PROOFS

"INFERENTIAL ARGUMENT SHOWING A STATEMENT IS TRUE"

EXAMPLE: PUZZLES

OUTLAWS & NOBLEMEN. OUTLAWS ALWAYS LIE, ALWAYS TELL THE TRUTH

A: "~~~~"

B: A SAID "I AM AN OUTLAW"

C: DON'T TRUST B! WHAT HE SAID IS A LIE.

$\Rightarrow$ WHO IS AN OUTLAW, WHO NOBLEMAN?

$\Leftarrow$ PROVE THAT

(1) A is ___

(2) B is ___

(3) C is ___

A COLLECTION OF STATEMENTS.

(S_1) ALL OUTLAWS ALWAYS LIE

(S_2) ALL NOBLEMEN ALWAYS TELL TRUTH

(S_3) A SAID X (SOMETHING) [EITHER X = NOBLEMAN OR
X = OUTLAW]

(S_4) B SAID A SAID "I AM AN OUTLAW"
B SAID Y. Y = X IS "I AM AN OUTLAW"

(S_5) C SAID Z = "WHAT B SAID IS A LIE!"

(S_6) EVERYONE IS AN OUTLAW OR A NOBLEMAN

(S1) ALL OUTLAWS ALWAYS LIE
(S2) ALL NOBLEMEN ALWAYS TELL TRUTH
(S3) A SAID X (SOMETHING)
(S4) B SAID A SAID "I AM AN OUTLAW"   B SAID Y. Y = X IS "I AM AN OUTLAW"
(S5) C SAID Z = "WHAT B SAID IS A LIE!"
(S6) EVERYONE IS AN OUTLAW OR A NOBLEMAN

$S = \{ (S_1) \ldots (S_6) \} \Rightarrow (C1) \quad (?)$

(C1) = "B IS A NOBLEMAN."

$(C_1) \& (S_2) \Rightarrow (C_2)$ = B TELLS THE TRUTH.

$(C_2) \& (S_4) \Rightarrow (C_3)$ = A SAID "I AM AN OUTLAW"

$$
\begin{tikzcd}
S_1 \arrow[d] & C_1 \arrow[dl] & \leftarrow \text{UNSURE} \\
C_2 \arrow[d] & S_4 \arrow[dl] & \\
C_3 & &
\end{tikzcd}
$$

"REASONING DIAGRAM" (NOT FOR EXAM)

ARROWS: "IMPLICATION"

"IT FOLLOWS THAT"

$$(C_3) \,\&\, (S_6) \Rightarrow (C_4) = \underbrace{\text{A NOBLE SAID HE IS AN OUTLAW}}_{C_{4A}}$$

