Exercises for Foundations of Computer Science

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Chapter 1

Sets

Exercise 1.
Use Venn diagrams to simplify $(A \cup B) - (B - A)$.

Exercise 2.
Use Venn diagrams to show that

a) $(A \cap B) \cup (A \cap B^c) = A$

b) $(A - B) \cap (B - A) = \emptyset$

Exercise 3.
Use Venn diagrams to show that

a) $B \cap [(A \cap B) \cup (A \cap B^c)] = A \cap B$

b) $(B^c \cap A) \cap A = (A^c \cup B)^c$

Exercise 4.
A hospital has admitted 50 patients, 25 with pneumonia, 30 with bronchitis, 10 with both.

a) How many patients have pneumonia, bronchitis or both?

b) How many do not have either?

Exercise 5.
A software company has 100 programmers. The following table lists the proficiencies:

Proficient in	# of employees
Java	45
C#	30
Python	20
C# and Java	6
Java & Python	1
C# & Python	5
C# & Python & Java	1

How many programmers are not proficient in any of the three langauges Java, Python, C#?

Exercise 6.
Schaum 1.41:
A survey on a sample of 25 new cars being sold at a local auto dealer was conducted to see which of three popular options, air conditioning (A), radio (R), and power windows (W), were already installed. The survey found: 15 had air-conditioning, 12 had radio, 11 had power windows, 5 had air-conditioning and power windows, 9 had air-conditioning and radio, 4 had radio and power windows, 3 had all three options. Find the number of cars that had

a) only W;

b) only A;

c) only R;

d) R and W, but not A;

e) A and R, but not W;

f) only one of the options;

g) at least one option;

h) none of the options.

Exercise 7.
Consider the following languages over binary strings:

$$\begin{aligned} K &= \{x \mid (x)_2 \text{ is prime}\} \\ L &= \{x \mid x \text{ does not contain two consecutive ones}\} \end{aligned}$$

$(x)_2$ signifies that the bitstring x is to be understood as an integer represented in binary. 1 is not prime. By convention binary represented numbers do not have leading zeros (i.e., 01 is not a valid binary number, but 1 is).

a) Determine the first 5 elements of the two languages, ordered in length, and alphabetically when equal length.

b) Draw a Venn diagram of the languages and identify one string for each of the three Venn diagram areas.

Exercise 8.
Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by proving two inclusions.

Exercise 9.
Prove that $(A \cup B)^c = A^c \cap B^c$ by proving two inclusions.

Exercise 10.
Prove that the following are equivalent:

a) $A \subseteq B$,

b) $A \cap B = A$, and

c) $A \cup B = B$.

Exercise 11.
Recall that with $\mathcal{P}(V)$ we denote the power set (set of all subsets) of the set V. Let $V = \{\{\emptyset\}, x, \{y\}\}$. Which of the following are true?

a) $\emptyset \in \mathcal{P}(V)$

b) $\{\emptyset\} \in \mathcal{P}(V)$

c) $\{\{\emptyset, x\}\} \in \mathcal{P}(V)$

d) $\{\{\emptyset\}, y\} \subseteq \mathcal{P}(V)$

e) $\emptyset \subseteq \mathcal{P}(\mathcal{P}(\mathcal{P}(V)))$

f) $x \in \{V\}$

Exercise 12.
An ordered pair can be defined using sets via $(a,b)_K := \{a, \{a,b\}\}$. However, the first set theoretical definition was given by Wiener: $(a,b)_W := \{\{\{a\}, \emptyset\}, \{\{b\}\}\}$

a) Prove that $(a,b)_K = (c,d)_K$ if and only if $a = c$ and $b = d$.

b) Prove that $(a,b)_W = (c,d)_W$ if and only if $a = c$ and $b = d$.

Exercise 13.

a) List the elements of $\mathcal{P}(\{\emptyset, \mathcal{P}(\emptyset)\})$.

b) Determine $\mathcal{P}((a,b)_K)$. (see Exercise 12).

Exercise 14.
Schaum 1.59:
Prove the following properties of the symmetric difference of sets $\oplus$:

a) associativity: $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

b) commutativity: $A \oplus B = B \oplus A$.

c) cancellation: if $A \oplus B = A \oplus C$, then $B = C$.

d) distributivity w.r.t. intersection: $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$.

Exercise 15.
Using the laws of set algebra, prove that $(V \cup W) \cap (V \cap W) = (V \cap W)$, and name the rules used.

Exercise 16.
Using the laws of set algebra, simplify $[(A \cup B^c) \cap C] \cup [(B - A) \cap C]$ as much as possible and name the rules used.

Exercise 17.
Prove that $(A \cup B)^c = ((A \cap B^c) \cup (A^c \cap B) \cup (A \cap B))^c$.

Exercise 18.
Prove or give a counterexample: $(A - B) - (C - D) = (A - C) - (B - D)$.

