Lecture 9

Exam regulations:

https://www.universiteitleiden.nl/en/science/computer-science/organisationfolder/board-of-examiners

check *Surveillance guidelines (in Dutch and English)*

-Students must be present at least 15 minutes before the start of the examination.
In case of calamities, students can be admitted to the examination room up to 45 minutes after the start of the examination.

-Students come in and do not go outside (until 45 minutes after the examination starts).

-Media carriers such as smart phones, smart watches, earpieces,
smart glasses are strictly forbidden during exams, must be out of reach and disabled.

-Toilet visit is only allowed after 45 minutes after the startof the examination.
-The toilet can no longer be visited during the last 30 minutes of the examination (calculated from the official end time).

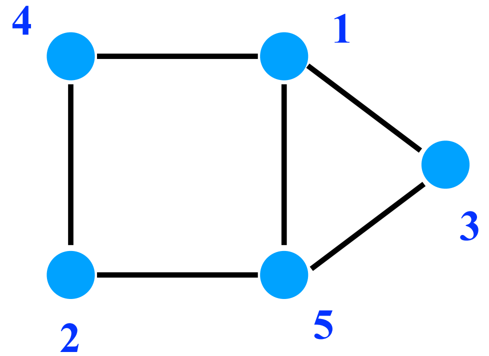
**-If you do not hand in your exam to invigilators with name,
id information, your exam will not be evaluated (evaluated zero!)**

Basics of Graph Theory

Definition. A graph *G* is an ordered pair *(V,E) where*

- ***V = V(G) is the set of vertices***
- ***E=E(G) is the set of edges***

G



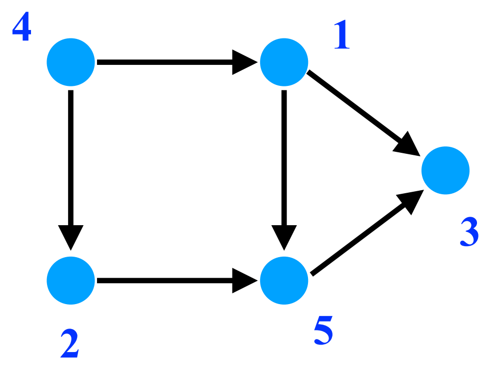
V = {1,2,3,4,5}

E= {{1,3},{1,4},{1,5},
{2,4},{2,5},{3,5}}

Definition. A graph *G* is an ordered pair *(V,E) where*

- ***V = V(G) is the set of vertices***
- ***E=E(G) is the set of directed edges (arrows, arcs)***

G



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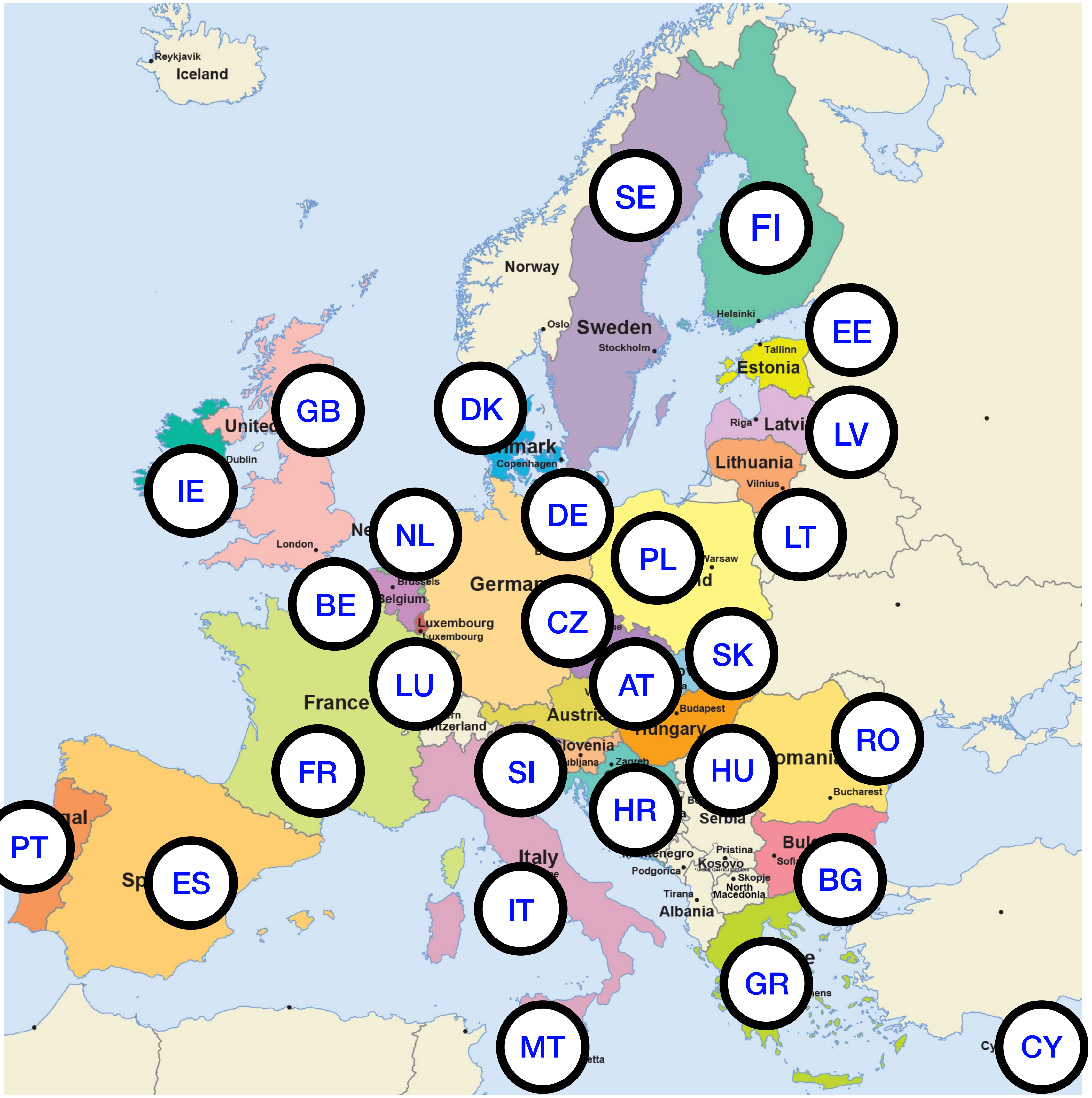

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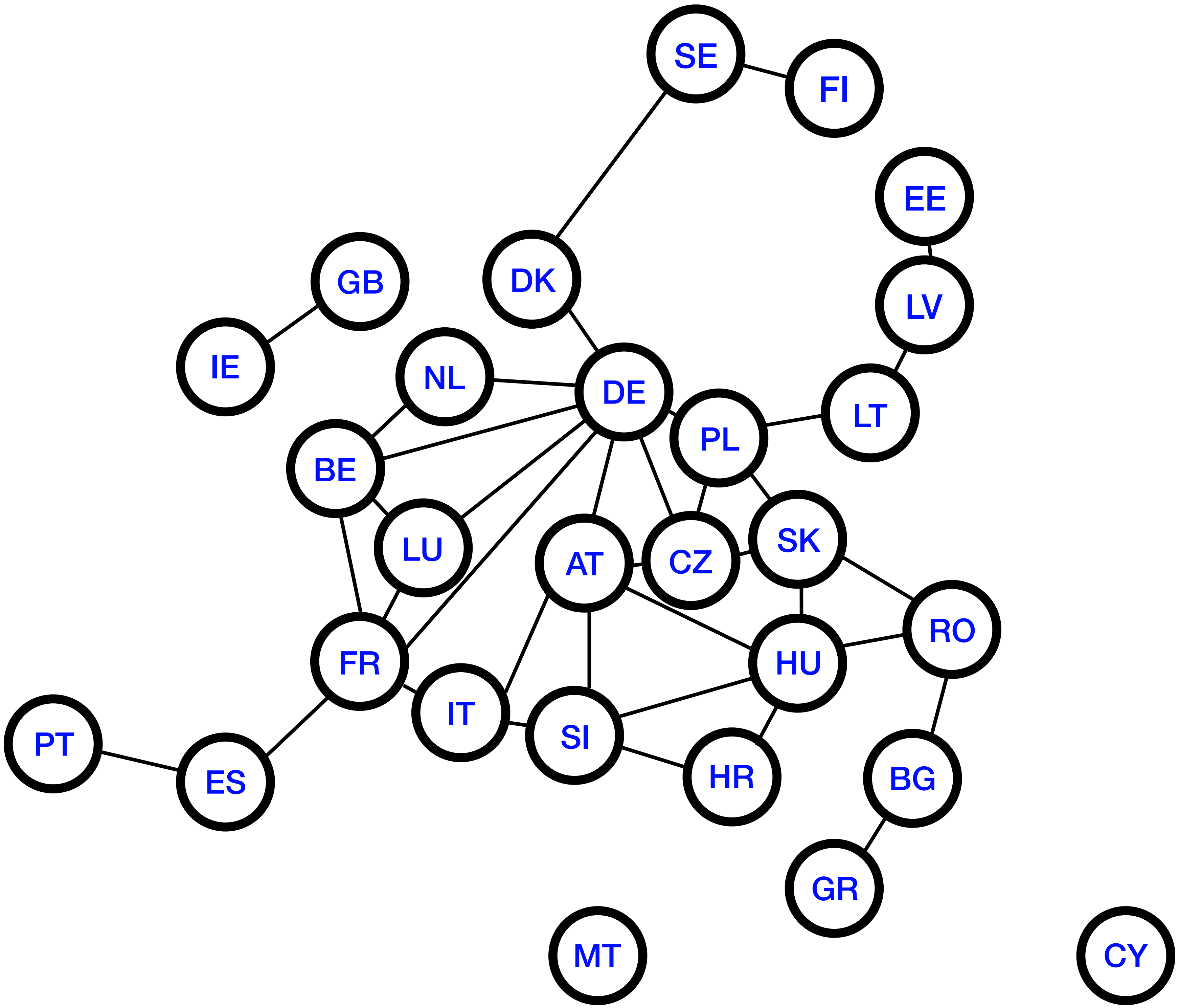
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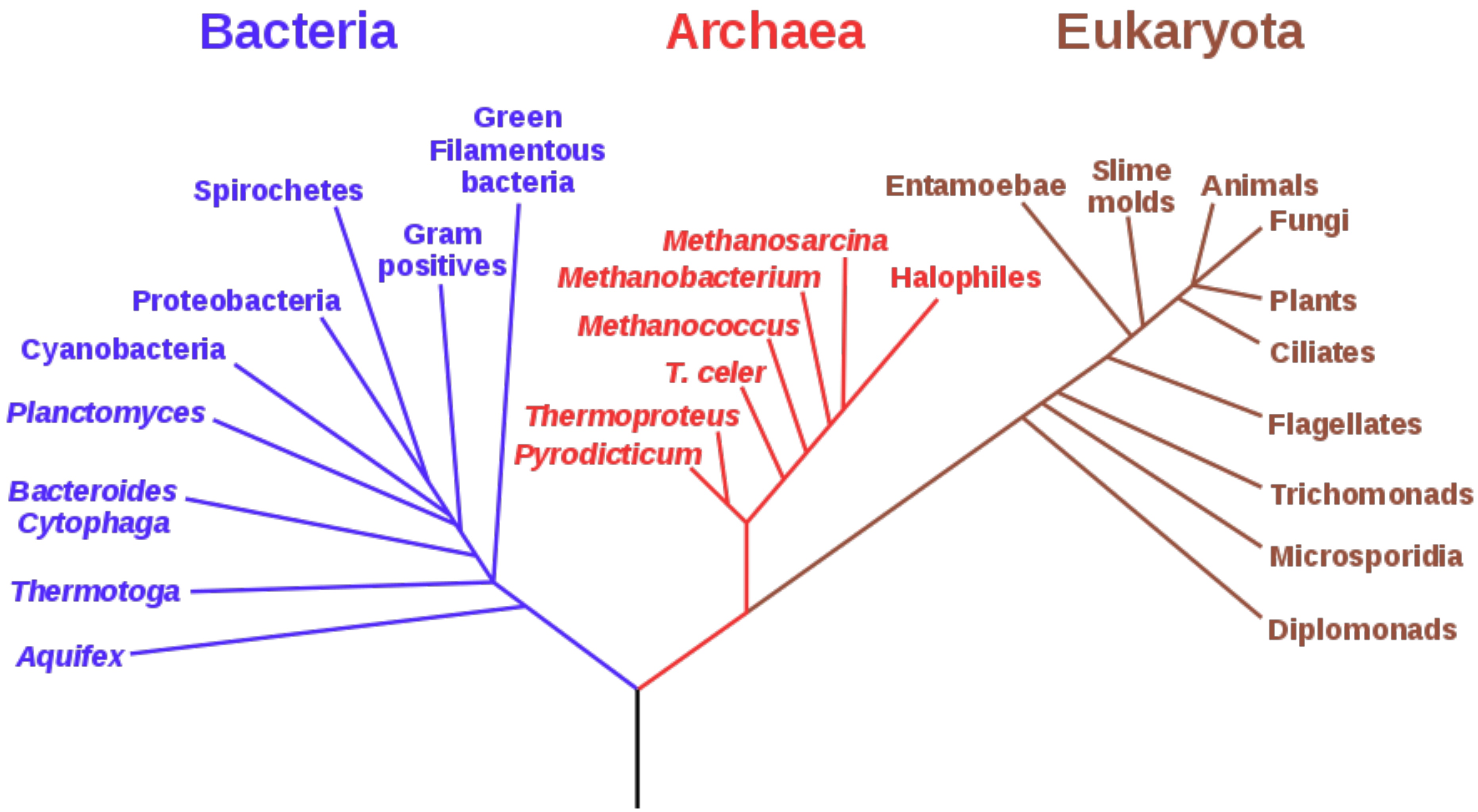
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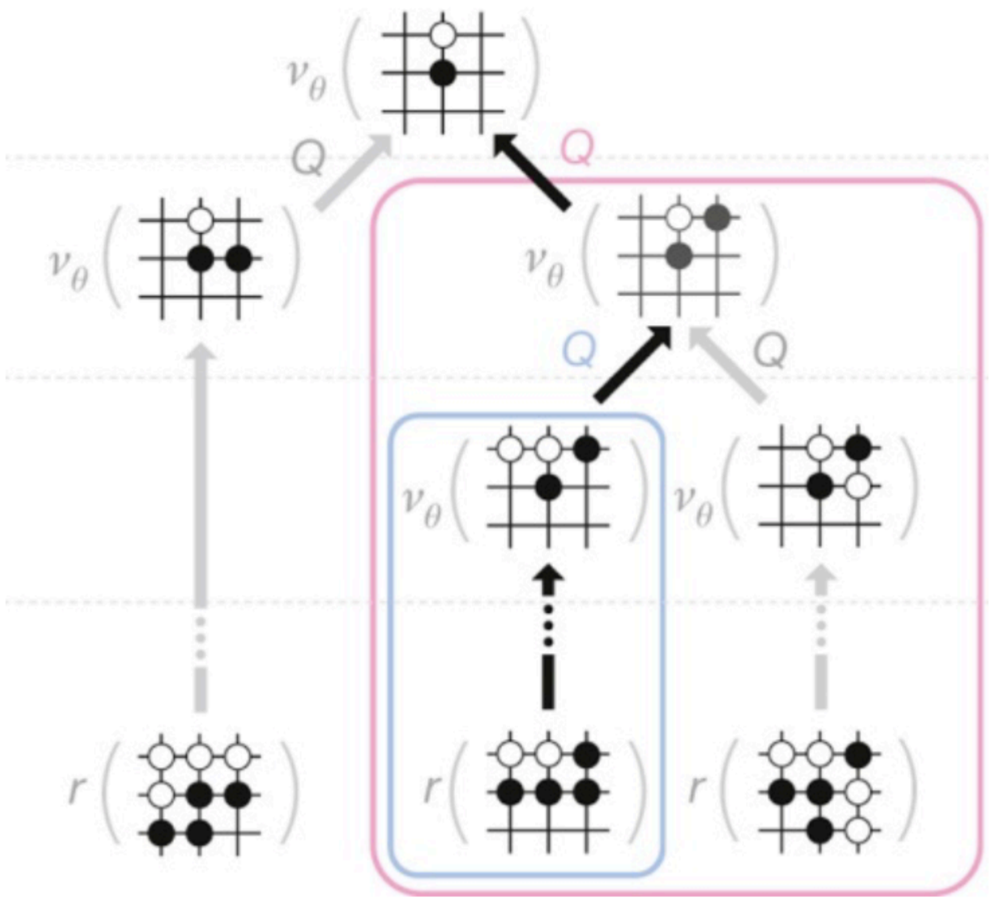
Microsporidia
Diplomonads



