Unit: Multicriteria Landscape Analysis
Learning Goals

1. Correct definition related to multiobjective optimization: Efficient set, Pareto front, weak efficient set, strict efficient set, strictly non-dominated set, weakly non-dominated set.

2. Shapes of Pareto fronts: Classification convex/concave and invariances

3. Identification of efficient sets based on contour plots and level sets
Pareto optimization: All Definitions

Decision space $\mathcal{S}$, Feasible decision space $\mathcal{X}$

Objective functions $f_1 : \mathcal{S} \rightarrow \mathbb{R}, f_2 : \mathcal{S} \rightarrow \mathbb{R}, \ldots, f_m : \mathcal{S} \rightarrow \mathbb{R}$.

Or as a vector valued function: $f(\mathcal{X}) \rightarrow \mathbb{R}^m$

Image of $\mathcal{X}$ under $f$:
$\mathcal{Y} = f(\mathcal{X}) = \{y \in \mathbb{R}^m \mid \text{exists } x \in \mathcal{X} : f(x) = y\}$

Pareto dominance:
$\forall y^1, y^2 \in \mathbb{R}^m : y^1 \prec y^2 \iff y^1 \leq y^2 \land y^1 \neq y^2$.

We define a preorder in the feasible decision space $\mathcal{X}$:
$\forall x^1, x^2 \in \mathcal{X} : x^1 \preceq x^2 :\iff f(x^1) \leq f(x^2)$
$x^1 \prec x^2 :\iff f(x^1) \prec f(x^2)$

Note, that the antisymmetry gets lost in the feasible search space, and hence $(\mathcal{X}, \preceq)$ is not a partially order but only a preordered set. Why?
Pareto optimization: All Definitions

Efficient point: A point $x \in \mathcal{X}$ is called efficient, iff not exists $x' \in \mathcal{X}$ with $x' \prec x$

Efficient set $\mathcal{X}_E$: Set of all efficient points in $\mathcal{X}$

Nondominated point: A point $y \in \mathcal{Y}$ is called nondominated (or Pareto optimum), iff not exists $y' \in \mathcal{Y}$ with $y' \prec y$

Nondominated set or Pareto front $\mathcal{Y}_N$: The set of all nondominated points in $\mathcal{Y}$ is called the Pareto front or nondominated set.
Weakly efficient and nondominated set

A point $x$ is weakly efficient, if it there is no other point $x'$ in $\mathcal{X}$ with $f_1(x') < f_1(x) \land \ldots \land f_m(x') < f_m(x)$.

A point $x$ is strictly efficient, if it there is no other point $x'$ in $\mathcal{X}$ with $x' \leq x$.

The weakly (strictly) efficient set $\mathcal{X}_{wE}$ ($\mathcal{X}_{sE}$) is the set of all weakly (strictly) efficient points.

A point in $y \in \mathcal{Y}$ is called weakly non-dominated, iff there is no point in $y' \in \mathcal{Y}$ such that $y_1' < y_1 \land \ldots \land y_m' < y_m$.

The weakly non-dominated set $\mathcal{Y}_{wN}$ is the set of all weakly nondominated solutions in $\mathcal{Y}$.

The weakly non-dominated set $\mathcal{Y}_{wN}$ is the image of $\mathcal{X}_{wE}$ under $f$, that is $\mathcal{Y}_{wN} = f(\mathcal{X}_{wE})$.
Weak non-domination vs. non-domination

Consider the set \( \mathcal{Y} = \{ y \in \mathbb{R}^2 | 0 < y_1 < 1, 0 \leq y_2 \leq 1 \} \): The non-dominated set \( \mathcal{Y}_N \) is empty, while \( \mathcal{Y}_{wN} \) is not.