Exercise 19.
The symmetric difference has a number of properties proven in Exercise 14. Here we are looking for similar laws.

a) Does cancellation hold for intersections? That is, does $A \cap B = A \cap C$ imply $B = C$?

b) Is union distributive with respect to symmetric difference? That is, does $A \cup (B \oplus C) = (A \cup B) \oplus (A \cup C)$ hold?

c) Is intersection distributive with respect to symmetric difference? That is, does $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$ hold?

d) Is symmetric difference distributive with respect to intersection? That is, does $A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C)$ hold?

Exercise 20.
Schaum 1.40:
Use Theorem 1.9 $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ to prove Corollary 1.10: If A, B and C are finite sets, then so is $A \cup B \cup C$, and $n(A \cup B \cup B)$ is

$$n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Exercise 21.
We are working in the universe $\mathcal{U}$. Given a set $A \subseteq \mathcal{U}$, we can define the *characteristic function* $\kappa_A : \mathcal{U} \to \{0, 1\}$ such that $\kappa_A(x) = 1$ if and only if $x \in A$.

a) Express the function κ_{A^c} in terms of κ_A.

b) Express the functions $\kappa_{A \cup B}$ and $\kappa_{A \cap B}$ in terms of the functions κ_A and κ_B.

Chapter 2

Relations

Exercise 22.
Schaum 2.4:
Given $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Let R be the following relation from A to B: $R = \{(1, y), (1, z), (3, y), (4, x), (4, z)\}$.

a) Determine the matrix of the relation R.

b) Draw the arrow diagram of R.

c) Find the inverse relation R^{-1}.

d) Determine the domain and range of R.

Exercise 23.
Schaum 2.7:
Let R and S be the following relations on $A = \{1, 2, 3\}$:

$$\begin{aligned} R &= \{(1,1),(1,2),(2,3),(3,1),(3,3)\}, \\ S &= \{(1,2),(1,3),(2,1),(3,3)\}. \end{aligned}$$

Find:

a) $R \cup S$,

b) $R \cap S$,

c) R^c (in the universe $A \times A$),

d) $R \circ S$,

e) $S^2 = S \circ S$.

Exercise 24.
Consider the relation

$$R = \{(1,2),(2,1),(3,4),(4,2),(4,5)\}$$

on $\{1,2,3,4,5\}$.

a) Draw the directed graphs corresponding to R and $R \circ R$.

b) Determine the transitive closure $R^+ = \bigcup_{n\geq 1} R^n$, and write R^+ as a set of pairs.

Exercise 25.
For each of following definitions of relation R, determine if R is reflexive, irreflexive, symmetric, antisymmetric, transitive.

a) xRy iff $x \cdot y$ is odd (choose appropriate universe).

b) xRy iff $x \cap y = \emptyset$ (on $\mathcal{P}(\mathbb{N}_0)$)

c) xRy iff $x + 4y = 10$ (choose appropriate universe).

Exercise 26.
Let R, S be relations on a set A of size $|A| \geq 3$. For every property $P \in$ {reflexive, irreflexive, symmetric, antisymmetric, transitive}, prove or give a counterexample for the following statements:

a) If R and S have property P, then $R \cap S$ has property P.

b) If R and S have property P, then $R \cup S$ has property P.

Exercise 27.
Let $R \subseteq A \times B$ be a relation, and $\mathrm{id}_B = \{(b,b) \mid b \in B\}$ the identity relation on B. Prove or give a counter example for the following statements:

a) R is functional $\Leftrightarrow R^{-1} \circ R \subseteq \mathrm{id}_B$.

b) R is surjective $\Leftrightarrow \mathrm{id}_B \subseteq R^{-1} \circ R$.

Exercise 28.
For the following relations on $\mathbb{N}^+$, check whether they are reflexive, irreflexive,symmetric, antisymmetric or transitive. Explain your answer.

a) xRy if $x \mid y$.

b) xRy if $x = 2y$.

c) xRy if $x^2 \geq y$.

Exercise 29.
Schaum 2.12:
Let R be a relation on a set A, and let P be a property of relations, such as symmetry and transitivity. Then P will be called R-closable if P satisfies the following two conditions:

1) There is a P-relation S containing R.

2) The intersection of P-relations is a P-relation.

a) Show that symmetry and transitivity are R-closable for any relation R.

b) Suppose P is R-closable. Then $P(R)$, the P-closure of R, is the intersection of all P-relations S containing R, that is,

$$P(R) = \bigcap \{S \mid S \text{ is a } P\text{-relation and } R \subseteq S\}$$

Exercise 30.
Let R be a relation on $\mathbb{R}^2$, given as $(x, y)R(p, q)$ iff $x < p$ or both $x = p$ and $y \leq q$, for all $x, y, p, q \in \mathbb{R}$. Is R a partial order on $\mathbb{R}^2$? Explain your claims. (Recall a partial order is a reflexive, antisymmetric, transitive relation).

Chapter 3

Functions

Exercise 31.
Which of the following functions is injective?

a) $f : \mathbb{R} \to \mathbb{R}$, defined as

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ (x+1)/x & \text{if } x > 0 \end{cases}.$$

b) $f : \mathbb{R} \to \mathbb{R}$, defined as $f(x) = \sin(x)$.

c) $f : \text{Countries} \to \text{Cities}$, defined as $f(\text{country}) = \text{capital}(\text{country})$.