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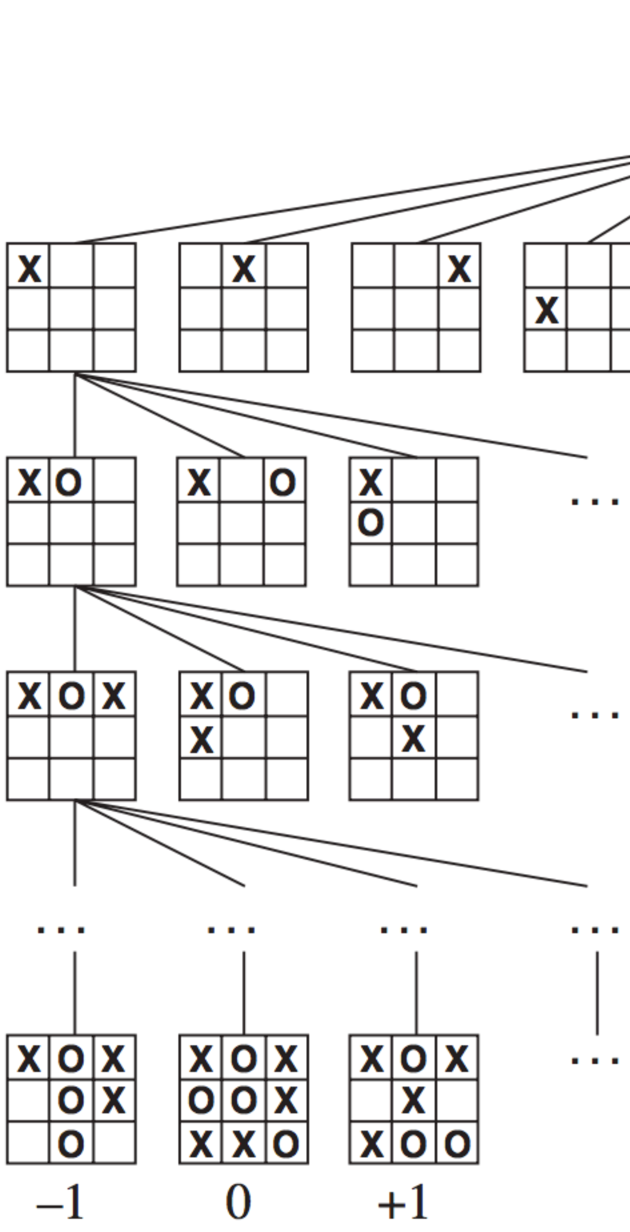
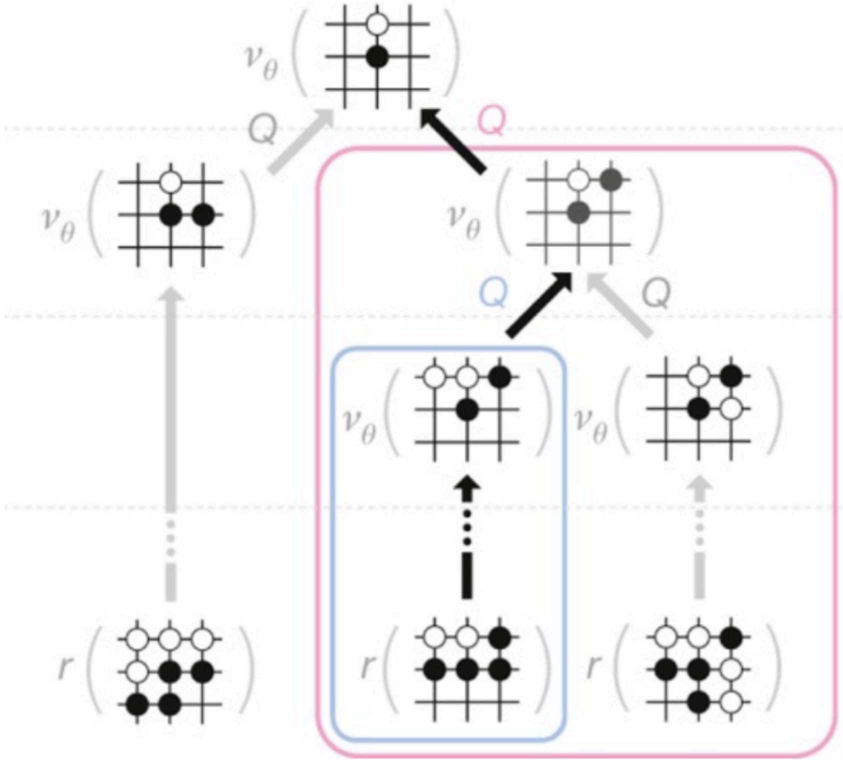
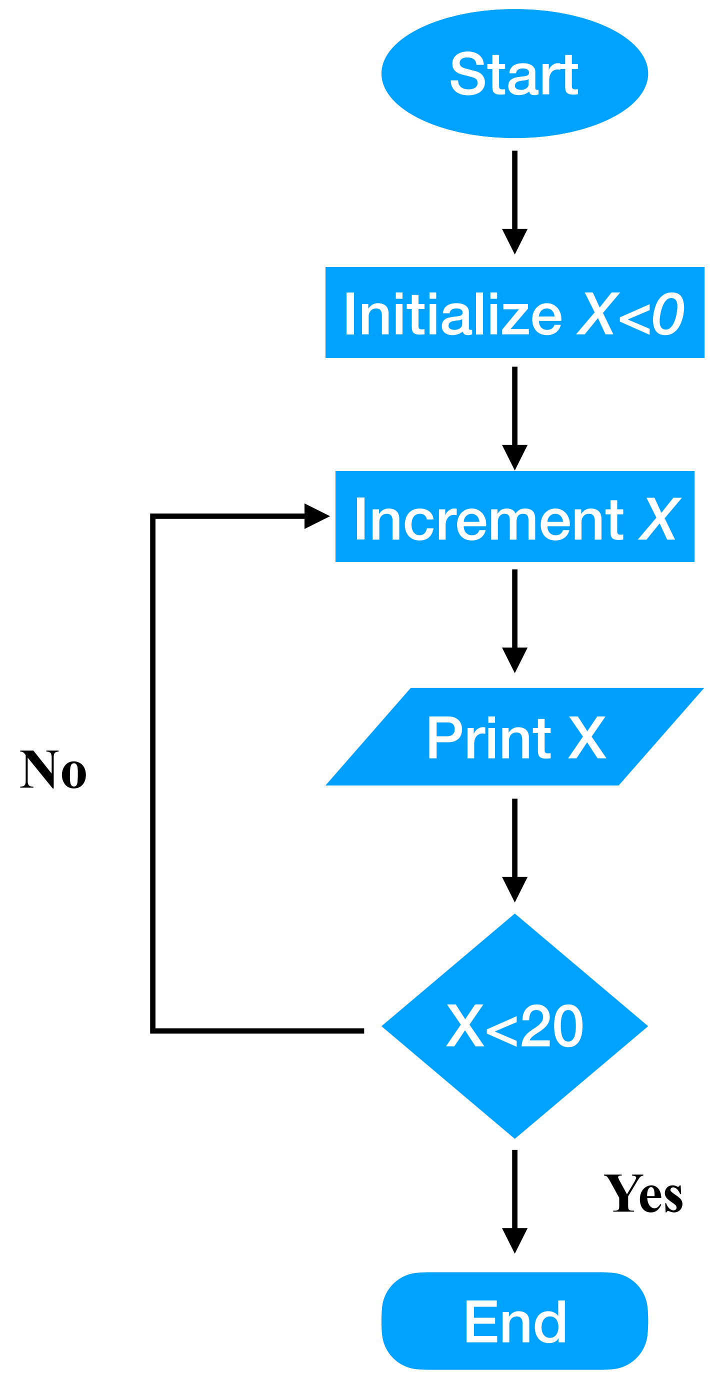


https://commons.wikimedia.org/wiki/File:Game_Tree_for_Tic_Tac_Toe.png



Nature, volume 529, pages 484–489 (2016)
Go board: Wikipedia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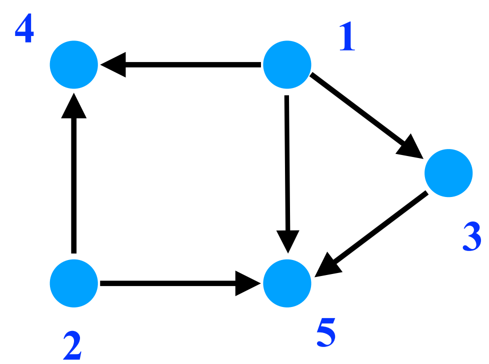
Schaum:
Undirected graphs (Chapter 8)
Directed graphs (Chapter 9)

Trees (Ch. 10) ... subsequent lectures

Directed graphs and relations



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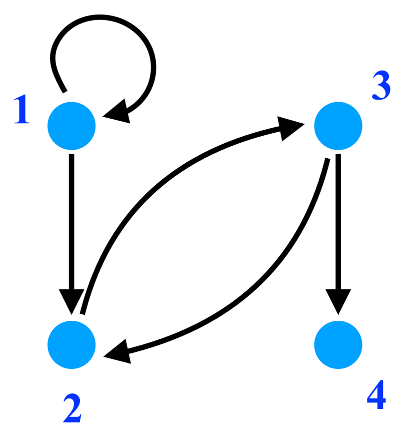
A binary relation in *A*
can be represented
by a directed graph or a matrix:
Schaum Ch 9



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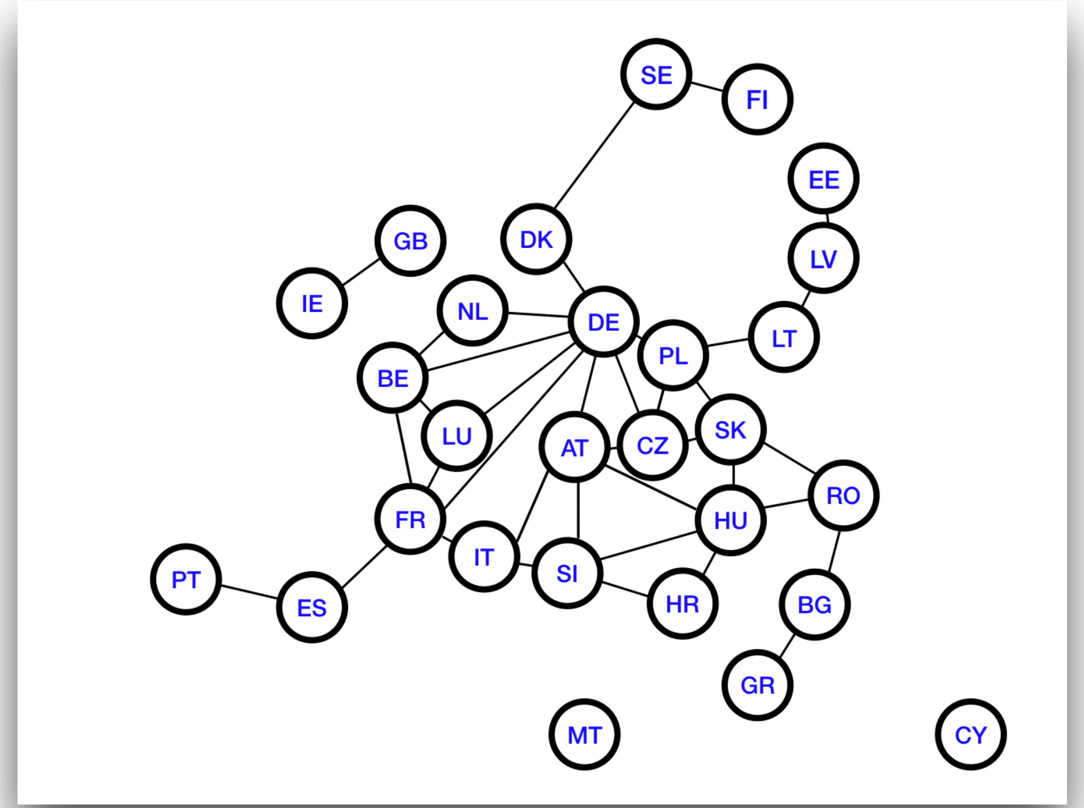
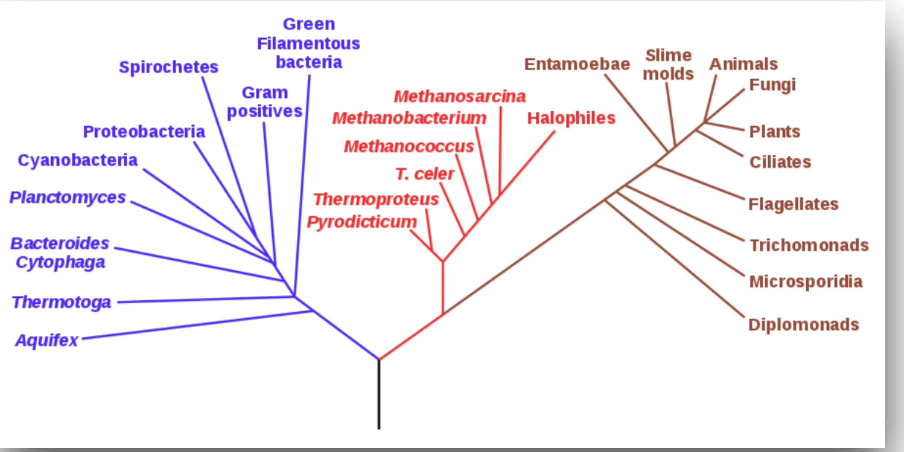
(Directed graphs *are relations*)