OR

$$\underbrace{\text{AN OUTLAW SAID HE IS AN OUTLAW}}_{C_{4B}}$$

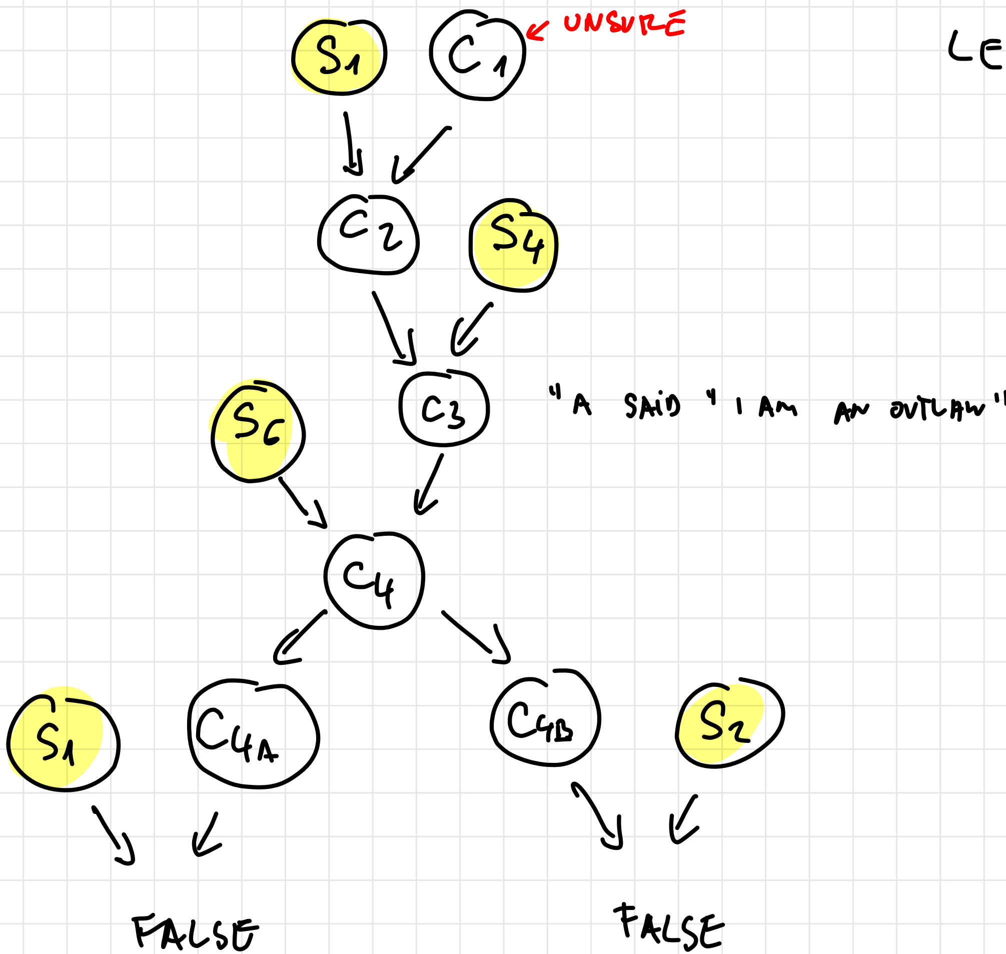
$$(C_4) = (C_{4A}) \text{ OR } (C_{4B})$$

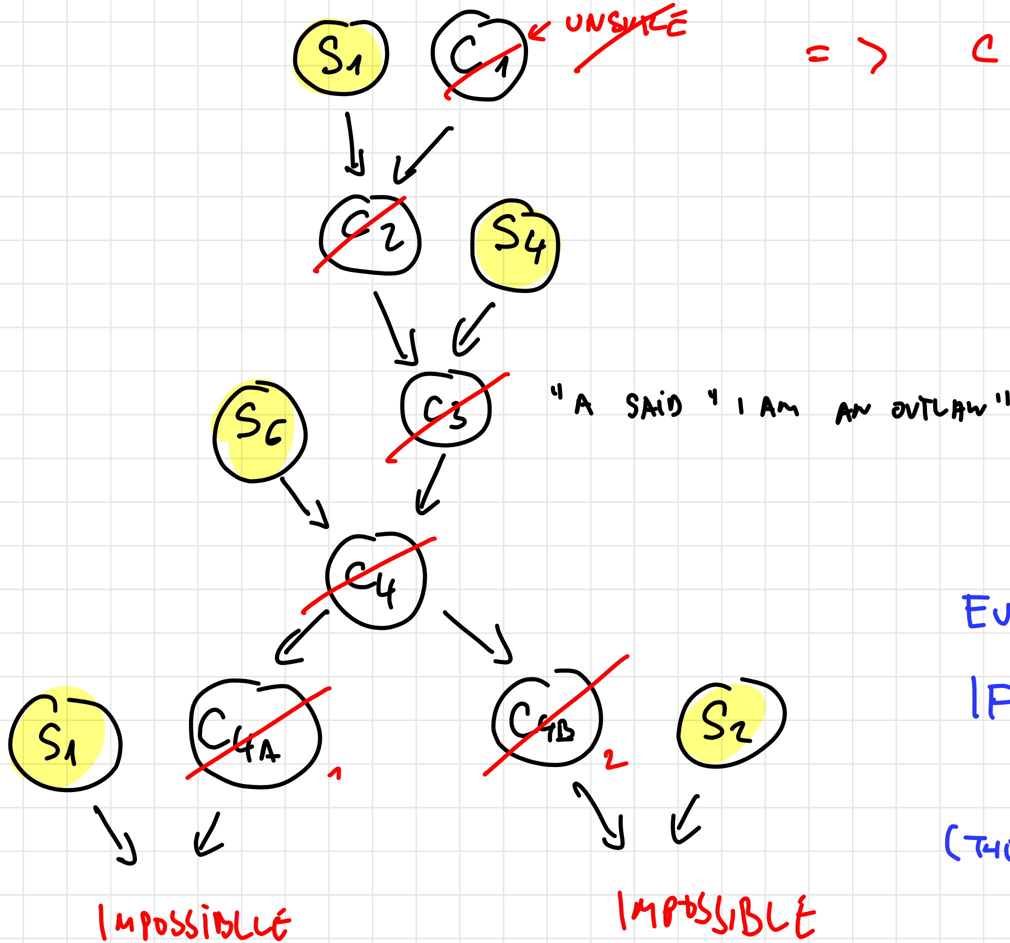
$(C_{4A}) \& (S_1)$ = A LIAR TOLD THE TRUTH $\Rightarrow$ IMPOSSIBLE

