$R \subseteq A \times B$

$S \subseteq B \times C$

$T \subseteq C \times D$

CLAIM: $\underbrace{(R \circ S) \circ T}_{P} = \underbrace{R \circ (S \circ T)}_{Q}$

(1) NOTE "=" IS EQUALITY OF SETS;

NEED:

$x \in P \Rightarrow x \in Q$

$x \in Q \Rightarrow x \in P$.

$x \in P \Rightarrow x = (a, b)$ $\quad (a, b) \in R$

$(a \in A, b \in B)$

$R \subseteq A \times B$

$S \subseteq B \times C$

$T \subseteq C \times D$

RECALL $\underline{R' \circ S'} \subseteq A' \times C'$

$(a, b) \in R' \circ S' \iff$

there exists "a middle" y,

such that $(a, y) \in R'$ & (y, b)

$\exists y \in B$ S.T. $(a, y) \in R'$ AND $(y, b) \in S'$.

"PRIMES" (R', S') TO AVOID CONFUSION, THIS HOLDS ALWAYS

WE HAVE TO SET UP OUR SET OF "PREVIOUS KNOWLEDGE" STATEMENTS

$R \subseteq A \times B$, $S \subseteq B \times C$, $T \subseteq C \times D$; ONE DIRECTION

$$\underbrace{(a,d) \in (R \circ S) \circ T}_{(c_1)} \Rightarrow \underset{(c_2)}{(a,d) \in R \circ (S \circ T)}$$

$(c_1) \Rightarrow \exists\, y \in C$ ST

$$\underbrace{(a,y) \in R \circ S}_{(c_{11})} \quad \text{AND} \quad \underbrace{(y,d) \in T}_{(c_{12})}$$

$(c_{11}) \Rightarrow \exists\, z \in B$ ST $\underbrace{(a,z) \in R}_{(c_{111})}$ & $\underset{(c_{112})}{(z,y) \in S}$

FROM $(C_{111}), (C_{112}), (C_{12})$ WANT TO INFER (CONCLUDE) (C_2)

RECALL

(C_2) $\quad (a,c) \in R \circ (S \circ T)$

$$\exists \tilde{z} \in B, \ \overbrace{(a,\tilde{z}) \in R \ \& \ (\tilde{z},d) \in (S \circ T)}^{D_1'} \qquad (D_1)$$

$$\exists \tilde{y} \in C, \ \underbrace{(\tilde{z},\tilde{y}) \in S \ \& \ (\tilde{y},d) \in T}_{D_2'} \qquad (D_2)$$

$(D1) \equiv \exists \tilde{z} \in B$ s.t. $D'1$

$(D2) \equiv \exists \tilde{y} \in C$ s.t. $D'2$

$\boxed{\text{CLAIM}}$. IF $(C_{111}), (C_{112}), (C_{12})$ HOLD THEN $\tilde{y} := y$ AND $\tilde{z} := z$ SATISFY D_1' & D_2' SO (D_1) & (D_2) ARE TRUE, AND HENCE SO IS $(C2)$

USING

$$\left[\begin{array}{l} y \in C \quad \text{AND} \quad \overbrace{(a,y) \in R \circ S}^{(S_3)} \quad \text{and} \quad \overbrace{(y,d) \in T}^{(S4)}. \\ z \in B \quad \text{AND} \quad \underbrace{(a,z) \in R}_{(S_1)} \quad \text{and} \quad \underbrace{(z,y) \in S}_{(S2)} \end{array}\right.$$

PROVE $(D'1)$:

$$\underbrace{(a,z) \in R}_{\equiv (S_1) \Rightarrow \text{TRUE}} \quad \& \quad \underbrace{(z,d) \in S \circ T}_{\text{REMAINS}} \qquad (D'1) \quad \Big]$$

$(z,d) \in S \circ T \iff \exists g \; (z,g) \in S \;\&\; (g,d) \in T$

choose $g := y$, SO THEN BY $(S2)$ & $(S4)$

$(z,y) \in S \;\&\; (y,d) \in T$ IS TRUE.

SO $D'1$ IS TRUE FOR THAT z, SO $D1$ IS ALSO TRUE

NEXT WE USE

$$\left[\begin{array}{llll} y \in C & \text{AND} & \overbrace{(a,y) \in R \circ S}^{(S_3)} & \text{and} \quad \overbrace{(y,d) \in T}^{(S_4)} . \\ z \in B & \text{AND} & \underbrace{(a,z) \in R}_{(S_1)} & \text{and} \quad \underbrace{(z,y) \in S}_{(S_2)} . \end{array}\right]$$

TO PROVE $(D'2)$: $(z,y) \in S$ & $(y,d) \in T$

BUT THESE ARE EXACTLY $(S2)$ & $(S4)$ SO $D'2$ IS TRUE, WITH THAT CHOICE OF y.

BUT THEN $D2$ IS TRUE AS SUCH A y EXISTS.