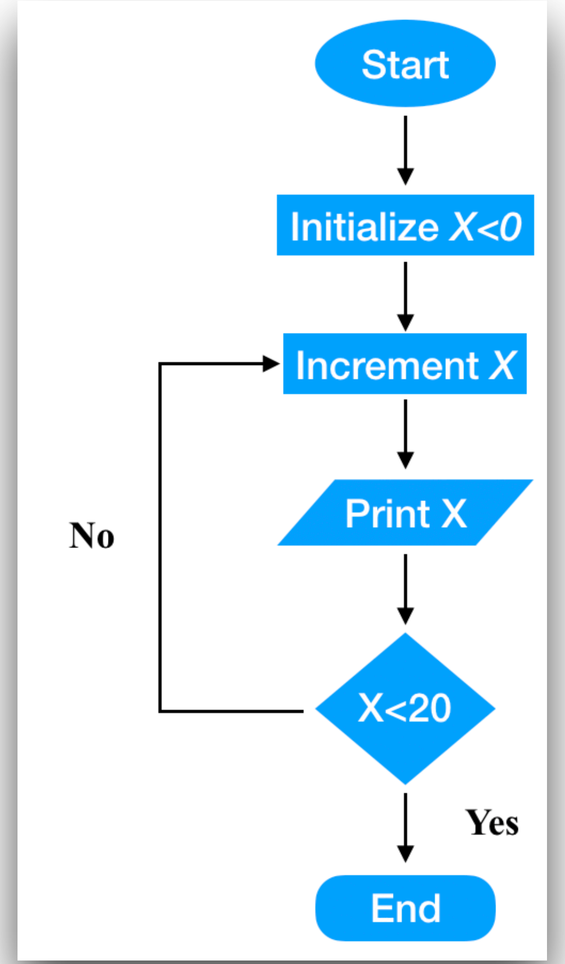
Undirected graphs, directed graphs and trees

Convention:

"graph" = undirected graph







Graphs and multigraphs

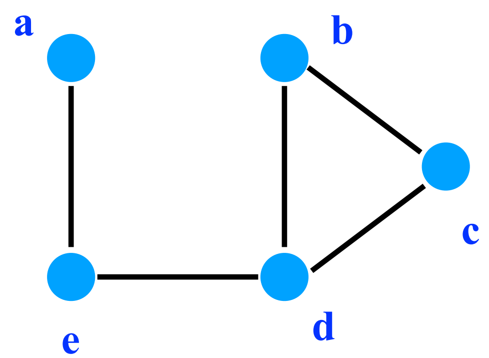
Definition. A graph *G* is an ordered pair *(V,E) where*

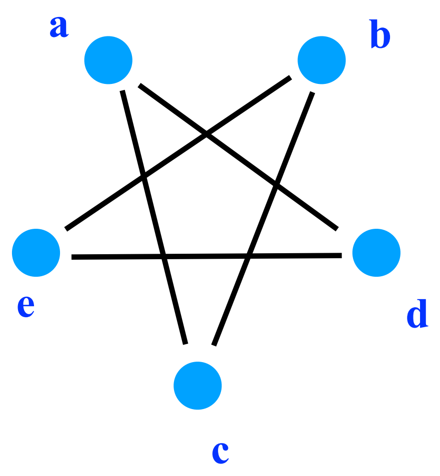
- ***V = V(G) is the set of vertices***
- ***E=E(G) is the set of edges***

for undirected graphs, an edge e is a set of two vertices: *e = {u,v}*.

We say *e* is an edge between vertices *u* and *v*.
u~v denotes that an edge between u & v exists

Different graphs over the same set of vertices…





$$\begin{aligned}
&G = G(V, E)\\
&V = \{a, b, c, d, e\}\\
&E = \{\{a, c\}, \{a, d\},\\
&\{b, e\}, \{b, c\},\\
&\{e, d\}, \{a, b\}\}
\end{aligned}$$

Graphs…

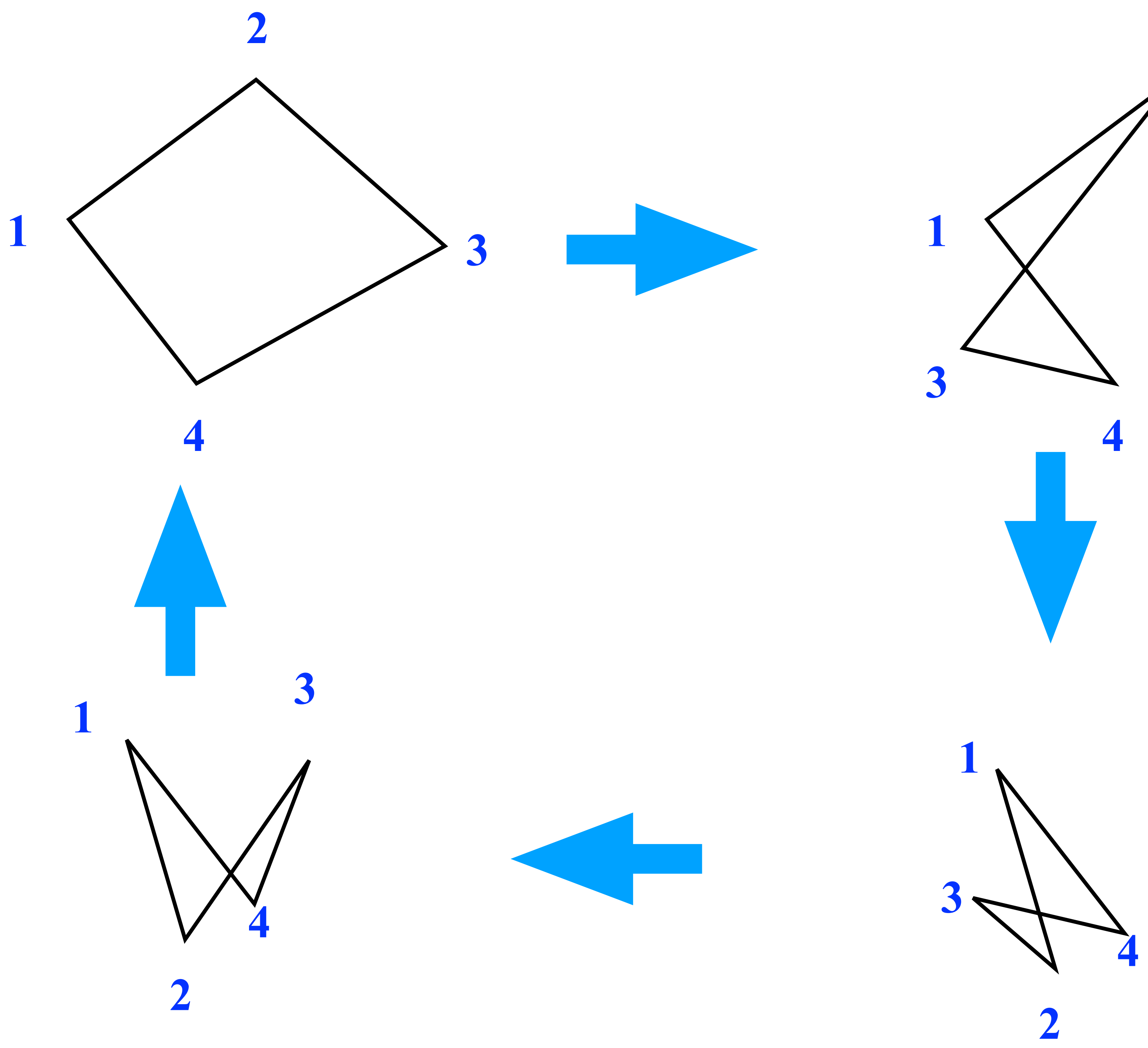
Here, only finite graphs (|V|, |E| $\leq \infty$)

Certain claims we make only hold for finite graphs

By definition, Graphs have no multiple edges (*E is a set, not multiset*), and no loops

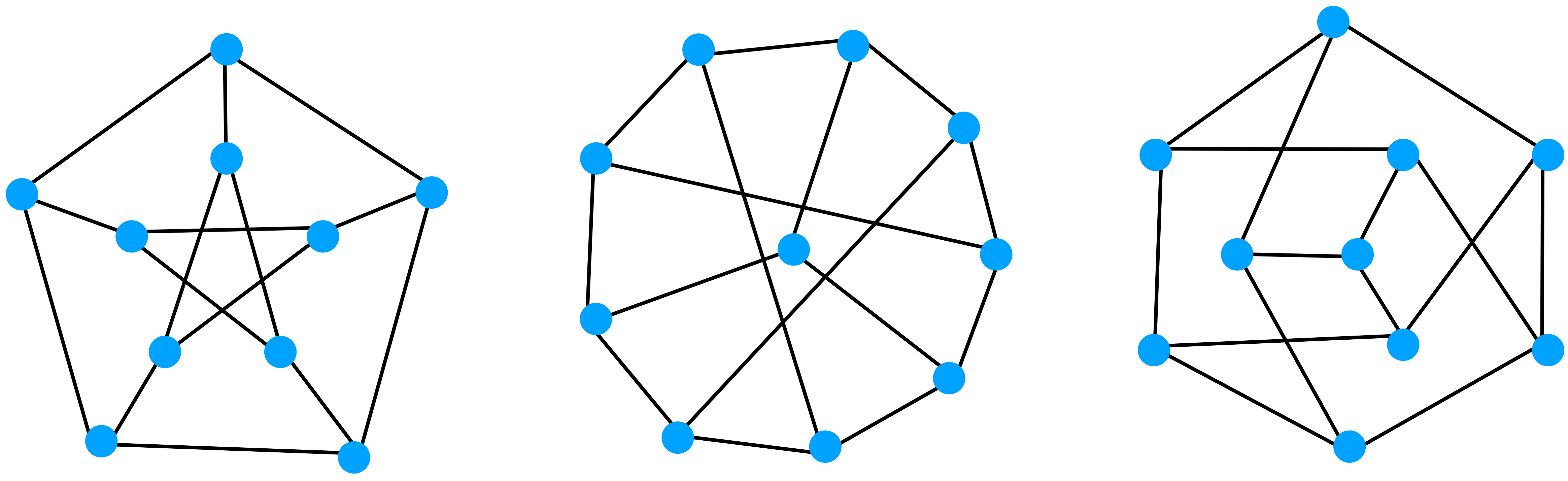
(a Graph is a Relation on Vertices! What kind of relation is it?)

Graphs… how the vertices are positioned does not matter (visual representation…)