Exercise 32.
Which of the functions in Exercise 31 is surjective?

Exercise 33.
Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions. Prove or give a counterexample for the following statements:

a) If f and g are bijective, then $f + g$ is bijective.

b) If f and g are bijective, then $f \cdot g$ is bijective.

Note $f + g$ is the short-hand for the function $(f + g)(x) := f(x) + g(x)$, and analogously for the product.

Exercise 34.
Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and $f : A \to B$. Show that

a) $V \subseteq A \Rightarrow f^{-1}(f(V)) \supseteq V$.

b) $W \subseteq B \Rightarrow f(f^{-1}(W)) \subseteq W$.

Exercise 35.
Prove that the composition of bijections is a bijection.

Exercise 36.

a) Provide a bijection between $\mathbb{N}^+$ and $\mathbb{N}^+ - \{1, 2, 3, 4, 5\}$.

b) Provide a bijection between $\mathbb{N}^+$ and $\mathbb{Z}$. (See Schaum 3.11).

Exercise 37.
Suppose $f : A \to B$ is bijective. Let $f|_V : V \to B$ be defined as $f|_V(x) = f(x)$, for all $x \in V$ (this is called a restriction of f to V).

a) Show that $f|_V$ is injective.

b) Prove by counterexample that it need not be surjective.

Exercise 38.
Find a generic formula (without proof) for:

a) $1 = 1, 1 - 4 = -(1 + 2), 1 - 4 + 9 = 1 + 2 + 3, 1 - 4 + 9 - 16 = -(1 + 2 + 3 + 4), \ldots$

b) $\frac{1}{1\cdot 2} = \frac{1}{2}, \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{2}{3}, \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{3}{4}, \ldots$.

Exercise 39.

a) The Fibonacci sequence is defined as $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n > 1$. Let a_k be the k^{th} *prime* Fibonacci number (1 is not prime, so $a_1 = F_3 = 2$). Compute $\sum_{k=1}^{3} a_k$.

b) Compute $\sum_{k=0}^{n} 2^k$, $\sum_{k=2}^{n}(-3^k)$, $\sum_{k=0}^{n} \pi$, $\sum_{k=0}^{n}((-3)^k + 2^k + 1)$,

c) In the universe $\mathbb{N}^+$, let V_n denote the set of multiples of n. Express the set of prime numbers in terms of V_n

d) Let $A = (a_{ij})$ be an $n \times n$ matrix. Express the sum of all the elements below the diagonal (not including the diagonal).

Chapter 4

Graphs

Exercise 40.
Consider the adjacency matrix

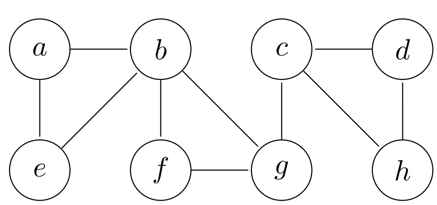
$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

a) Provide the graph whose adjacency matrix is M, where the vertices are enumerated 1, 2, 3, 4.

b) Provide the graph associated with the relation $M \circ M$, and with $M^{-1} \circ M$. (Here, M is understood as a relation, not matrix, and the operations are relation operations.)

c) Just by looking at the adjacency matrix, how can we see if the graph is undirected? That it has no self-loops? That the associated relation is functional?

Exercise 41.
Consider the following graph G:



a) Specify the degree of each vertex, and verify Theorem 8.1.

b) Find all simple paths from a to g.

c) All trails (a walk with distinct edges) from b to c.

d) the distance $d(a, c)$ from a to c.

e) $diam(G)$, the diameter of G.

Exercise 42.
Find all cycles, cut points, and bridges of the graph in Exercise 41.

Exercise 43.
A clique is an induced complete subgraph of a graph. Find all cliques in the graph in Exercise 41.

Exercise 44.
Consider the graph G in Exercise 41. Find the subgraph $G(V', E')$ induced by:

a) $V' = \{b, c, d, e, f\}$,

b) $V' = \{a, c, e, g, h\}$,

c) $V' = \{b, d, e, h\}$,

d) $V' = \{c, f, g, h\}$.

Which of them are isomorphic?

Exercise 45.
How many non-isomorphic connected graphs are there over 4 vertices? 5? 10?

Exercise 46.
Suppose that an undirected graph G contains two distinct simple paths from a vertex u to a vertex v. Show that G has a cycle. Does this claim hold if

a) the graph is directed?

b) the path is not simple?

Exercise 47.
Let G be a connected graph. Prove:

a) If G contains a cycle C that contains an edge e, then $G - \{e\}$ is still connected.

b) If $e = \{u, v\}$ is an edge such that $G - \{e\}$ is disconnected, then u and v belong to different components of $G - \{e\}$.