PROVEN: $(a,d) \in (R \circ S) \circ T \Rightarrow (a,d) \in R \circ (S \circ T)$

REMAINS: $\underbrace{(a,d) \in R \circ (S \circ T)} \Rightarrow \underbrace{(a,d) \in (R \circ S) \circ T}$

$\exists y \in B,\ \overbrace{(a,y) \in R}^{(C_1')}\ \&\ (y,d) \in S \circ T,$

$\exists z \in C\quad \underbrace{(y,z) \in S}_{(C_2')}\ \&\ \underbrace{(z,d) \in T}_{(C_3')}$

$C_1', C_2', C_3' \Rightarrow$

$(D_1')\ \exists\ \tilde{z}\quad \overbrace{(a,\tilde{z}) \in R \circ S}^{D_{11}}\ \&$

$(\tilde{z},d) \in T \} D_{12}$

$(D_2)\ \exists\ \tilde{y}\quad \underbrace{(a,\tilde{y}) \in R}_{D_{21}}\ \&\ \underbrace{(\tilde{y},\tilde{z}) \in S}_{D_{22}}$

TAKE $\tilde{y} = y$ & $\tilde{z} = z$

[ASSUME C_1', C_2', C_3')]

$C_1' \Rightarrow \underline{D_{21}},\ C_2' \Rightarrow \underline{D_{22}},\ C_3' \Rightarrow \underline{D_{12}}$

D_{11} remains

LASTLY: $\underline{D_{11}} \Leftrightarrow \underline{(a,z) \in R \circ S}$

$\Rightarrow \exists y \; \underbrace{(a,y)}_{\text{NOTE} = C_1'} \in R \;\&\; \underbrace{(y,z)}_{\text{NOTE } C_2'} \in S$

So $C_1', C_2' \Rightarrow D_{11}$.

So $C_1' \ldots C_3' \Rightarrow$ "ALL" D's.

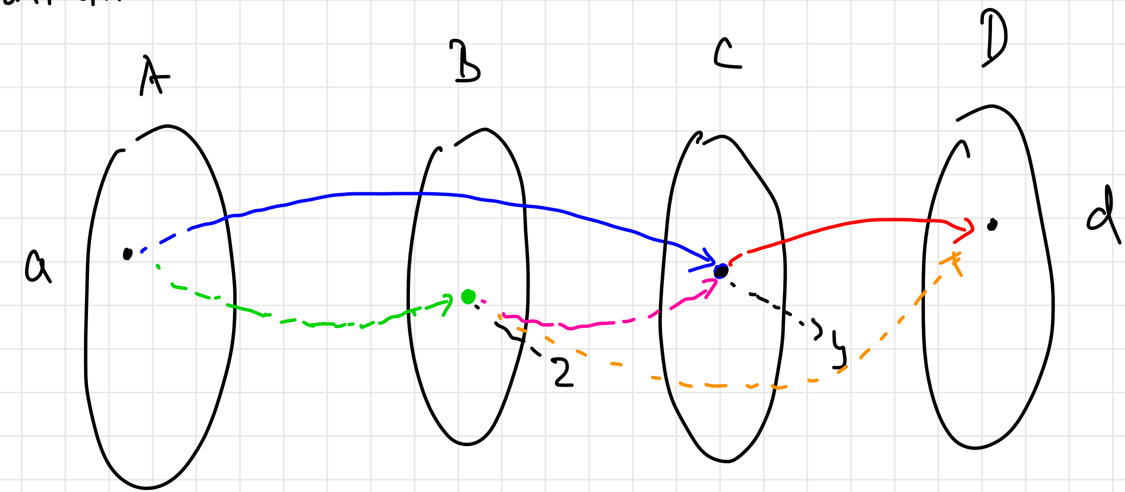
$$\boxed{R \circ (S \circ T) = (R \circ S) \circ T.}$$

NEXT: IN PICTURE.

PROOF IN PICTURE: $R \subseteq A\times B$, $S \subseteq B\times C$, $T \subseteq C\times D$ $\Rightarrow$ $(R\circ S)\circ T = R\circ(S\circ T)$

ASSOCIATIVITY OF COMPOSITION

ONE DIRECTION: $(a,d)\in(R\circ S)\circ T \Rightarrow (a,d)\in R\circ(S\circ T)$.



$(a,d)\in(R\circ S)\circ T \Rightarrow \exists y \quad (a,y)\in R\circ S \;\&\; (y,d)\in T$

$(a,y)\in R\circ S \Rightarrow \exists z \quad (a,z)\in R, (z,y)\in S$

$\Rightarrow$ y is st. $(z,y)\in S$ & $(y,d)\in T \Rightarrow (z,d)\in(S\circ T)$

z is st $(a,z)\in R$ & $(z,d)\in S\circ T \Rightarrow (a,d)\in R\circ(S\circ T)$