G={{1,2},{2,3}, {3,4}, {4,1}}

Graphs… how the vertices are positioned does not matter

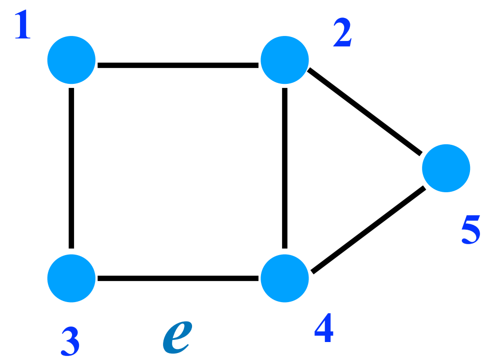


Petersen graph

Graphs: main concepts

$G = G(V,E)=(V,E)$

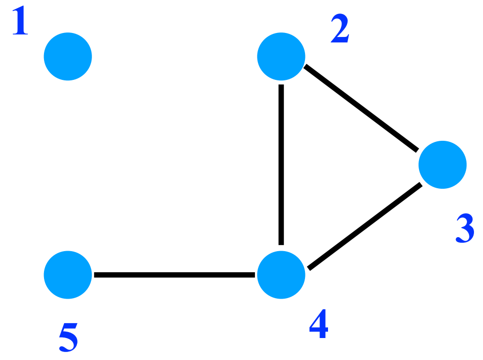
$e = \{u, v\}, u, v \in V;\ e \in E$



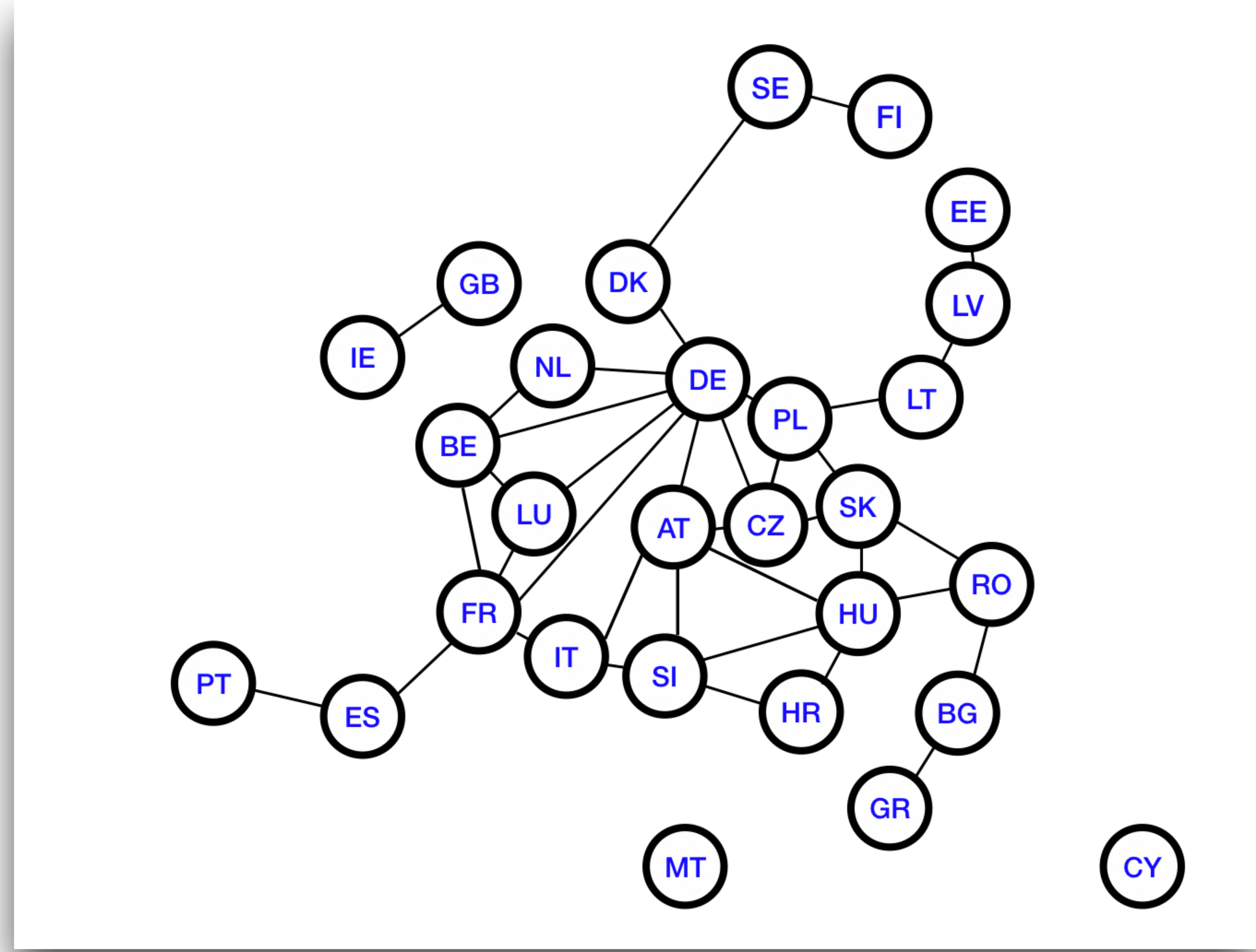
- e **connects** u and v
- **adjacency** - between two vertices (1,2)
- **incidence** - between edge and vertex [(*1,e*) - not
  (3,e) - are]
- **neighbour**(hood)

Vertex degree (degree of vertex v): number of incident edges : *deg(v)*

Isolated vertex = degree 0



Practice



Try it out:
isolated?
degree of NL?
Max degree?

Handshaking lemma (or sum-degree formula)

Theorem 8.1. The sum of all degrees of a graph G(V,E) is two times the number of edges:

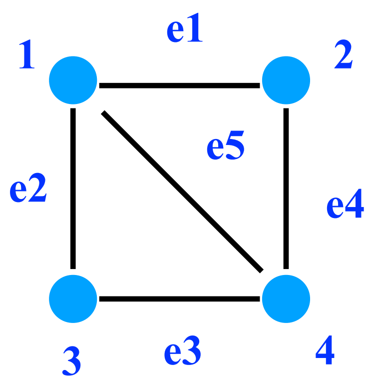
$$\sum_{v \in V} deg(v) = 2|E|$$

Degree-sum formula & Handshaking lemma

Theorem 8.1. The sum of all degrees of a graph G(V,E) is two times the number of edges:

$$\sum_{v \in V} deg(v) = 2|E|$$

Why? Consider the table (matrix) connecting edges to vertices ("incidence matrix")



$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} e1 \\ e2 \\ e3 \\ e4 \\ e5 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Handshaking lemma

Theorem 8.1. The sum of all degrees of a graph G(V,E) is two times the number of edges:

$$\sum_{v \in V} deg(v) = 2|E|$$

Corollary. The number of edges with odd degree is even.

Why?

$$\sum_{v \in V} deg(v) = \overbrace{\sum_{v \in EvenV} deg(v)}^{A} + \overbrace{\sum_{v \in OddV} deg(v)}^{B} = \underbrace{2|E|}_{\text{even!}}$$

$EvenV = \{v \mid deg(v) \text{ is even}\}$

$OddV = \{v \mid deg(v) \text{ is odd}\}$

A+B is even. A is Even

=> B must be even.

B is a sum of $|OddV|$ odd nubers

But then $|OddV|$ must be even

Note.

Let $s_1 \dots s_n$ be all odd numbers.

Let $\sum_{j=1}^{n} s_j$ be even

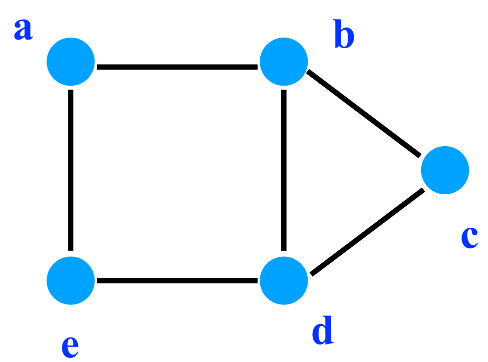
$$\underbrace{\sum_{j=1}^{n} (s_j - 1)}_{\text{even}} + \underbrace{n}_{\text{even}} \text{ is even}$$

n is even

Handshaking lemma

Theorem 8.1. The sum of all degrees of a graph G(V,E) is two times the number of edges:

$$\sum_{v \in V} deg(v) = 2|E|$$



The number of people who have shaken hands with an odd number of people is ALWAYS even…

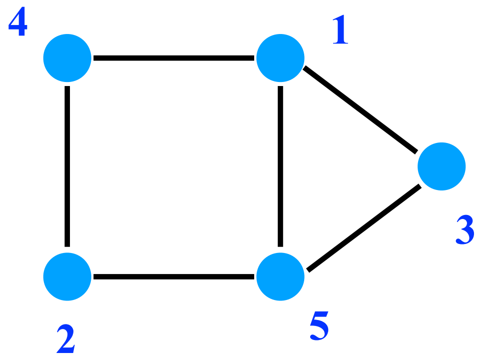
The adjacency matrix

Definition. Let *G(V,E)* be a graph with $V = \{v_1, v_2, \ldots, v_n\}$.

The adjacency matrix $A = (a_{ij})$ ***of G is an*** $n \times n$ ***defined with***

$$a_{ij} = \begin{cases} 1, & \textbf{\textit{if}}\ \{v_i, v_j\} \in E \\ 0, & \textbf{\textit{otherwise}} \end{cases}$$

adjacency matrix of a(n) (undirected, simple) graph is symmetric, with null diagonal



$$\textbf{\textit{from}} \quad \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \end{array} \overset{\textbf{\textit{to}}}{\begin{array}{c} \begin{array}{ccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}}$$

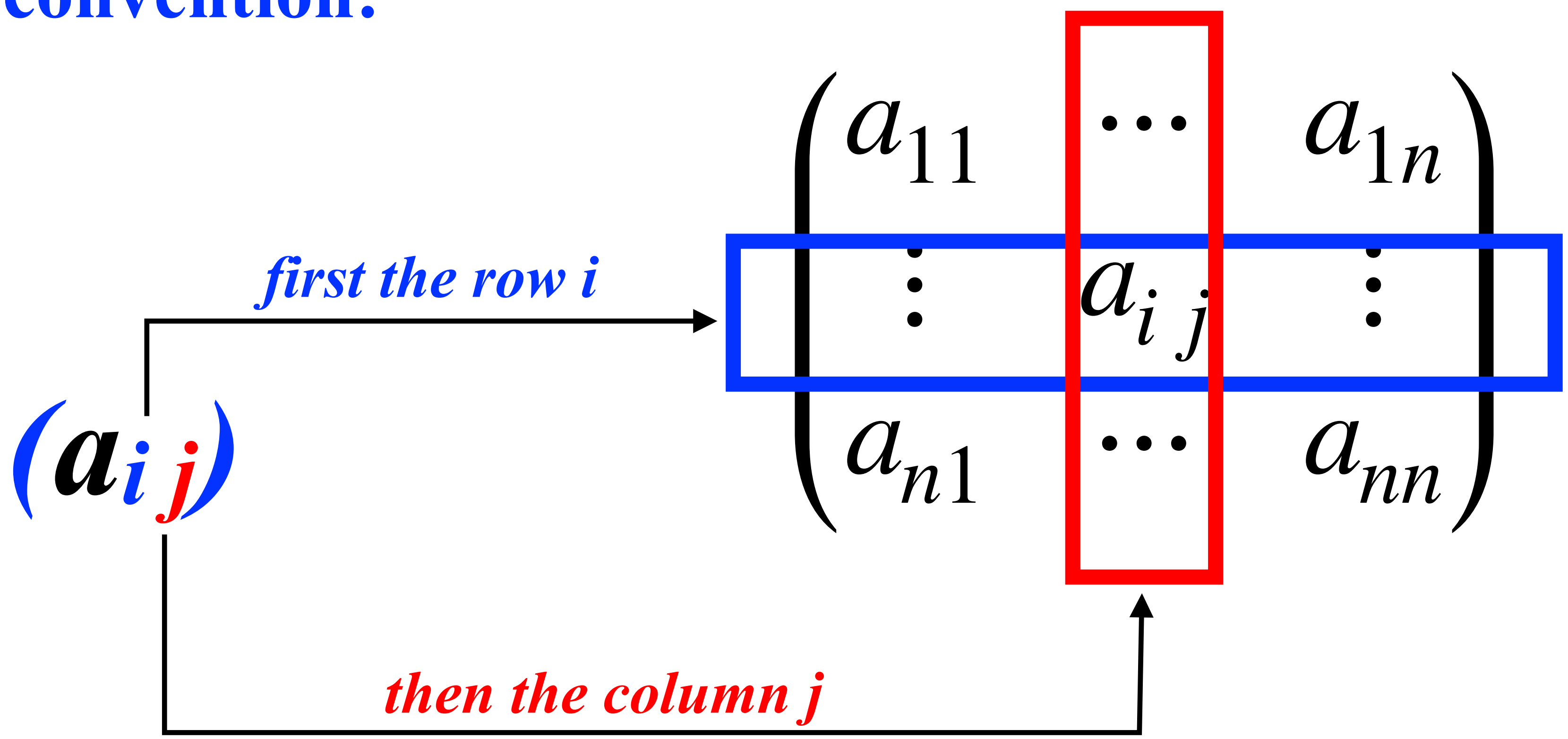
The adjacency matrix

Definition. Let *G(V,E)* be a graph with $V = \{v_1, v_2, \ldots, v_n\}$.

The adjacency matrix $A = (a_{ij})$ ***of G is an*** $n \times n$ ***defined with***

$$a_{ij} = \begin{cases} 1, & \textit{if } \{v_i, v_j\} \in E \\ 0, & \textit{otherwise} \end{cases}$$

Matrix convention:



The adjacency matrix

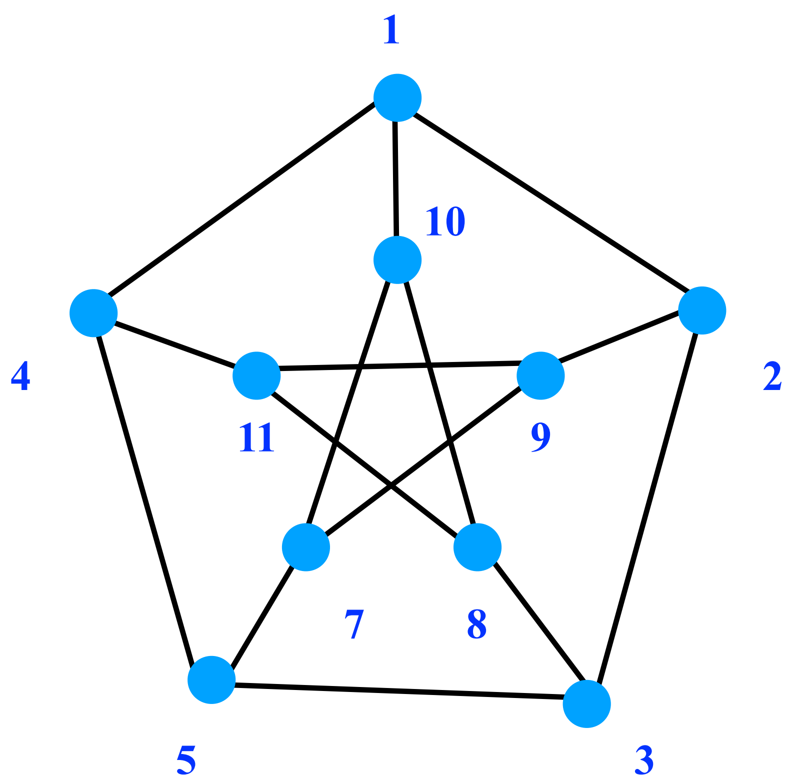
Definition. Let *G(V,E)* be a graph with $V = \{v_1, v_2, \ldots, v_n\}$.