Exercise 48.
Prove Theorem 8.11: The following are equivalent for a graph G:

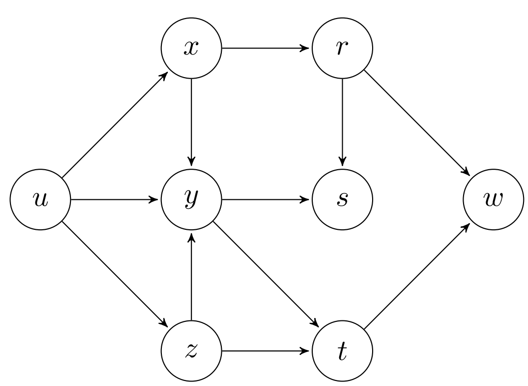
a) G is 2-colorable.

b) G is bipartite.

c) Every cycle of G has even length.

Exercise 49.
A d-regular graph, $d \geq 0$, has only vertices with degree d. Bipartite graphs allow a bipartition of the edges into two sets such that there is no edge between vertices of the same set. Construct a 3-regular graph of order at least 5, and check if it is bipartite.

Exercise 50.
Consider the following grap G:



A topological order of a directed graph $G = (V, E)$ is a sequence $v_1, \dots, v_n$ of all nodes of G so that if $(v_i, v_j) \in E$, then $i < j$. (You can use the nodes of the count on a line where only arrows of the count run from left to right.)

a) Determine a topological order of the directed graph above.

b) Prove Theorem 9.8 (Schaum): Let S be a finite directed cycle-free graph. Then there exists a topological sort of the graph S.

Chapter 5

Combinatorics

Exercise 51.

a) We have a row of 11 seats. In how many ways can you accommodate 6 people to take a seat?

b) In how many ways can you seat 10 people around a round table with 10 chairs around it.

Exercise 52.

a) How many different sequences of 8 binary digits exist with exactly 5 ones?

b) How many different sequences of n binary digits exist with exactly k ones?

c) How many different sequences of length n, with elements from $\{0, 1, 2\}$ exist, with exactly k ones?

Exercise 53.

a) How many different edges (so unordered pairs $\{u, v\}$) can an undirected graph with n nodes have?

b) How many different directed edges (so ordered pairs (u, v)) can a directed graph with n nodes have?

c) How many different functions are there from $\{1, 2, 3\}$ to $\{1, 2, 3\}$?

d) How many different functions are there from $\{1, 2, ..., n\}$ to $\{1, 2, ..., n\}$?

e) How many different bijections are there from $\{1, 2, ..., n\}$ to $\{1, 2, ..., n\}$?

Exercise 54.

a) Put a complete set of parentheses in the expression $1 - 2 - 3 - 4$. How many ways of doing this are there?

b) The same for $1 - 2 - 3 - 4 - 5 - 6 - 7$.

Exercise 55.

Determine the coefficient of x^3y^4z in the expansion of

a) $(x + y^2 + z)^6$

b) $(2x - y - 3z)^8$

Exercise 56.

A palindrome is a word that can be read the same way in either direction (such as RACECAR). How many 9-letter palindromes (not necessarily meaningful) can be formed using the letters A-Z?

Exercise 57.

How many three-digit numbers abc have the property that $a \leq b \leq c$?

Exercise 58.

Consider a knockout tournament for a two-player game. For each game, the winner continues to the next round, while the loser is knocked out. How many games are played if the tournament starts with $n \geq 1$ players.

Exercise 59.

Prove Pascal's identity: for all $n \in \mathbb{N}_0$ and $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Hint: use that the binomial coefficient $\binom{n}{k}$ equals the number of subsets with k elements of a set with n elements.

Exercise 60.

A *derangement* of a set S is a bijection $f : S \to S$, such that $f(x) \neq x$, for all $x \in S$. Informally, it is a permutation where every element moved. Let $!n$ be the number of derangements on a set of n elements.

a) Show that $!1 = 0$, $!2 = 1$, and $!n = (n-1)(!(n-1) + !(n-2))$, for $n \geq 2$.

b) Prove that $!n = n \cdot !(n-1) + (-1)^n$, for $n \geq 0$.

Exercise 61.

Programmeerwedstrijd competition, spring 2003

In a prison for life-threatening criminals, the prisoners have to be brought into the cell block one at a time. This is done by forming a long line with criminals (C) and guards (B). The prisoners are so dangerous that two cannot be allowed to stand next to each other, because anarchy would break out immediately. But, the guards so lazy that as soon as four are next to each other, they immediately start playing cards. This must also be prevented. For example, the following sequences may not be formed: C-B-B-C-C-B: 2 criminals side by side: anarchy! C-B-B-B-B-C: cards! The following sequence is possible: C-B-C-B-B-B-C-B-C-B-B-C-B-C-B-B-B-C The prison warden has ordered to make a sequence of length N. "Yes, but," one of the guards asks, "in how many different ways of doing that are there?"

The input is in the `imprisoned.in` file and the output is written in the `imprisoned.out` file. Every line of `imprisoned.in` contains is a value N (with $4 \leq N \leq 50$). Write a program that given this N, always calculates how many different rows of that length can be made. This answer is then always printed in a separate line in `imprisoned.out`. You do not have to take symmetry into account, i.e., C-B-B and B-B-C are counted as different rows.

