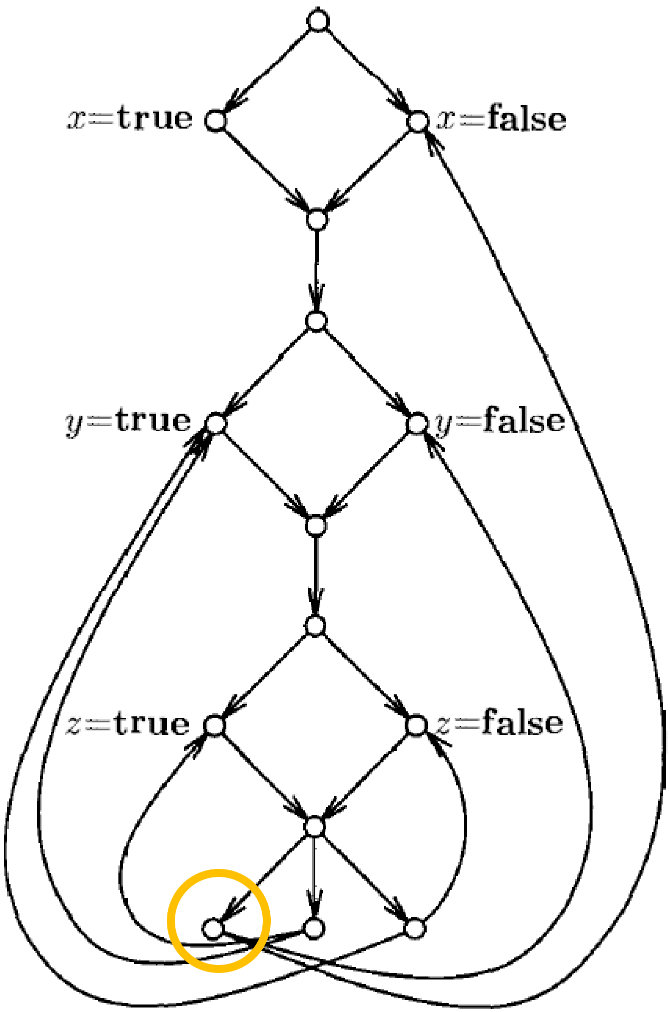
conclusion

roots can be found in the literature
(see Geography)

take care: game of life (what is the ‘goal’?)
is PSPACE, it also is undecidable ☺
(on infinite grid)

example of P-complete:
the domino topling simulation

geography



[Schäfer, J.CSS, 1978]

THM. Geography is PSPACE-complete

two players on directed graph alternately pick next vertex, without repetition

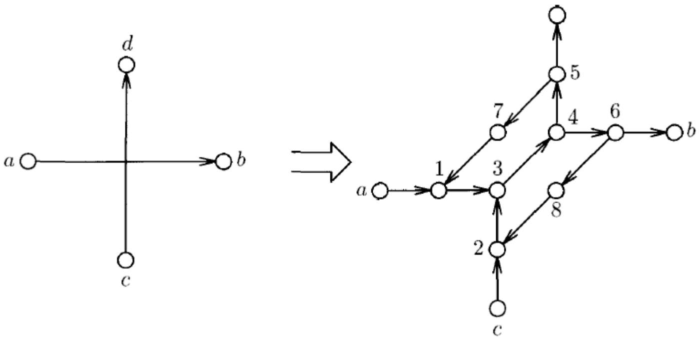
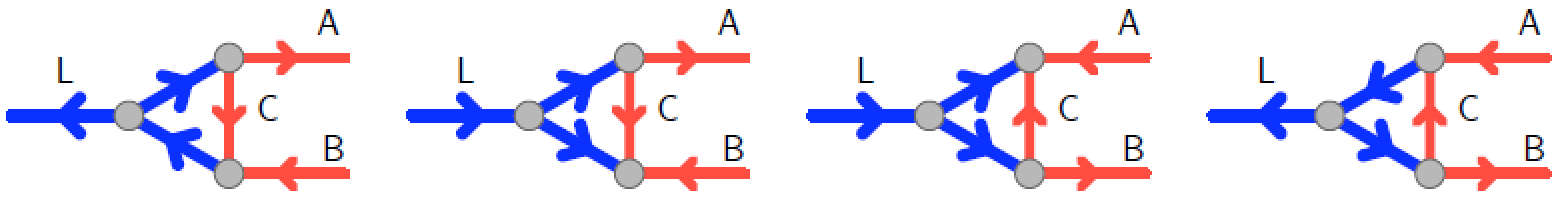


Figure 19-4. Crossing edges in GEOGRAPHY.

application to GO
[Lichtenstein&Sipser, J.ACM, 1980]

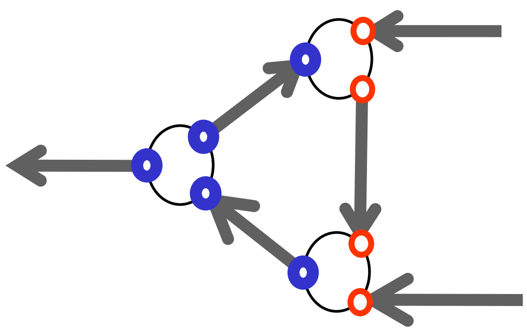
$$\exists x \forall y \exists z [(\neg x \vee \neg y) \wedge (y \vee z) \wedge (y \vee \neg z)].$$