The adjacency matrix $A = (a_{ij})$ ***of G is an*** $n \times n$ ***defined with***

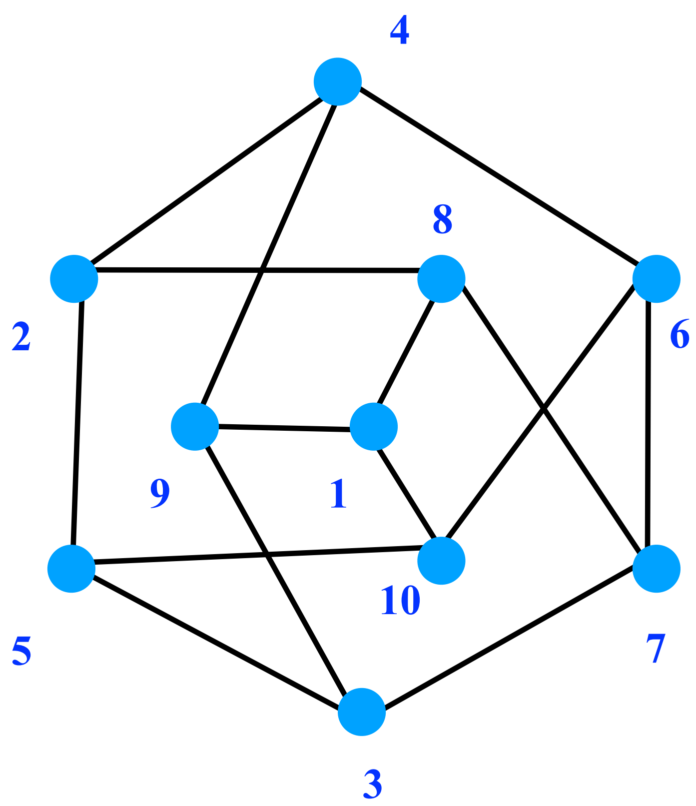
$$a_{ij} = \begin{cases} 1, & \textit{\textbf{if}}\ \{v_i, v_j\} \in E \\ 0, & \textit{\textbf{otherwise}} \end{cases}$$

Comment: no a-priori ordering on vertices… V is a *set and* not an ordered list (tuple)

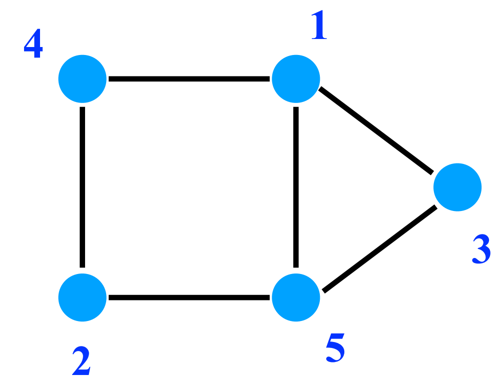
Graphs… "the same" or “isomorphic"



∃? ***isomorphism***



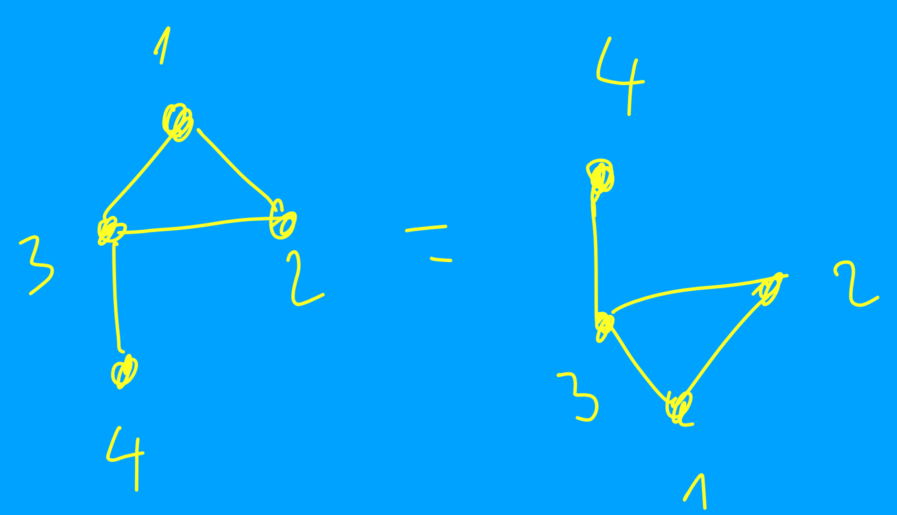
Graphs… "the same" (*equal*) or "isomorphic"

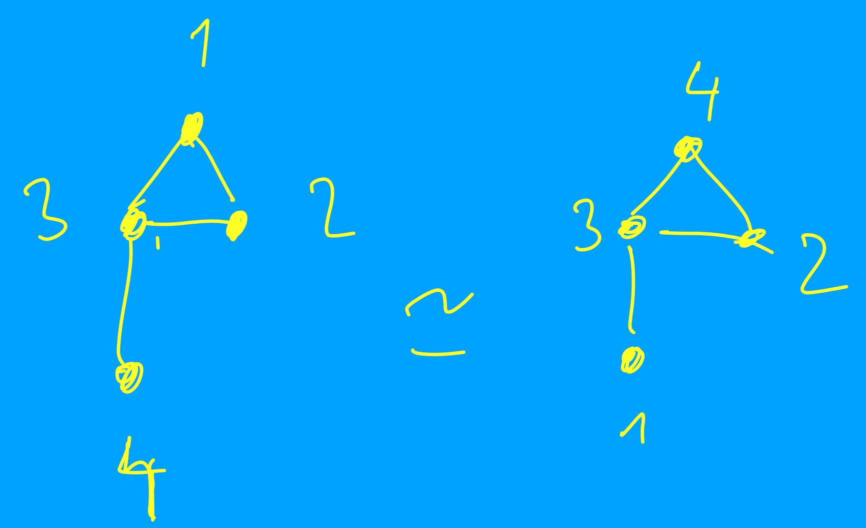


Examples

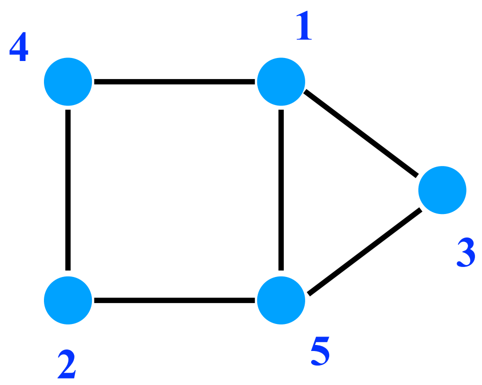
Isomorphic

Equal





Graphs… "the same" (*equal*) or "isomorphic"

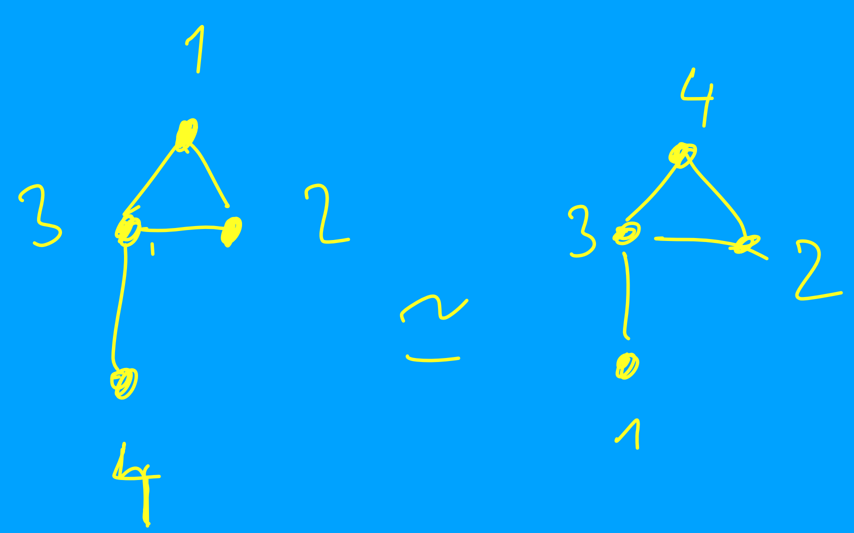


Definition. G(V,E) and G'(V',E') are isomorphic if there exists an isomorphism: $f : V \to V'$ **such that**

f is a bijection

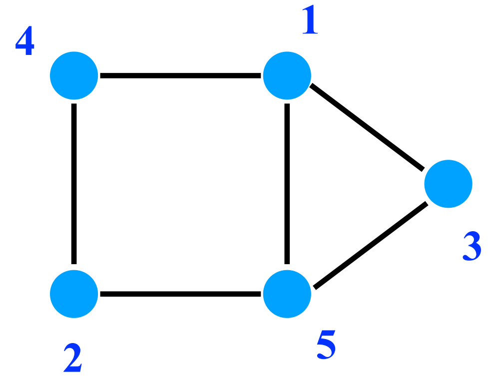
f preserves edges ($\{u, v\} \in E \Leftrightarrow \{f(u), f(v)\} \in E'$ ***)***

Example of isomorphism:



$f = \{(1,4), (2,2), (3,3), (4,1)\}$

Graphs… "the same” (*equal*) or “isomorphic"



Definition. G(V,E) and G’(V’,E’) are isomorphic if there exists a graph isomorphism: a mapping from *V to V’*

$f : V \to V'$ **such that**

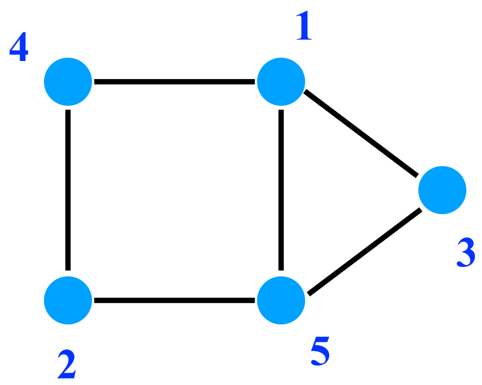
f is a bijection

f preserves edges ($\{u, v\} \in E \Leftrightarrow \{f(u), f(v)\} \in E'$ ***)***

Isomorphism: preserves *main numbers (properties)* (|V|, |E), set of degree values, set of path lengths…

3

Graphs… "the same" (*equal*) or "isomorphic"



Definition. G(V,E) and G'(V',E') are isomorphic if there exists an isomorphism: $f : V \to V'$ **such that**

***f* is a bijection**

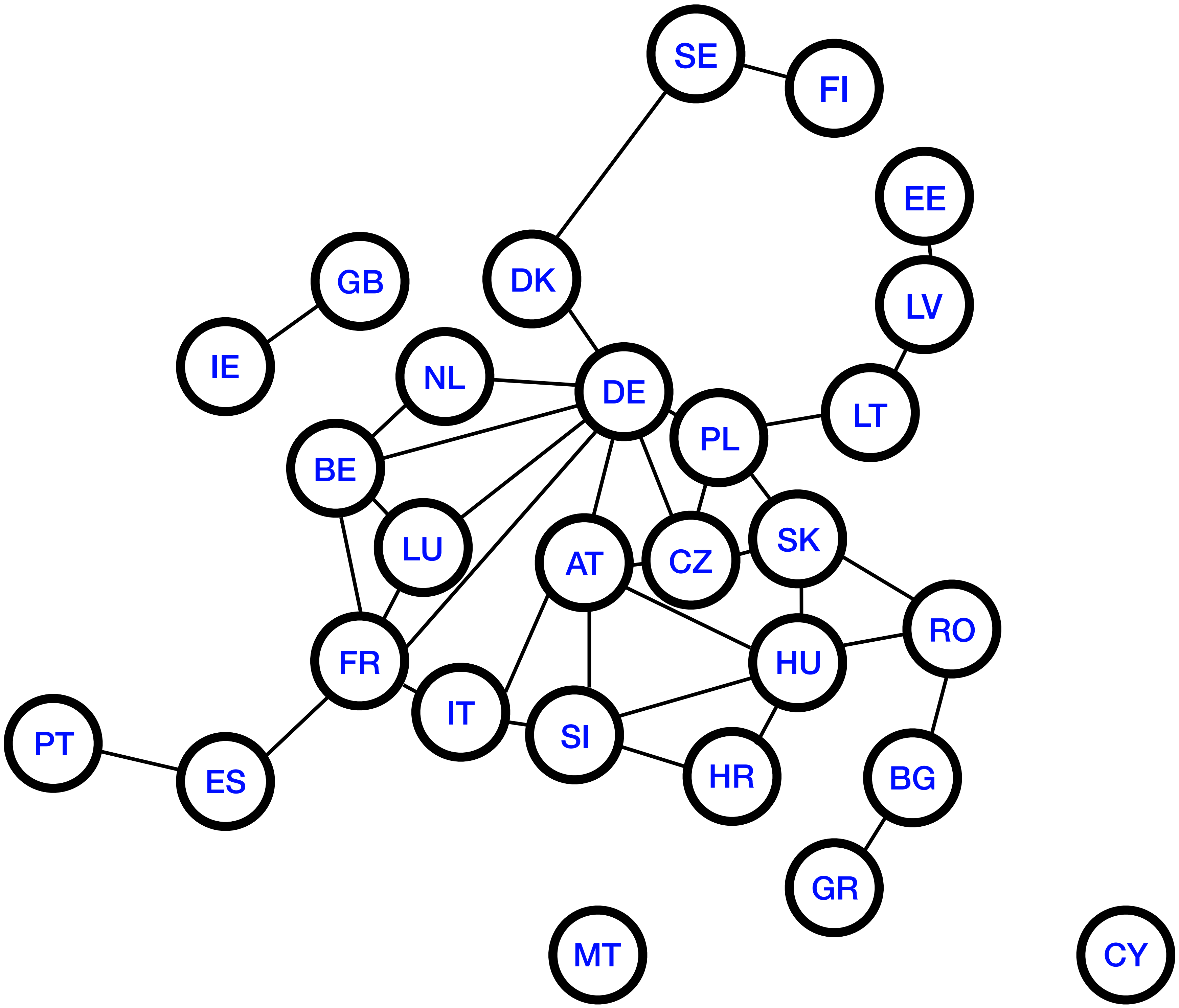
***f* preserves edges (** $\{u, v\} \in E \Leftrightarrow \{f(u), f(v)\} \in E'$ **)**

Isomorphism: preserves *main numbers* (|V|, |E),
set of degree values, set of path lengths…

3

Other relationships between graphs: homeomorphism, see Schaum..