$(C_{4B}) \& (S_2)$ = A TRUTH SPEAKING PERSON LIED $\Rightarrow$ IMPOSSIBLE

$\Rightarrow$ (C_4) IS FALSE

OUR REASONING LED TO FALSEHOOD





$\Rightarrow$ C_1 IS FALSE

EVERY STEP:

IF $A \to B$

NOT B

(THEN) NOT A

(C_5) = "B IS A NOBLEMAN IS UNTRUE"

$(C_5) \& (S_6) \Rightarrow (C_6) :=$ $\boxed{\text{B IS AN OUTLAW}}$

(S1) ALL OUTLAWS ALWAYS LIE
(S2) ALL NOBLEMEN ALWAYS TELL TRUTH
(S3) A SAID X (SOMETHING)
(S4) B SAID A SAID "I AM AN OUTLAW"
B SAID Y. Y = X IS "I AM AN OUTLAW"
(S5) C SAID Z = "WHAT B SAID IS A LIE!"
(S6) EVERYONE IS AN OUTLAW OR A NOBLEMAN

SPEEDING UP A BIT:

$(C_6) \& (S_5) \Rightarrow$ C SAID THE TRUTH (C_8)

$(C_8) \& (S_1) \& (S_2) \& (S_6) \Rightarrow$ $\boxed{\text{C IS A NOBLEMAN}}$

$(C_6) \& (S_4) \Rightarrow$ A DID NOT SAY "I AM AN OUTLAW" (C_9)

$(C_9) + (S_3) \Rightarrow$ A SAID I AM A NOBLEMAN. $\boxed{\text{A IS INCONCLUSIVE}}$

$\Rightarrow$ INCONCLUSIVE

PROOF: A NETWORK OF CLAIMS (PROPOSITIONS & CONCLUSIONS), AND LOGICAL CONNECTIVES

PROVING: ACTIVITY OF CONSTRUCTING A PROOF

LOGICAL CONNECTIVES:

\begin{tikzcd}
S_1 := A \to B \arrow[dr] & & S_2 := A \arrow[dl] \\
& S_3 := B &
\end{tikzcd}

MODUS PONENS

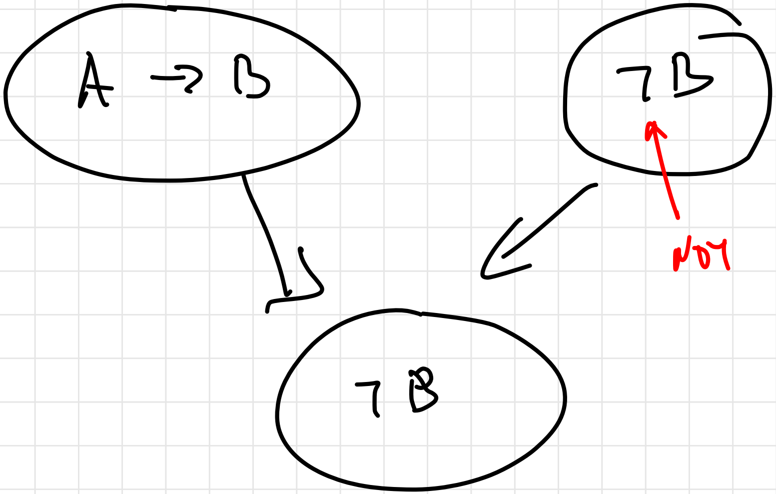
$$\frac{\begin{array}{c} A \to B \\ A \end{array}}{B}$$