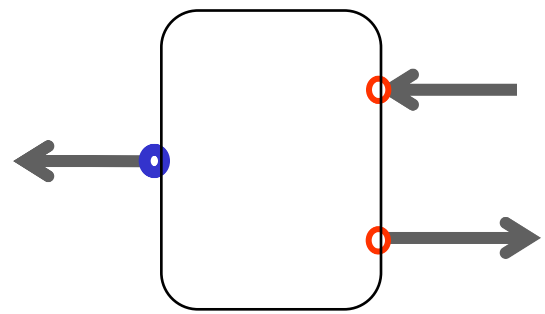
latch / protected OR



(a) Locked, A active (b) Unlocked, A active (c) Unlocked, B active (d) Locked, B active

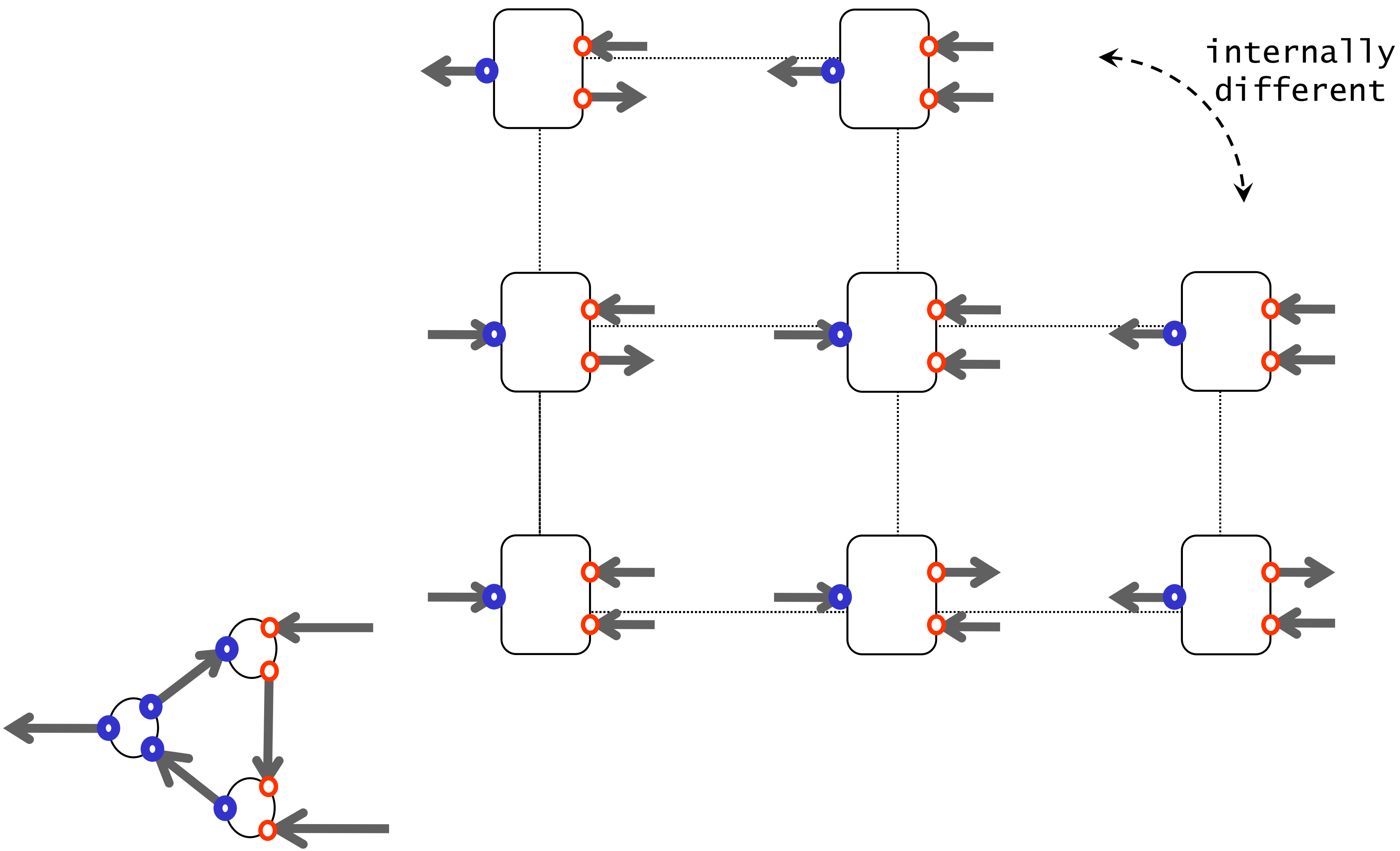
Figure 5-6: Latch gadget, transitioning from state A to state B.





we are not (really) interested in the internal states
only in the external connections:
input-output behaviour
synchronization, silent actions & simulation

latch behaviour



Game Complexity

IPA Advanced Course on Algorithmics and Complexity

Eindhoven, 25 Jan 2019

Walter Kosters
Hendrik Jan Hoogeboom

LIACS, Universiteit Leiden