Subgraphs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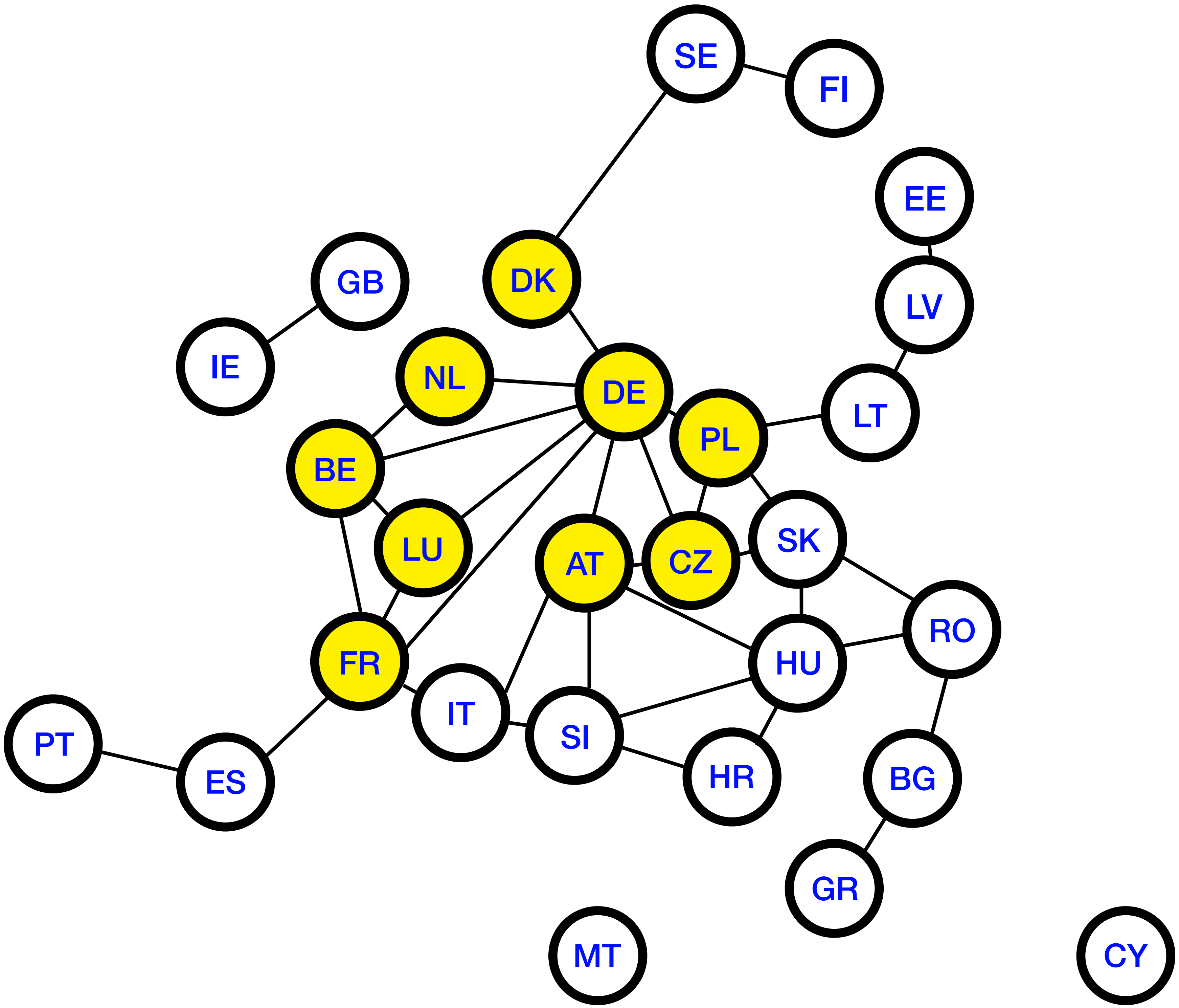
Subgraphs



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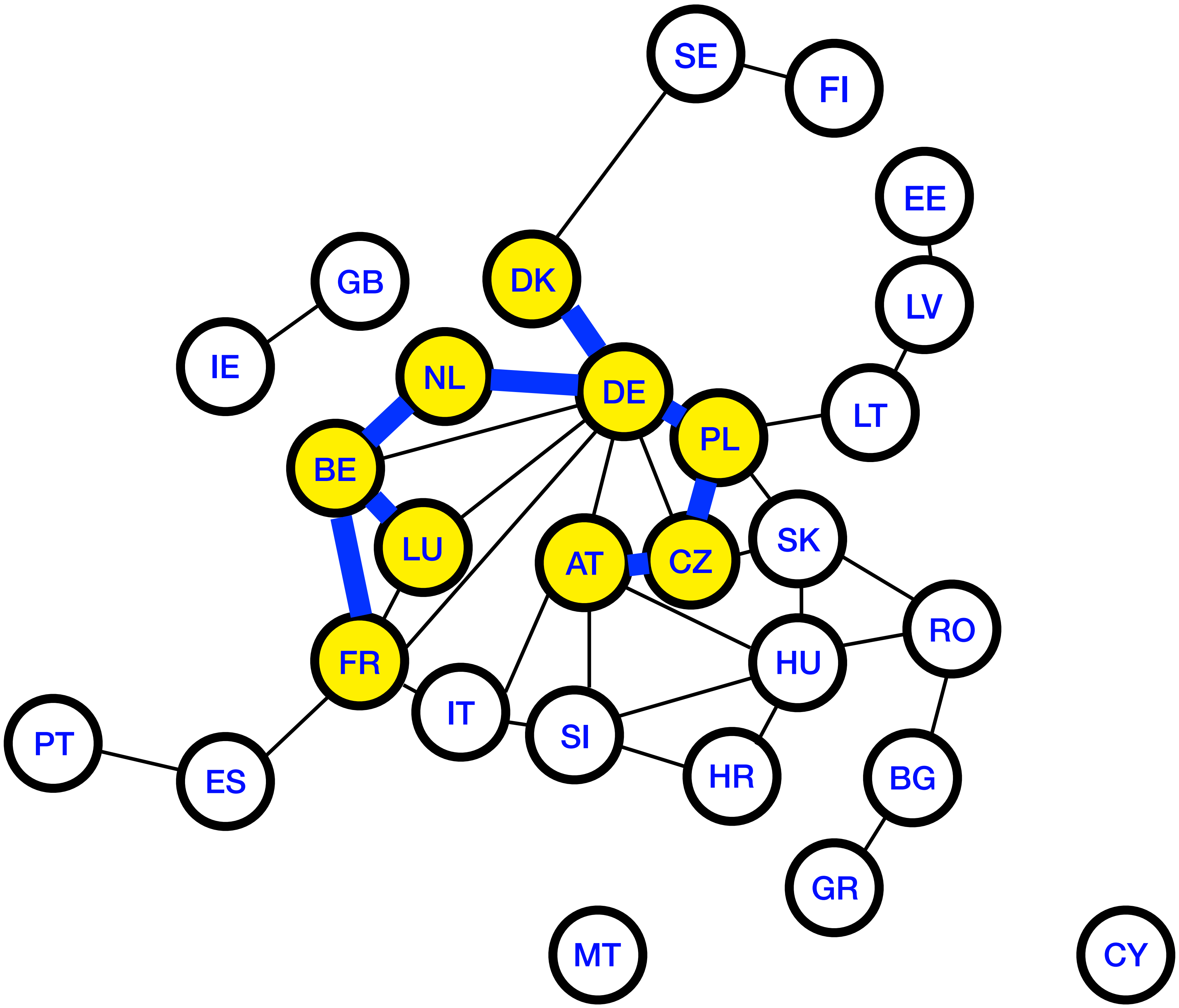
Subgraphs



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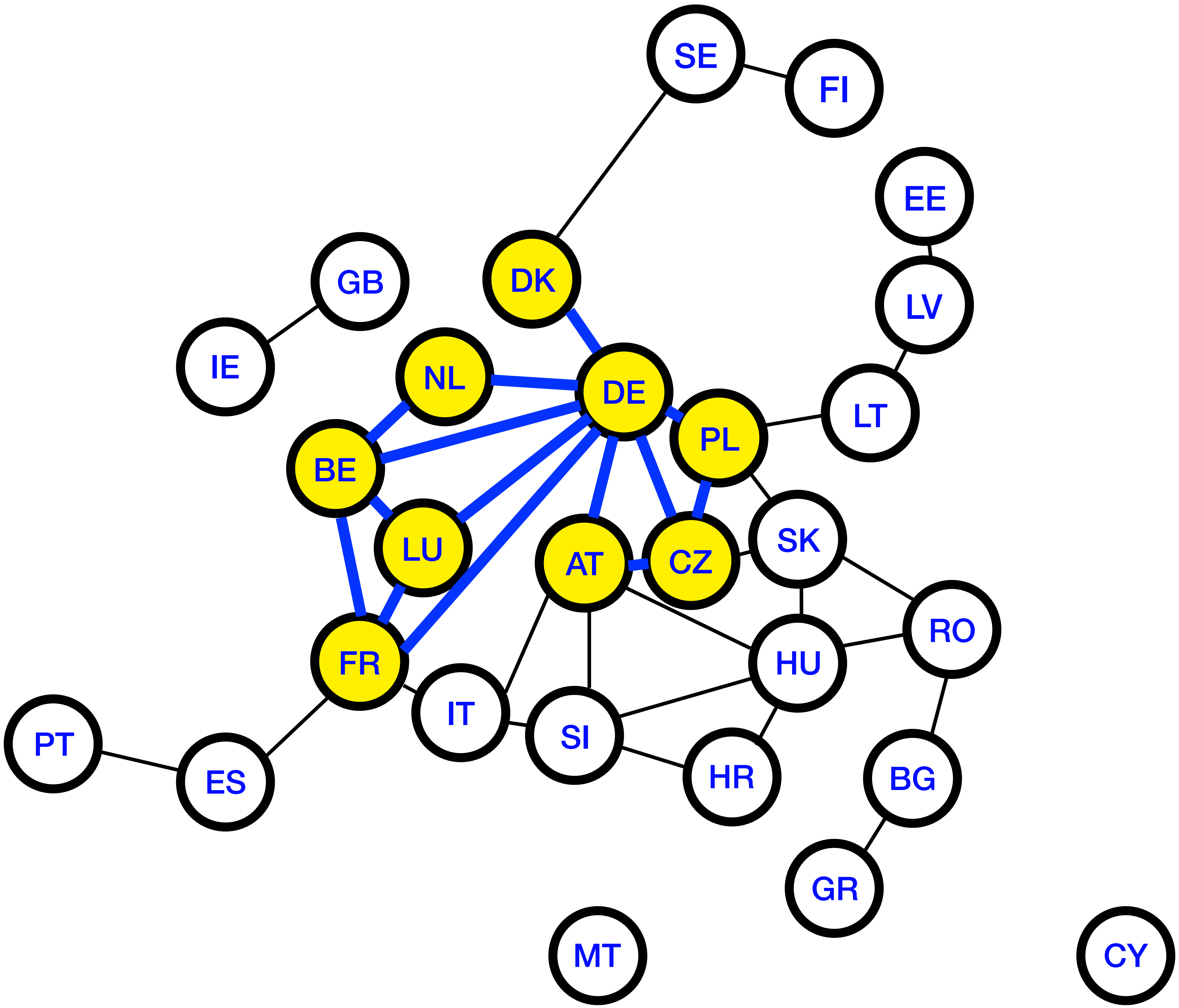
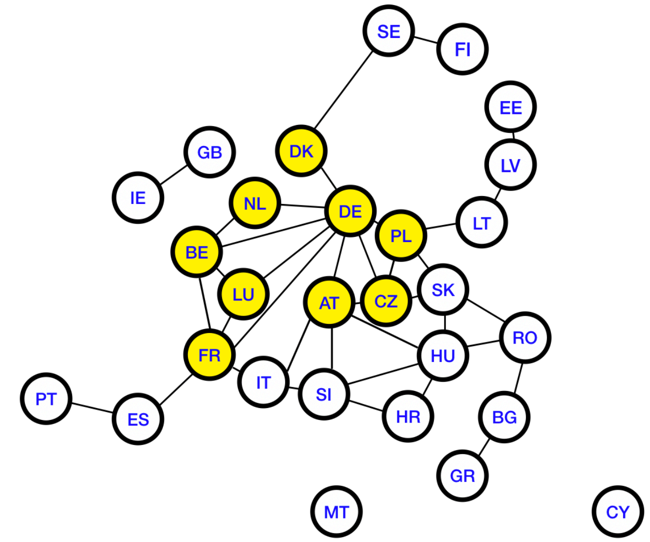
Induced subgraph



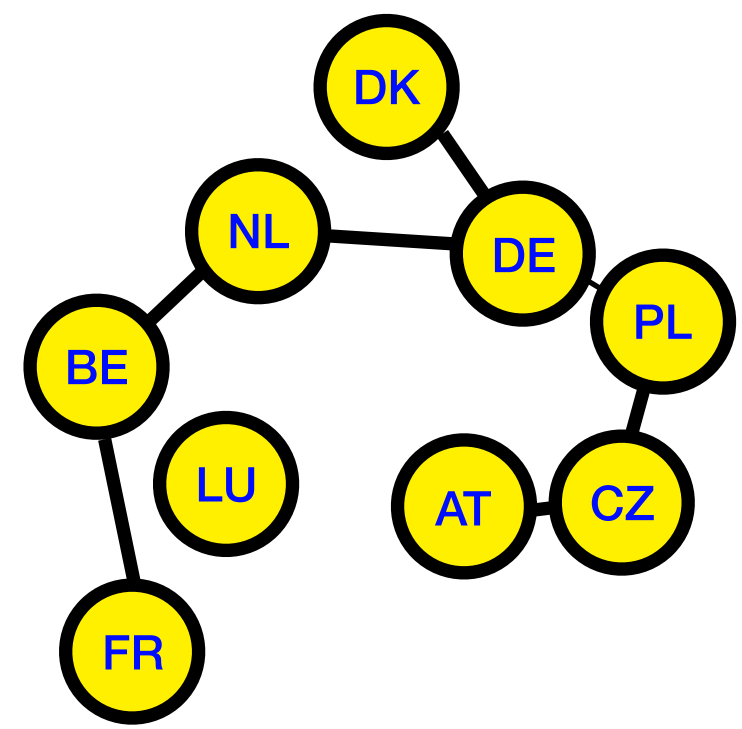
Subgraphs general



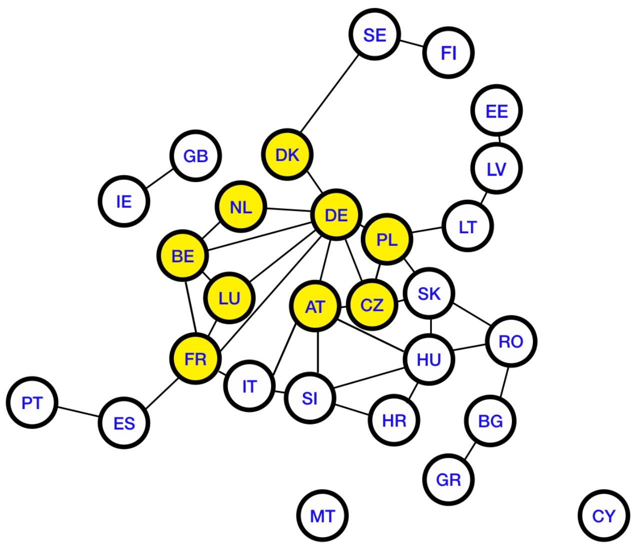
Definition. Given graph G(V,E) the graph G'(V',E') over $V' \subseteq V$, is a subgraph of G if

$$v, u \in V', \{v, u\} \in E' \Rightarrow \{v, u\} \in E\,.$$

subgraph: subset of vertices, no new edges
(but some may be "lost")