Consider the closed square \( \mathcal{Y} = \{ y \in \mathbb{R}^2 | 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1 \} \)
We have \( \mathcal{Y}_N = \{0\} \) and \( \mathcal{Y}_{wN} = \{ y \in \mathcal{Y} | y_1 = 0 \lor y_2 = 0 \} \)
Convex and concave PF: precise definition

A Pareto front $\mathcal{Y}$ is said to be convex, if $\mathcal{Y} \oplus \mathbb{R}^m_>$ is a convex set.

A Pareto front $\mathcal{Y}$ is said to be concave if $\mathcal{Y} \oplus \mathbb{R}^m_<$ is a convex set.
Different shapes of Pareto fronts

Convex pareto front
\[ f_2 \rightarrow \min \]
\[ \mathcal{V} \]
\[ \mathcal{V}_N \]

Concave pareto front
\[ f_1 \rightarrow \min \]
\[ \mathcal{V} \]
\[ \mathcal{V}_N \]

PF that is neither convex nor concave.
\[ \mathcal{V} \]
\[ \mathcal{V}_N \]

Disconnected Pareto front
\[ \mathcal{V} \]
\[ \mathcal{V}_N \]
Special points

Ideal vector: $y_k^I := y_k := \min_{y \in \mathcal{Y}} y_k$
Maximal point: $\bar{y}_k = \max_{y \in \mathcal{Y}} y_k$
Nadir point: $y_N^k = \max_{y \in \mathcal{Y}_N} y_k$

Computation of ideal point can be reduced to the solution of $m$ single-objective optimization problems

The computation of the Nadir point is a very difficult problem and no efficient method for computing $y_N^N$ is known for $m > 2$, yet.
3-D Attainment surface, dominated space

3D Attainment surface: Useful for visualizing finite non-dominated sets in 3-D
'Steps' into direction north to east.
3-D Attainment Surface, Continuous

The slope of the attainment surface is always in the direction north-northeast-east
Pareto front in three dimensions

Visualization of finite PF with 5 points.

Here maximization is considered: Dominance cones are the negative orthants

3-D continuous Pareto fronts and approximations to them with 70 points.
Optima seeking using contour plots

Contour plots help to localize optimizers of single-objective problems.

Often, they provide an intuition for reasoning about optima for higher dimensional functions.

A level set is informally defined as a set of arguments (variable settings) for which the function obtains the same value.

A contour is a connected part of a level set of a 2-dimensional function.

$$\sin(4 \times xy) + |x| + |y|$$
Finding efficient set using level sets (contours):
Single objective optimization, linear case

\[ z = 4x + 3y \]

Infeasible subspace
Feasible space
Optimal Solution

Level curve of \( f \) = contour line of equal height (\( f = \text{const} \))

Draw constraint boundaries \( g_i(x) = 0 \) and contours for \( f(c) = C' \) for different constants \( C' \).
Finding efficient points using contour plots

Contour plots can sometimes be used to find efficient points in bi-objective optimization graphically.

Is $p_1$ an efficient point? What about $p_2$?
Level sets and curves

Level sets can be used to visualize $\mathcal{X}_E$, $\mathcal{X}_{wE}$ and $\mathcal{X}_{sE}$ for continuous spaces:

\[
\mathcal{L}_\leq(f(\hat{x})) = \{x \in \mathcal{X} : f(x) \leq f(\hat{x})\} : \text{Level set}
\]

\[
\mathcal{L}_\equiv(f(\hat{x})) = \{x \in \mathcal{X} : f(x) = f(\hat{x})\} : \text{Level curve}
\]

\[
\mathcal{L}_<(f(\hat{x})) = \{x \in \mathcal{X} : f(x) < f(\hat{x})\} : \text{Strict level set}
\]

Draw the level set $\mathcal{L}_\leq(f(x_0))$ for $f(x) = |1 - x|^2 = (x_1 - 1)^2 + (x_2 - 1)^2$ and $x_0 = (1, 0)$ in the $x_1, x_2$ plane!
Finding Efficient Points by Level Sets: Example 1

Level sets can be used to determine whether $\hat{x} \in X$ is (strictly, weakly) non-dominated or not.