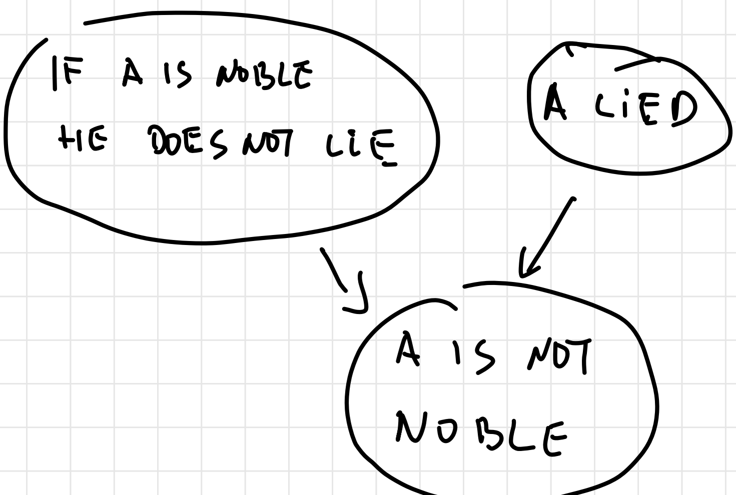
M.P.

IF A IS NOBLE
A TELLS TRUTH

A IS NOBLE

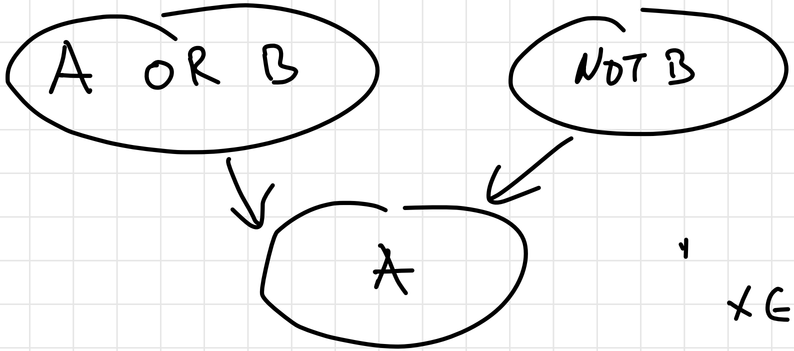
A TELLS
TRUTH





MODUS TOLLENS

$$\frac{\begin{array}{c} A \to B \\ \text{NOT } B \end{array}}{\text{NOT } A}$$

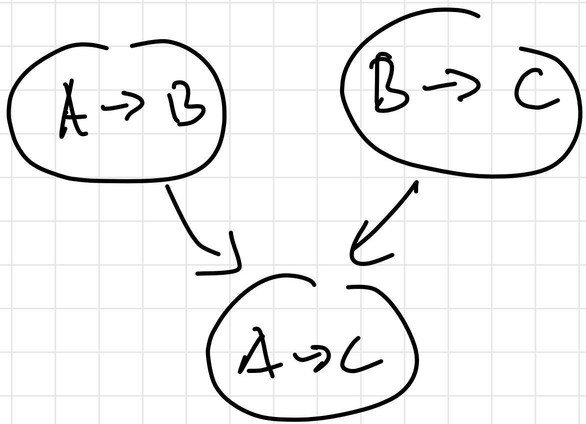


DISJUNCTIVE SYLLOGISM

SIMILAR TO

"$x \in (A \cup B - B)$"