Chapter 6

Recursion

Exercise 62.

a) What does the following function compute (for $y > 0$)?

```
int func(int x, int y) {
    if (y) return x * func(x, y - 1);
    return 1;
}
```

b) What does $fibo(4)$ output? Create a tree structure that displays the function calls. What is the length of the string printed by $fibo(8)$? What is the number of a's and b's in that string?

```
void fibo(int depth) {
    switch (depth) {
    case 0: cout << 'b'; return;
    case 1: cout << 'a'; return;
    default: fibo(depth - 1); fibo(depth - 2);
    }
}
```

Exercise 63.

a) Give a (possible) recursive definition of $V = \{1, -3, 5, -7, -9, 11, -13, \ldots\}$.

b) Give a (possible) recursive definition of $W = \{1, 4, 13, 40, 121, \ldots\}$.

Note, since definitions which use the ellipsis ("...") symbol are not unambiguous, there are many different recursive definitions consistent with the given specification, and any will do.

Exercise 64.

a) Give a recursive definition of $f(n) = 2^n$, for $n \geq 0$.

b) Give a recursive definition of $g(n) = 2^n + 1$, for $n \geq 1$.

c) Give a recursive definition of $h(n) = \sum_{i=1}^{n} i(i+1)$, for $n \geq 1$.

Exercise 65.

a) Describe a recursive function for the number of nodes of a binary tree, by it giving a base $f(\text{leaf})$ and a recursion $f(\text{node})$ expressed in $f(\text{left})$ and $f(\text{right})$.

b) Using the same approach, provide a function that determines the height of a binary tree.

c) Using the same approach, provide a function for the maximum of value stored in the nodes of a binary tree.

Exercise 66.

The Blurpsen set is the smallest set (language) so that:

a) Δ is a Blurps.

b) If x is a Blurps, then $x\Delta\Delta$ and $\diamond xx\diamond$ are Blurps.

c) If x and y are Blurps, then $x\Delta y$ is also a Blurps.

Show that all Blurps have an odd number of triangles or at least one diamond.

Chapter 7

Induction

Exercise 67.

Express using the summation symbols, and prove using induction on n.

a) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, for $n \geq 1$.

b) $2 + 6 + 18 + \cdots + (2 \cdot 3^n) = 3^{n+1} - 1$, for $n \geq 0$.

Exercise 68.

Prove by induction on n:

a) $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, for $n \geq 0$.

b) $\sum_{k=1}^{n} (-1)^k k^2 = (-1)^n \sum_{k=1}^{n} k$, for $n \geq 0$.

c) $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$, for $n \geq 0$.

For b) use the results of a).

Exercise 69.

Show that $(1 - a)^n \geq 1 - na$, for all $n \in \mathbb{N}_0$ and all $0 < a < 1$.

Exercise 70.

Prove Newton's binomial theorem: for all $x, y \in \mathbb{R}$ and $n \in \mathbb{N}_0$,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Hint: use induction and Pascal's identity in Exercise 59.

Exercise 71.
Fibonacci numbers are defined recursively as $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_n + F_{n+1}$, for $n \geq 0$. Prove, by induction on n, that

a) $\sum_{k=1}^{n} F_k = F_{n+2} - 1$, for all $n \geq 1$.

b) $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$, for all $n \geq 1$.

Exercise 72.
Binet's formula.

a) Prove that $x^n = xF_n + F_{n-1}$, for all $n \geq 1$ and $x \in \mathbb{R}$, with $x^2 = x + 1$.

b) Show that $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$, for all $n \geq 1$.

Exercise 73.
Fibonacci by matrix multiplication.

a) Show that $Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$, for $n \geq 0$, with $Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

b) Show that $F_{2n} = F_{n+1}^2 + F_{n-1}^2$, and $F_{2n+1} = F_{n+1}^2 + F_n^2$.

c) Show that $Q^2 = Q + I$, where I is the identity matrix.

d) Show that $F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k$

Exercise 74.
Lucas numbers are defined recursively as $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_n + L_{n+1}$, for $n \geq 0$. Prove that $L_{n+2}L_n - L_{n+1}^2 = 5(-1)^n$.

Note: Lucas numbers have the same recursive form as Fibonacci numbers, but different initial values.

Exercise 75.
How can we compute a desired Fibonacci number F_n using a number of operation which is proportional to $\log(n)$? Go through the fibo function from the lectures, it loops about n times so the number of operations there is proportional to n not $\log(n)$.

Hint: use that $\sum_{k=1}^{n} F_k = F_{n+2} - 1$.

Exercise 76.
Fibonacci numbers are defined recursively as $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_n + F_{n+1}$, for $n \geq 0$.

a) Compute $\sum_{k=1}^{5} F_{2k}$.

b) Prove by induction that $\sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1$, for all $n \geq 1$.

c) If we take initial values $F_0 = 1$ and $F_1 = 3$, what equality do we get in b)?

Exercise 77.
(Exam question) Prove:

$$1 + \sum_{i=1}^{n} (i \cdot i!) = (n+1)!,$$

for all natural numbers $n > 0$.