The point $\hat{x}$ cannot be non-dominated! Why?
Answer: Dominating solutions are in the area where the two strict level sets intersect.
Finding Efficient Points by Level Sets: Example 2

Is \( \hat{x} \) efficient?

Answer: It is not possible to improve \( f_1 \) and \( f_2 \) at the same time relative to their values in \( \hat{x} \). Therefore, \( \hat{x} \) is efficient.
The point $\hat{x}$ can only be efficient if its level sets intersect in level curves.

$$x \text{ is efficient} \iff \bigcap_{k=1}^{m} \mathcal{L}_{\leq}(f_k(x)) = \bigcap_{k=1}^{m} \mathcal{L}_{=}(f_k(x))$$

The point $\hat{x}$ can only be weakly efficient if its strict level sets do not intersect.

$$x \text{ is weakly efficient} \iff \bigcap_{k=1}^{m} \mathcal{L}_{<}(f_k(x)) = \emptyset$$

The point $\hat{x}$ can only be strictly efficient if its level sets intersect in exactly one point.

$$x \text{ is strictly efficient} \iff \bigcap_{k=1}^{m} \mathcal{L}_{\leq}(f_k(x)) = \{x\}$$
Proof: Theorem on efficient points

The point \( \hat{x} \) can only be efficient if its level sets intersect in level curves.

\[
\hat{x} \text{ is efficient } \iff \bigcap_{k=1}^{m} L_{\leq}(f_k(\hat{x})) = \bigcap_{k=1}^{m} L_{=} (f_k(\hat{x}))
\]

Proof:
\( \hat{x} \) is efficient

\( \iff \) there is no \( x \) such that both \( f_k(x) \leq f_k(\hat{x}) \) for all \( k = 1, \ldots, m \) and \( f_k(x) < f(\hat{x}) \) for at least one \( k = 1, \ldots, m \)

\( \iff \) there is no \( x \in \mathcal{X} \) such that both \( x \in \bigcap_{k=1}^{m} L_{\leq}(f(\hat{x})) \)
and \( x \in L_{<}(f_j(\hat{x})) \) for some \( j \)

\( \iff \bigcap_{k=1}^{m} L_{\leq}(f_k(\hat{x})) = \bigcap_{k=1}^{m} L_{=} (f_k(\hat{x})) \)
Finding the efficient set in IR²: Example

\[ f_1(x_1, x_2) = 2 + \frac{1}{3}x_2 - x_1 \rightarrow \min \]
\[ f_2(x_1, x_2) = \frac{1}{2}x_2 + \frac{1}{2}x_1 \rightarrow \max \]
\[ 2 - \frac{2}{3}x_1 - x_2 \leq 0 \]
\[ x_1 \in [0, 3], x_2 \in [0, 2] \]

Indicate region that is dominated by \( \hat{p}_1 \).
Finding the efficient set in IR^2: Example

\[ f_1(x) = \sqrt{\sum_{i=1}^{2} (x_i - c_i^1)^2} \rightarrow \min \]
\[ f_2(x) = \sqrt{\sum_{i=1}^{2} (x_i - c_i^2)^2} \rightarrow \min \]

\[ c^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad c^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

What is the Pareto front?
What about three dimensional decision spaces?
Multi-sphere Testproblem with local Pareto fronts

Hiking in multicriteria landscapes …

Orange mountains: \( f_1(x_1, x_2) \rightarrow \text{max} \)

Blue mountains: \( f_2(x_1, x_2) \rightarrow \text{max} \)
Take home messages

Important definitions in Pareto optimization are the (weakly, strictly) efficient set, Pareto front, ideal/nadir point, (feasible) decision/objective space

Pareto fronts can be convex or concave, connected or disconnected

Theorems on level sets can be used to identify (globally) efficient points analytically; they are useful for reasoning about the location of the efficient set;

Often optima occur at the constraint boundary; In particular, for linear problems this is the case. In 2-D contour plots can be used to identify efficient solutions at the boundary.