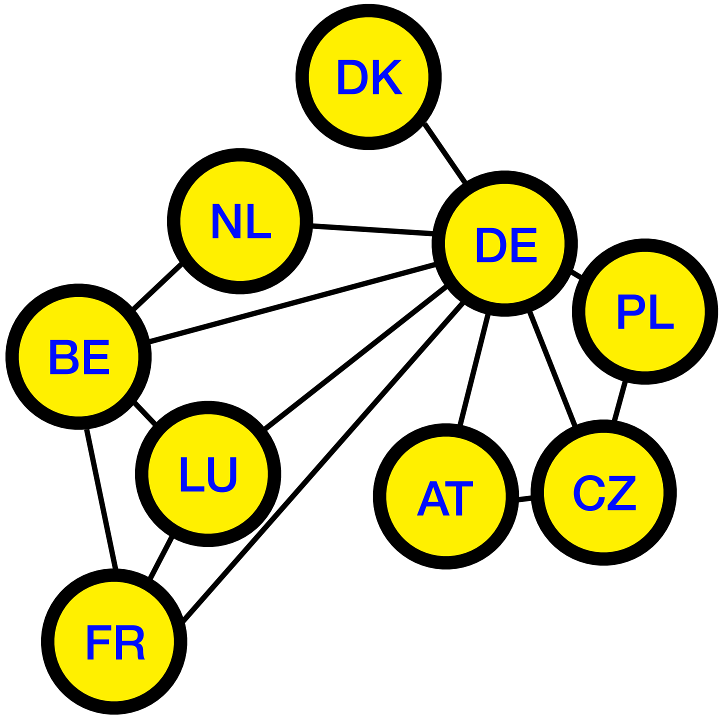
Subgraphs: (vertex) induced



Definition. Given graph G(V,E) the graph G'(V',E') over $V' \subseteq V$, is a vertex induced subgraph of G if

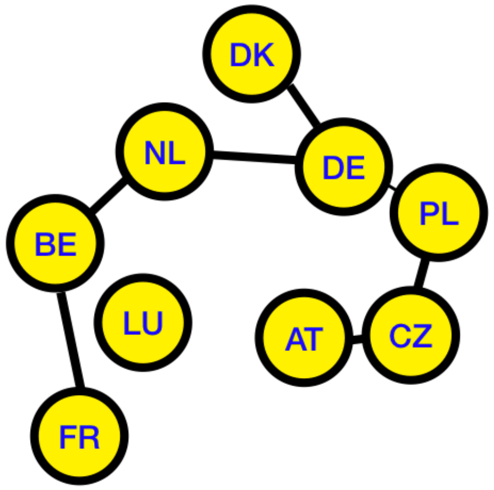
$$v, u \in V', \{v, u\} \in E' \Leftrightarrow \{v, u\} \in E.$$

Induced subgraph: all edges are "inherited"

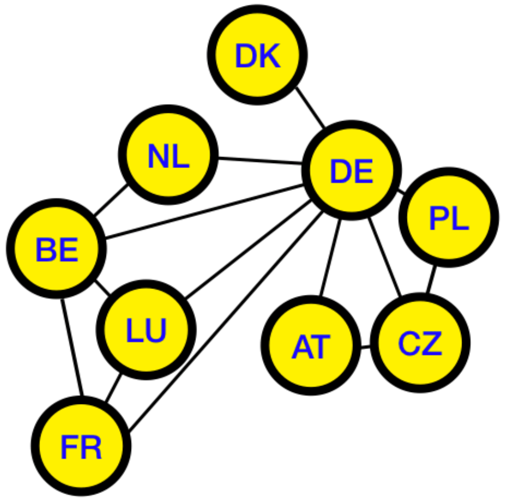


Subgraphs: (vertex) induced and general

Definition. Given graph G(V,E) the graph G'(V',E') over $V' \subseteq V$, is a **subgraph** of G if $v, u \in V', \{v, u\} \in E' \Rightarrow \{v, u\} \in E$.



Definition. Given graph G(V,E) the graph G'(V',E') over $V' \subseteq V$, is a vertex **induced subgraph** of G if $v, u \in V', \{v, u\} \in E' \Leftrightarrow \{v, u\} \in E$.



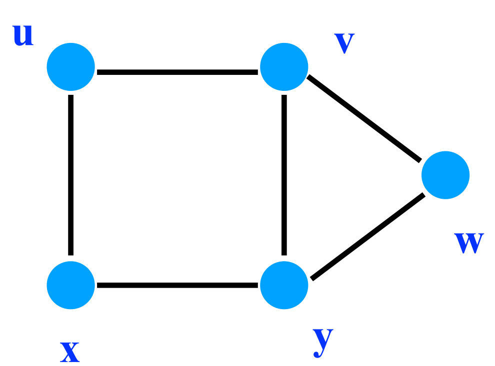
Question: what is the relationship between subgraphs and induced subgraphs over the same set of vertices?

Subgraphs: edge and vertex removal

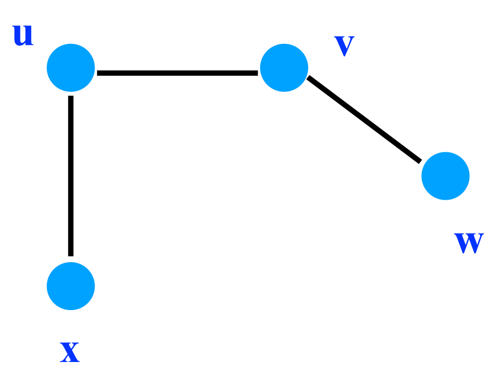
Notation. Given graph *G=G(V,E)*, for $v \in V$,
with $G - v$ (also $G - \{v\}$) we denote the induced subgraph over $V - \{v\}$.
In other words, the graph obtained by removing the vertex *v and all the incident edges*.

Notation. Given graph *G=G(V,E)*, for $e \in E$,
with $G - e$ (also $G - \{e\}$) we denote the subgraph $(V, E - \{e\})$.
In other words, the graph obtained by removing the edge *e*.

G=G(V,E)



G-y



Subgraphs: edge and vertex removal

Notation. Given graph G=$G(V,E)$, for $v \in V$,
with $G - v$ (also $G - \{v\}$) we denote the induced subgraph over $V - \{v\}$.
In other words, the graph obtained by removing the vertex v *and all the incident edges*.

Notation. Given graph G=$G(V,E)$, for $e \in E$,
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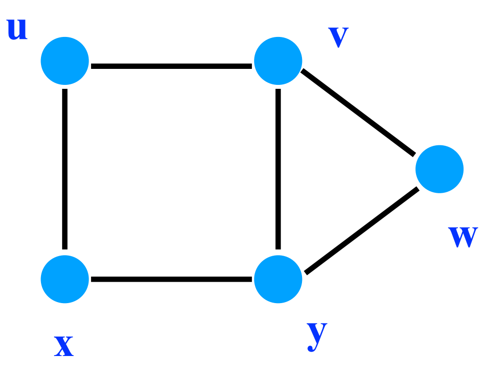
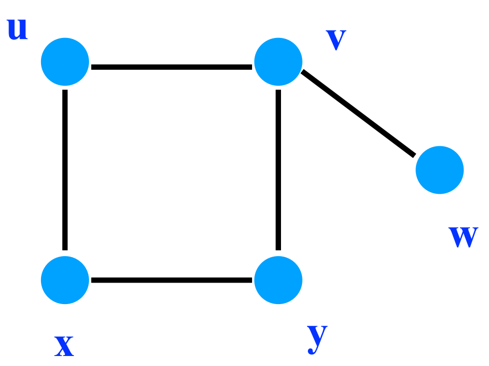
In other words, the graph obtained by removing the edge e.

G=$G(V,E)$

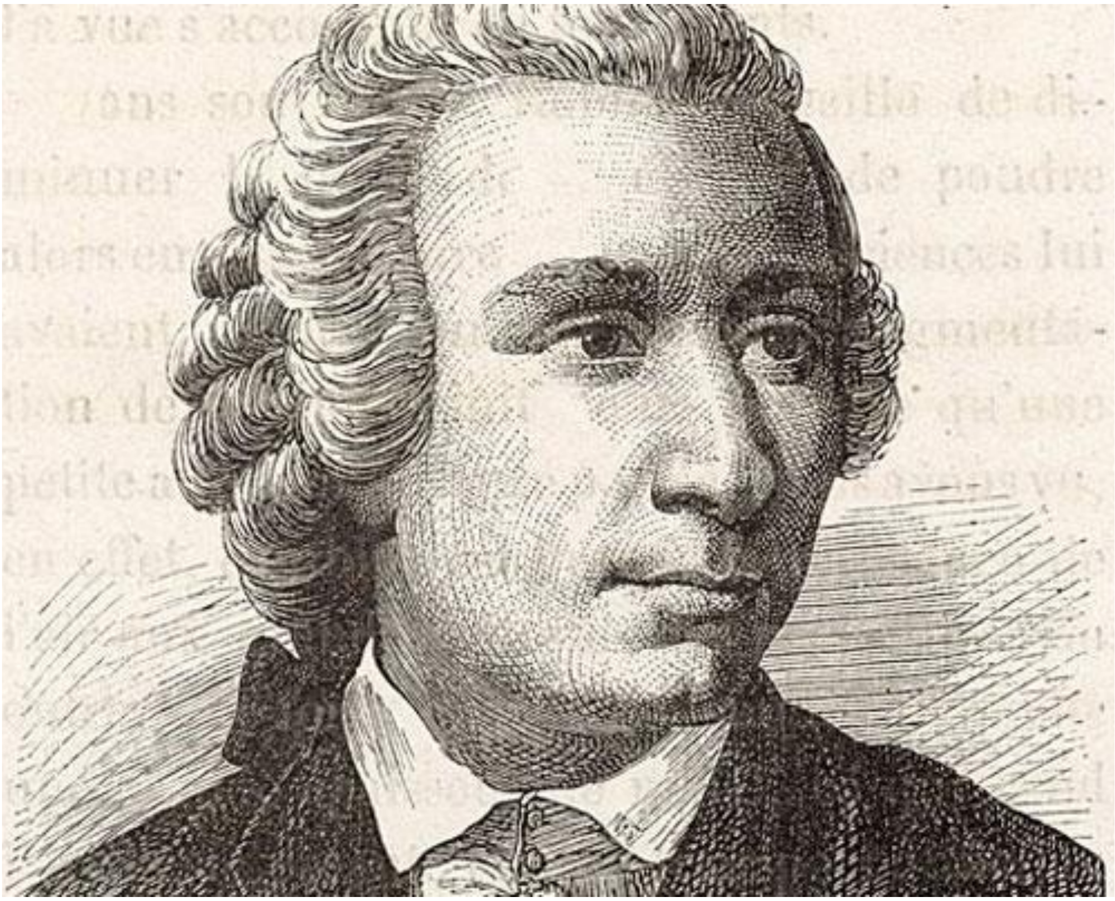


G-$\{y, w\}$



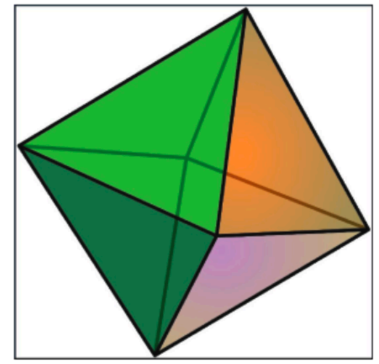
Stopped here.

Math culture: Euler, Seven Bridges of Königsberg and beginnings of graph theory.

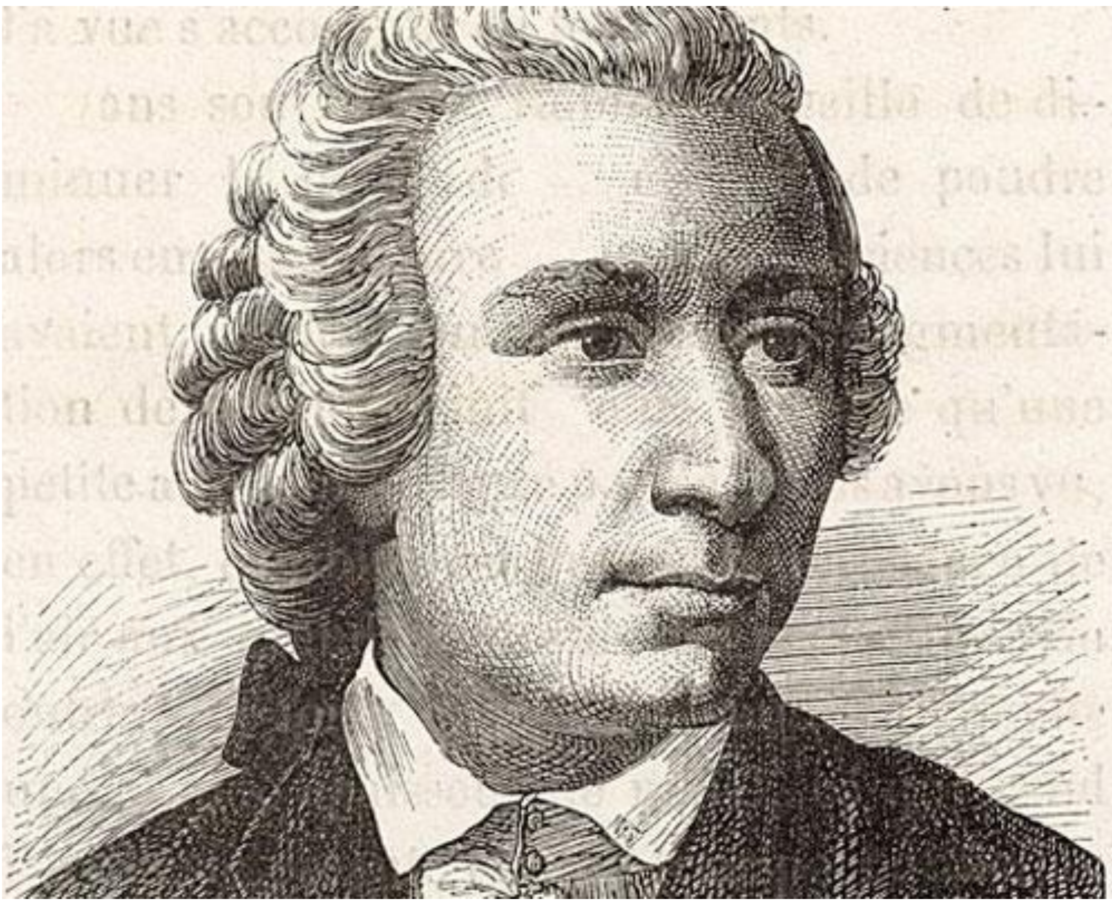


"Read Euler, read Euler, he is the master of us all.”