Exercise 78.
(Exam question) We define the sequence A_n, for $n \in \mathbb{N}_0$, by means of the recurrence $a_n = a_{n-1} + a_{n-2}$, with initial values $a_0 = 2$ and $a_1 = 1$. Prove by induction that $an = (-1)^n + 2^n$.

Exercise 79.
(Exam question) The language $L \subseteq \{a, b\}^*$ is defined recursively as (1) $a \in L$, (1); (2) if $x, y \in L$ then $bxy \in L$; (3) no other words are in L. Prove with induction that for every $z \in L$ it holds that that $\#_a(z) = \#_b(z) + 1$. Here, $\#_a(z)$ and $\#_b(z)$ denote the numbers of letters a and b in z, respectively. For example $\#_a(abbaabbabb) = 4$ and $\#_b(aa) = 0$.

Exercise 80.
(Exam question) Using induction, prove that for all $n \geq 1$ the number of edges $\updownarrow(n)$ of a complete graph K_n is $n(n-1)/2$.

Chapter 8

Trees

Exercise 81.
Draw all (undirected) trees with exactly six vertices.

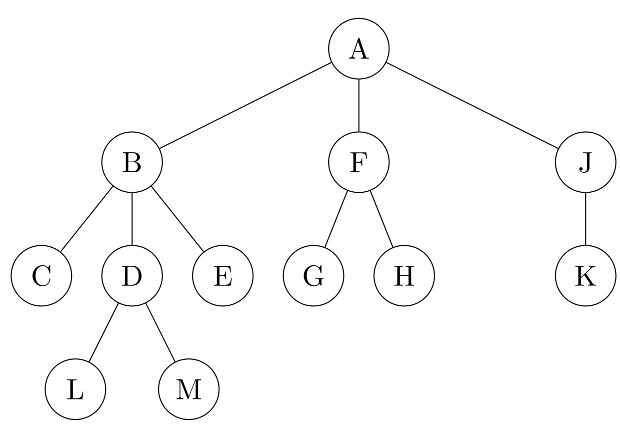
Exercise 82.
Let $G = (V, E)$ be an undirected, cycle-free graph with c components, so a forest with c trees.

a) Prove that $|V| = |E| + c$.

b) Show that Theorem 8.6, implication $(ii) \to (iii)$ follows from this result.

Exercise 83.

a) Consider the following tree:



Find the corresponding binary tree in left-child right-sibling representation.

b) Suppose T is a general tree with root R and subtrees $T_1, T_2, \ldots, T_M$. The postorder traversal of T is defined as follows:

   a) Traverse the subtrees $T_1, T_2, \ldots, T_M$ in postorder.

   b) Process the root R.

   Traverse the tree T (given with the graph above) in postorder.

c) Give the definition of 'preorder traversal' and illustrate it on T.

Exercise 84.

a) Draw the five binary trees with three vertices.

b) Let t_n be the number of binary trees with n vertices. Show that the following recurrence relation holds:

$$t_{n+1} = \sum_{k=0}^{n} t_k t_{n-k}, \quad t_0 = 1$$

c) How many binary threes with 6 vertices are there?

Chapter 9

Expressions

Exercise 85.
Assume that @ is an associative binary operation.

a) Prove that

$$(((((1 \mathbin{@} 2) \mathbin{@} 3) \mathbin{@} 4) \mathbin{@} 5) \mathbin{@} 6) = 1 \mathbin{@} (2 \mathbin{@} (3 \mathbin{@} (4 \mathbin{@} (5 \mathbin{@} 6)))).$$

b) Is $(1 \mathbin{@} 2) \mathbin{@} 3 = (3 \mathbin{@} 1) \mathbin{@} 2$?

Exercise 86.
Compute the following expressions in reverse Polish notation:

a) $5\ 1\ 2 + 4 \times + 3\ -$

b) $30\ 400 \times 15\ 60 \times +$

Exercise 87.
Consider the algebraic expression $E = \frac{(3x-5z)^4}{a(2b+c^2)}$.

a) Draw the ordered rooted parse tree T of E, using Knuth's uparrow ($\uparrow$) for exponentiation, an asterisk (*) for multiplication, and a slash (/) for division.

b) Use T to rewrite E in Polish notation and in reverse Polish notation.

Chapter 10

Formal languages

Exercise 88.
The language L over $\{a, b\}$ is defined recursively as follows:

1) $a \in L, b \in L$.
2) if $x \in L$, then $ax \in L$ and $xbb \in L$
3) L contains no other words.

a) Show that the following words are elements of L: aa, bbb, $abbb$, $abbbb$.
b) Explain that the following words are not an element of L: ba, bb, $bbbb$. Explain what properties of L-words you use to argue the claim.
c) Give a (non-recursive) specification of L (without proof).

Exercise 89.
The language L over $\{a, b\}$ is defined recursively as:

1) $\lambda \in L$, with λ the empty string,
2) if $x \in L$ then $ax \in L$ and $axb \in L$,
3) no other strings are in L.

a) Give all words from L with length less or equal to 5.
b) Give a general description of the language L. What do the words from L look like?

Exercise 90.
The language L is defined recursively as follows:

1) $\lambda \in L$ (empty word)
2) if $x \in L$, then axa and bxb are in L.