$\Rightarrow x \in A$



HYPOTHETICAL SYLLOGISM

EXTREMELY USEFUL

$$(A \Rightarrow B \;\&\; B \Rightarrow C) \Rightarrow (A \Rightarrow C)$$

PROVE: $A \subseteq B \Rightarrow A \cap B = A$

KNOW: $A \subseteq B \Leftrightarrow x \in A \Rightarrow x \in B$

KNOW: $A \cap B = A \Leftrightarrow [x \in A \Rightarrow x \in A \cap B \ \& \ x \in A \cap B \Rightarrow x \in A]$

ALSO STATEMENTS. (PROPOSITIONS)

$x \in A \Rightarrow x \in B \rightarrow$

$x \in A \Rightarrow x \in A \cap B$

$x \in A \cap B \Rightarrow x \in A$

ASSUMPTION $A \subseteq B \Rightarrow A \cap B = A$

$A \subseteq B$

"FREE"

$x \in A \Rightarrow x \in A$

$x \in A \Rightarrow x \in B$

"FREE"

$x \in A \cap B \Rightarrow x \in A \;\&\; x \in B$

"FREE"

$x \in A \;\&\; x \in B \Rightarrow x \in A$

$x \in A \Rightarrow x \in A \cap B$

$(A \subseteq A \cap B)$

IMPLIES

$x \in A \cap B \Rightarrow x \in A$

$(A \cap B \subseteq A)$

$A \cap B = A$

CLAIM + "ALL ALWAYS TRUE"

$\Rightarrow$ CLAIM.

ASSUMING $A \subseteq B$ PROVE $A \cap B = A$.

IN EXAM...

SLIGHTLY MORE DETAILED THAN NECESSARY...

(1) $A \subseteq B \Rightarrow \underbrace{[x \in A \Rightarrow x \in B]}_{(C1)}$

(2) $A \cap B = A$. $A \subseteq A \cap B$ & $A \cap B \subseteq A$

NOTE $A \cap B \subseteq A$ IS ALWAYS TRUE.

SO IT SUFFICES TO PROVE $A \subseteq A \cap B$.

(3) $A \subseteq A \cap B \Leftrightarrow$ $x \in A \Rightarrow x \in A$ & $x \in B$.

SINCE $x \in A$ & $x \in B \Rightarrow x \in B$ IT HOLDS THAT

$x \in A \Rightarrow x \in B \Leftrightarrow A \subseteq A \cap B$.

BUT $x \in A \Rightarrow x \in B \equiv (C1)$. SO $(C1) \Rightarrow A \subseteq A \cap B$

(is exactly the claim)

HENCE ALSO $(C1) \Rightarrow A \cap B = A$. $\square$

EXERCISE 9.

PROVE THAT

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

BY PROVING TWO INCLUSIONS.

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PROVE THAT

THE MATHEMATICAL STATEMENT $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ IS TRUE

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BY PROVING TWO INCLUSIONS.

ACTUALLY WHAT WE MEAN IS

"FOR ANY CHOICE OF SETS A, B, C,
THE SET $A \cap (B \cup C)$ AND THE SET $(A \cap B) \cup (A \cap C)$
ARE EQUAL AS SETS.

$\forall A, B, C \qquad [A \cap (B \cup C)] = [(A \cap B) \cup (A \cap C)]$. ← "MAIN CLAIM"

"...BY PROVING TWO INCLUSIONS". ?? WHICH INCLUSIONS?

NEED TO PROVE EQUALITY OF TWO SETS.

SETS P & Q ARE EQUAL IF

$$P \subseteq Q \quad \& \quad Q \subseteq P.$$

TWO INCLUSIONS!

SO : "PROVE EQUALITY" =

PROVE THAT $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ (1)

AND PROVE THAT $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ (2)

TO PROVE THE MAIN CLAIM YOU NEED TO PROVE CLAIMS (1) & (2)

PROVING "CLAIM", 'STATEMENT' (1)

$\forall\, A, B, C \quad A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Again RECALL what "$\subseteq$" MEANS. $(P \subseteq Q)$

IF $x \in P$ THEN (IT MUST BE THE CASE THAT) $x \in Q$.

NEED:

$$x \in A \cap (B \cup C) \Rightarrow x \in (A \cap B) \cup (A \cap C)$$

$x \in A \cap (B \cup C)$

$\Rightarrow$

$$\underbrace{x \in A \quad \& \quad x \in B \cup C}_{} \quad (c1)$$

$[x \in B \cup C \Rightarrow x \in B$ OR $x \in C]$

$x \in (A \cap B) \cup (A \cap C)$

$$\Rightarrow \left.\begin{array}{c} (x \in A \quad \& \quad x \in B) \\ \text{OR} \\ (x \in A \quad \& \quad x \in C) \end{array}\right\} (\$)$$

(c1) $x \in A \quad \& \quad (x \in B$ OR $x \in C)$

$\Rightarrow (x \in A \ \& \ x \in B)$ OR $(x \in A \ \& \ x \in C) \Leftrightarrow (\$)$ $\square$

AGAIN, CLEANER. ONE "LINE OF IMPLICATIONS"

MORE THEN SUFFICIENT FOR EXAM

$$\underbrace{x \in A \cap (B \cup C)}_{(1)} \Rightarrow x \in A \;\&\; x \in (B \cup C)$$

$$\Rightarrow x \in A \;\&\; (x \in B \text{ OR } x \in C) \Rightarrow$$

$$(x \in A \;\&\; x \in B) \text{ OR } (x \in A \;\&\; x \in C)$$

$$\Rightarrow \underbrace{x \in (A \cap B) \cup (A \cap C)}_{(2)}$$

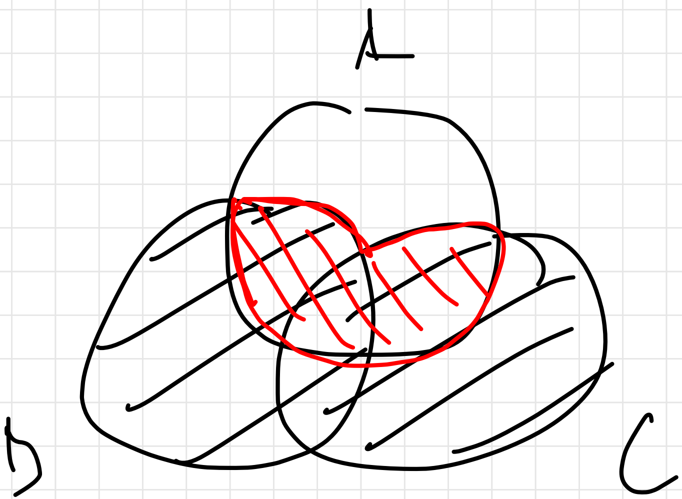
SO (1) $\Rightarrow$ (2)

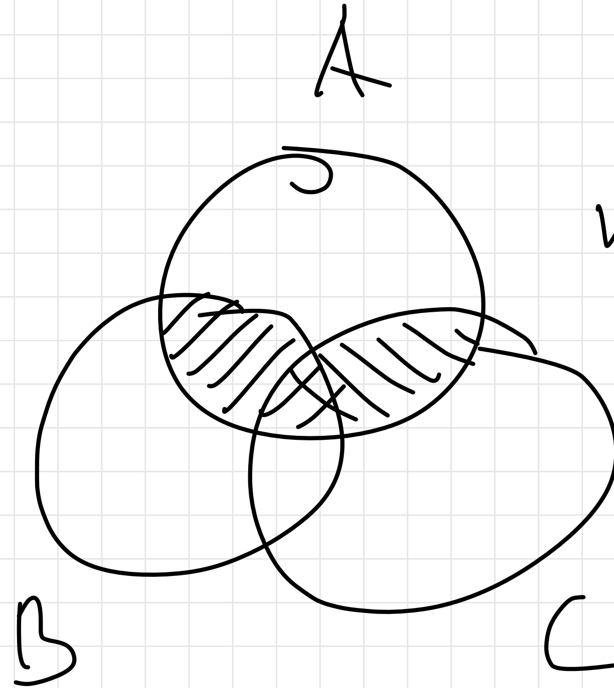
IN THIS CASE, EVERY IMPLICATION IS A BIMPLICATION

... GO BACKWARDS!

so (2) => (1), so $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

SANITY





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