For a word w, let w^R be its reverse.

a) Show that $L \subseteq \{ww^R \mid w \in \{a,b\}^*\}$.
b) Conversely, prove that $ww^R \in L$ for every $w \in \{a,b\}^*$

Exercise 91.
Determine a shortest word that is not in the language:

a) $1^*\{01\}^*0^*$
b) $\{0\}^* \cdot (\{10\} \cdot \{0\}^*) \cdot \{1\}^*$
c) $(0^* \cup 1^*)(0^* \cup 1^*)(0^* \cup 1^*)$
d) $1^*\{0,10\}^*1^*$

Exercise 92.
Compute L^2, for the following languages:

a) $L = \{\lambda\}$, with λ the empty word.
b) $L = \{\lambda, a\}$
c) $L = \{a, b, ab\}$
d) $L = \emptyset$
e) $L = \{aa\}^*$.

Exercise 93.
Prove or give a counterexample:

a) If $L = L^2$, then $\lambda \in L$, with λ the empty word.
b) If $L = L^2$, then $L = L^n$, for $n > 1$.
c) If $L = L^2$, then $L = L^*$.

d) If $K^2 = L^2$, then $K = L$.

e) If $K^* \subseteq L$, then $K \subseteq L$.

Exercise 94.
Let K, L, and M be languages over an alphabet Σ. Prove or give a counterexample:

a) $(K^*)^n = (K^n)^*$, for $n \geq 1$.

b) $(KL)^* = (LK)^*$.

c) $(K - L)M = KM - LM$.

d) $(K^*L^*)^* = (L^*K^*)^*$.

e) $(K^c)^* = (K^*)^c$.

f) $(K^*L^*)^*K^* = (K^* \cup L^*)^*$.

Exercise 95.
Let K be a language over an alphabet Σ. Prove that $(K^*)^2 = K^*$.

Exercise 96.
The language L is defined recursively as follows:

a) $a \in L$

b) If $x \in L$, then $xb, xba \in L$

c) L contains no other words.

Show that L is exactly the language of words without a subword aa.

Exercise 97.
Use language operations to get the following languages over $\Sigma = \{a, b\}$ from finite languages. For example, the set of words of even length is $\{aa, ab, ba, bb\}^*$.

a) The set of words of odd length.

b) The words with exactly one occurrence of the letter a.

c) The words that start with an a or end with two b's (or both).

d) The words with at least three consecutive a's.

e) The words with the subword $bbab$.

Exercise 98.
The language $L \subseteq \{0,1\}^*$ is defined recursively as follows:

1) $01, 10 \in L$,

2) If $w \in L$, then $ww^R \in L$, with w^R the reverse of w,

3) L contains no other words.

a) For the following words, determine whether or not they belong to L: 0110, 1010, 01010, 01011010.

b) Let $K = \{01, 10\} \cup \{w \in \{01, 10\}^+ \mid w = w^R\}$.

c) Prove by induction that $L \subseteq K$.

d) Is $K = L$? Explain your answer.

Chapter 11

Countability

Exercise 99.
Show that the following sets are countable:

a) $\mathbb{N}_0 \times \mathbb{N}_0 = \{(x, y) \mid x \in \mathbb{N}_0, y \in \mathbb{N}_0\}$.

b) The set of finite sequences of integers.

c) Σ^*, for some countable alphabet Σ.

Exercise 100.

a) Prove that A is countable, if there is an injective function $f : A \to \mathbb{N}_0$.

b) Prove that every subset of a countable set is countable.

Exercise 101.
Show that the following sets are uncountable.

a) The set consisting of all infinite sequences of zeros and ones; More formally that is the set of all functions from $\mathbb{N}_0$ to $\{0, 1\}$.

b) The set of languages over the alphabet $\{0, 1\}$.

Chapter 12

Modular arithmetic

Exercise 102.

In the Gregorian calendar, the standard calendar in most of the world, leap years are defined as follows: Every year that is exactly divisible by four is a leap year, except for years that are exactly divisible by 100, but these centurial years are leap years if they are exactly divisible by 400. For example, the years 1700, 1800, and 1900 are not leap years, but the years 1600 and 2000 are

Which day is February 29, 2020? (January 1, 2000 was a Saturday).

Exercise 103.

Schaum 2.15:

Let A be the set of nonzero integers and let $\approx$ be the relation on $A \times A$ defined as $(a, b) \approx (c, d)$ whenever $ad = bc$. Prove that $\approx$ is an equivalence relation.

Exercise 104.

Let $a(m)$ be the alternating sum of the digits of a number m given in decimal. That is, $a(m) = d_0 - d_1 + d_2 - d_3 + \cdots + (-1)^k d_k$, if $m = (d_k \cdots d_1 d_0)_{10}$. Prove that $m \equiv a(m)$ modulo 11.

Exercise 105.

A congruence class (residue class) $\bar{x} \in \mathbb{Z}_m$ modulo m, for $m \geq 1$, is called invertible if a congruence class $\bar{y}$ exists for which $\bar{x} \cdot \bar{y} = \bar{1}$.

Determine the invertible congruence classes modulo 7 and determine their inverse. Do the same for $\mathbb{Z}_{10}$.

Exercise 106.

a) Provide an addition and multiplication table for $\mathbb{Z}_4$.

b) Determine the invertible congruence classes of $\mathbb{Z}_4$.

c) Show that $\bar{x}^4 = \bar{x}^2$, for every $x \in \mathbb{Z}_4$.

d) Prove that $x^4 - x^2$ is divisible by 4, for every $x \in \mathbb{Z}$.

Exercise 107.

a) Show that $\bar{x}^{12} = \bar{1}$ for each $x \in \mathbb{Z}_{13}$ with $\bar{x} \neq \bar{0}$. Note: if you do not use calculators/computers, then it is handy to pre-compute $\bar{x}$, $\bar{x}^2, \bar{x}^3, \bar{x}^6, \bar{x}^{12}$ in succession.

b) Determine the remainder of $100^{100} + 1000^{1000}$ when divided by 13.

Chapter 13

Finite automata

Exercise 108.
For each automaton in Figures 13.1 to 13.7, determine

a) its accepted language,

b) if it deterministic,

c) if it can be made deterministic by to adding branches and states.

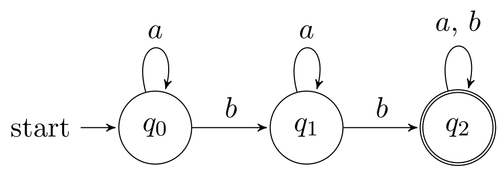


Figure 13.1: Automaton 1

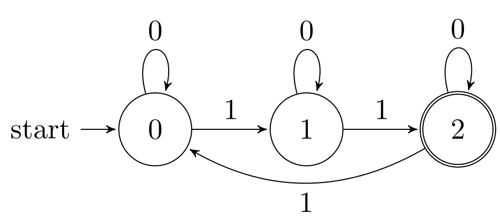


Figure 13.2: Automaton 2

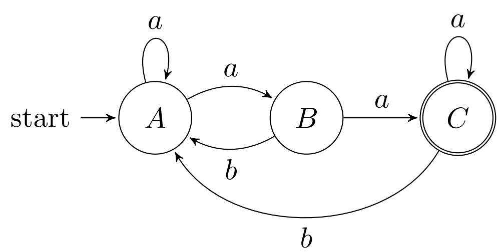


Figure 13.3: Automaton 3

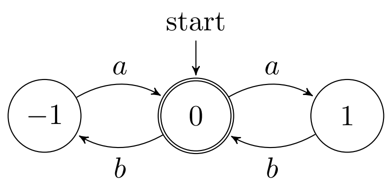


Figure 13.4: Automaton 4

Exercise 109.

For each of the following languages, determine a finite automaton with alphabet $\{0, 1\}$. Try (also) to give a deterministic finite automaton. The language consists of the words w, such that

a) w has exactly two 0s.

b) w does not end with 01.

c) w has no infix 00.

d) w has an even number of 0s.

e) every 0 in w is immediately followed by 11.

f) w contains both 11 and 010 as subwords.

g) the second to last letter of w is 0.

h) w has at most two occurrences of the subword 00.

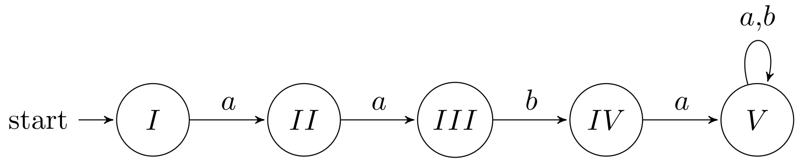


Figure 13.5: Automaton 5

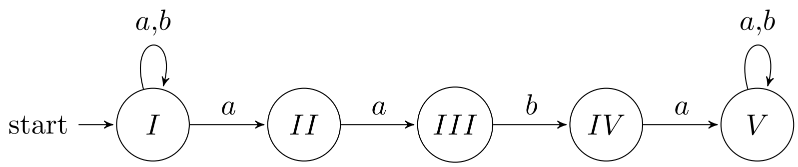


Figure 13.6: Automaton 6

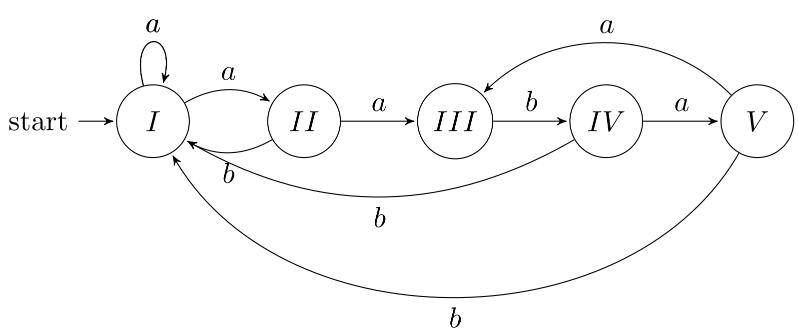


Figure 13.7: Automaton 7