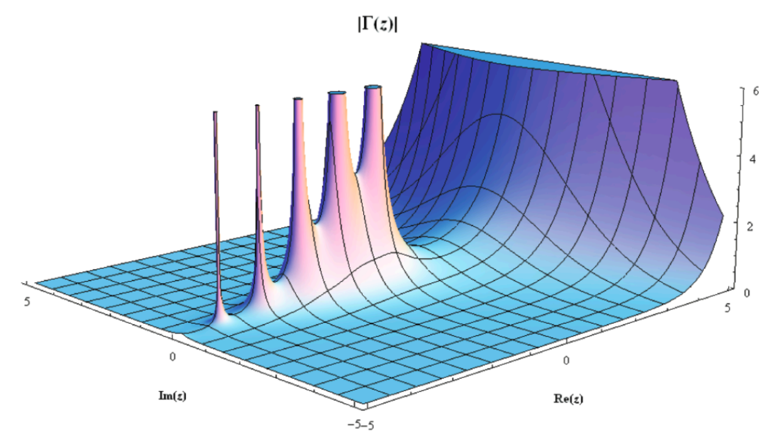
Polyhedral formula : $V - E + F = 2$



$$\prod_{p\in\mathcal{P}} \frac{1}{1-1/p^s} = \zeta(s), \quad s > 1,$$



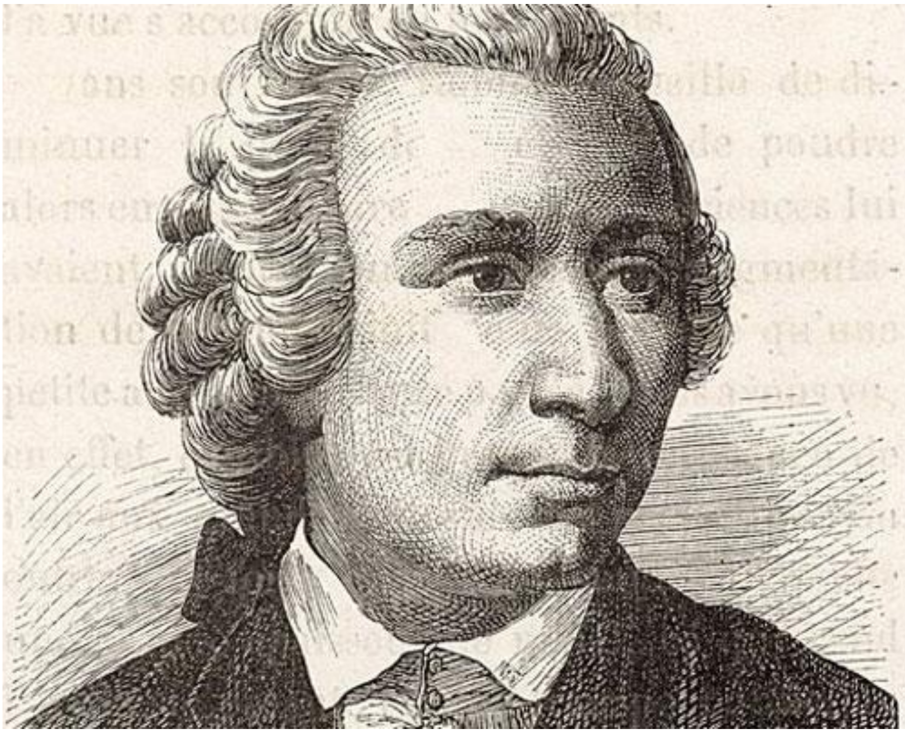
$$x! = \int_0^\infty \exp(-t)t^x \mathrm{d}t = \Gamma(x+1)$$

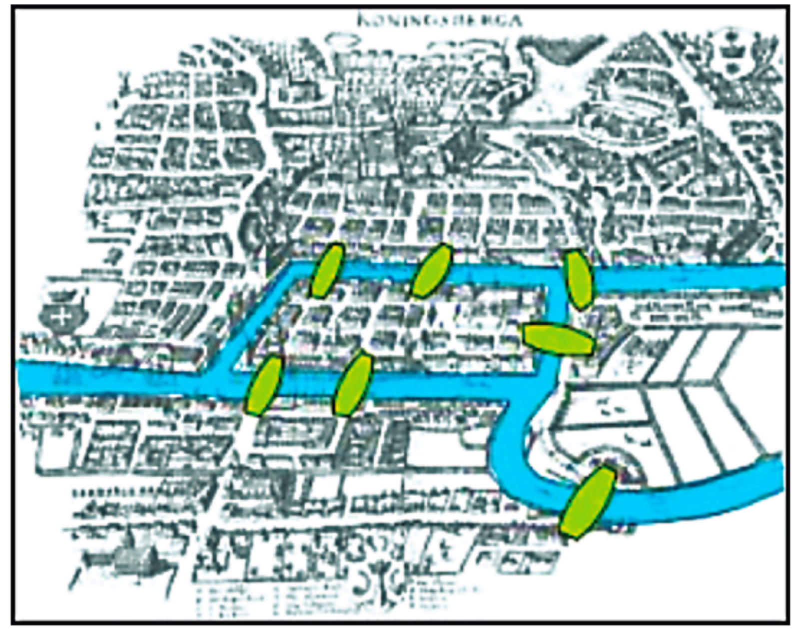


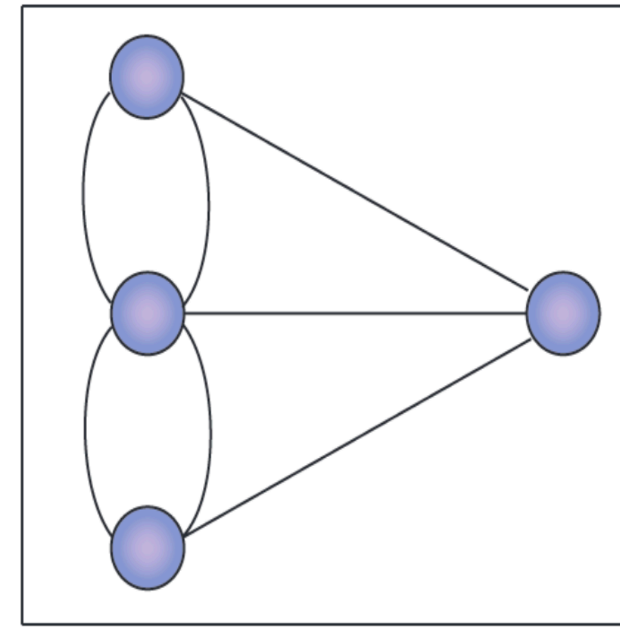
$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{n}-\ln n\right) = 0.57721\ldots.$$

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$

"Read Euler, read Euler, he is the master of us all." - Laplace





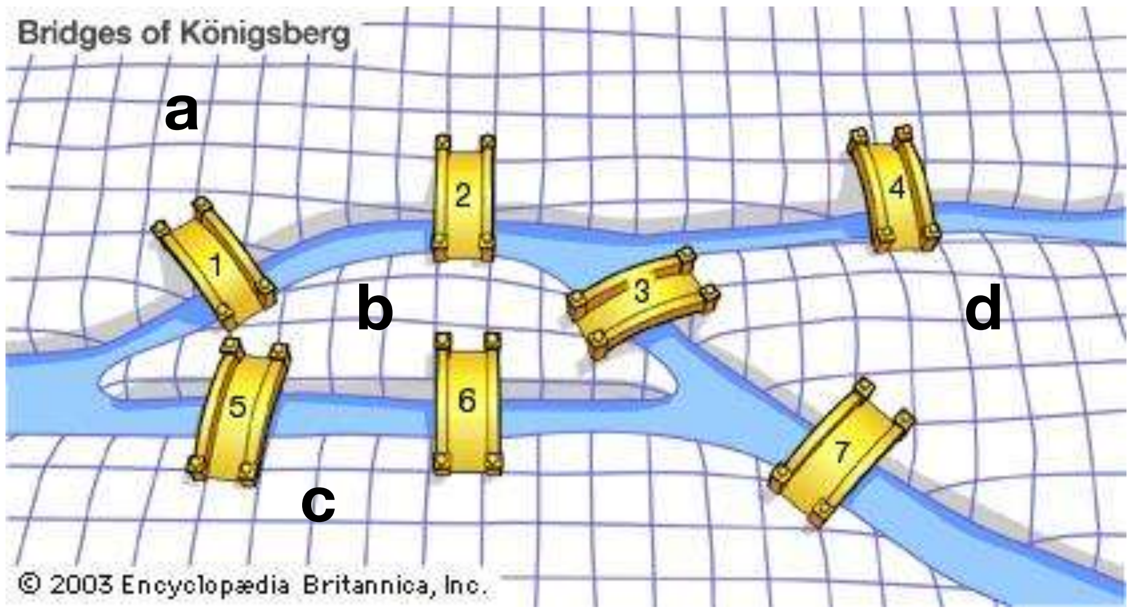


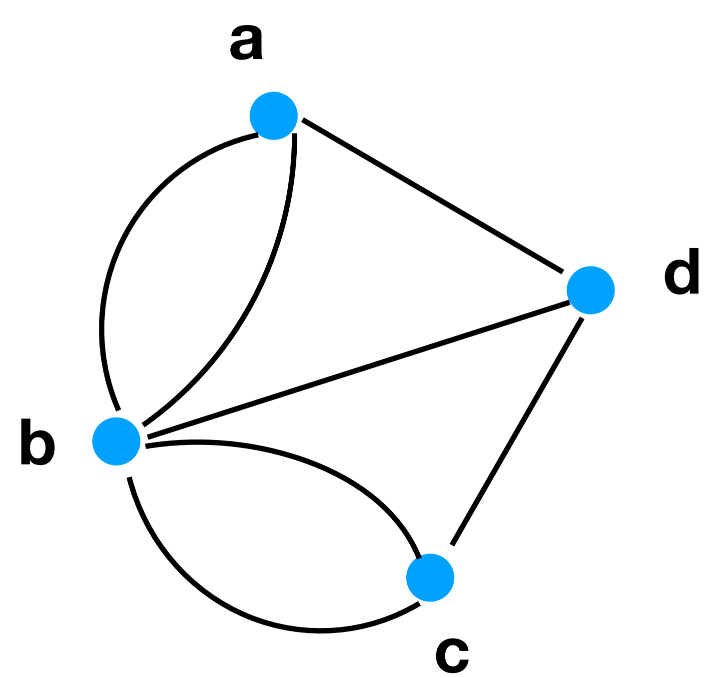
Can we cross that, and every other brige… but only once?

“This question is so banal, but seemed to me worthy of attention in that [neither] geometry, nor algebra, nor even the art of counting was sufficient to solve it.”

Seven Bridges of Königsberg

Mutigraph!



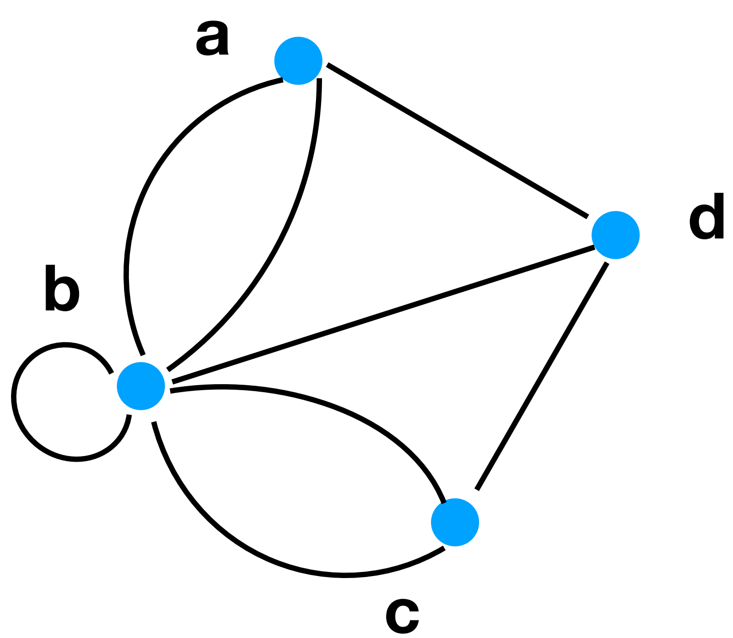


$$
\begin{array}{c c}
 & \begin{array}{cccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \end{array} \\
\begin{array}{c} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{array} & \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}
\end{array}
$$

Seven Bridges of Königsberg

Mutigraph!

can have multiple lines
and loops



Definition. A graph *G* is an ordered pair *(V,E)* where
- *$V = V(G)$ is the set of vertices*
- *$E=E(G)$ is the set of edges*

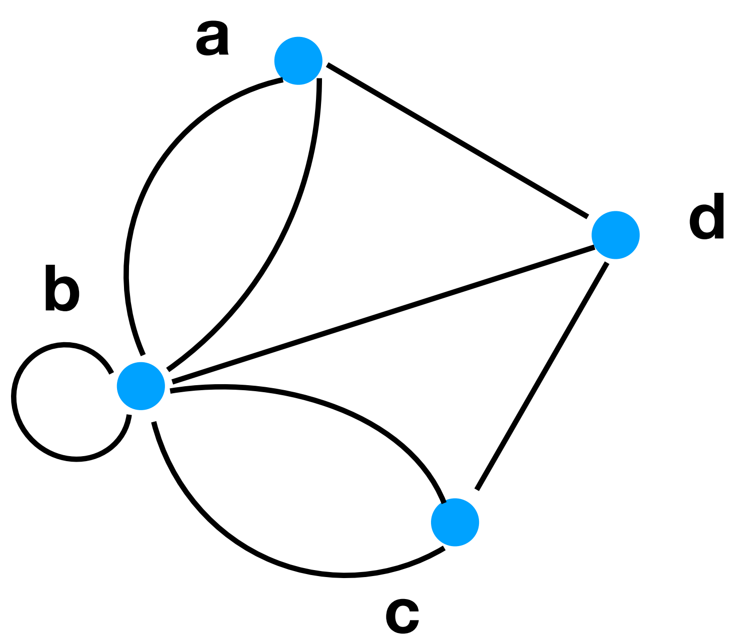
$$\begin{array}{c c} & \begin{array}{cccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \end{array} \\ \begin{array}{c} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{array} & \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & \boxed{1} & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} \end{array}$$

Why is this not a good definition?

Seven Bridges of Königsberg

Mutigraph!

can have multiple lines
and loops



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & \boxed{1} & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} \end{array}$$

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Seven Bridges of Königsberg

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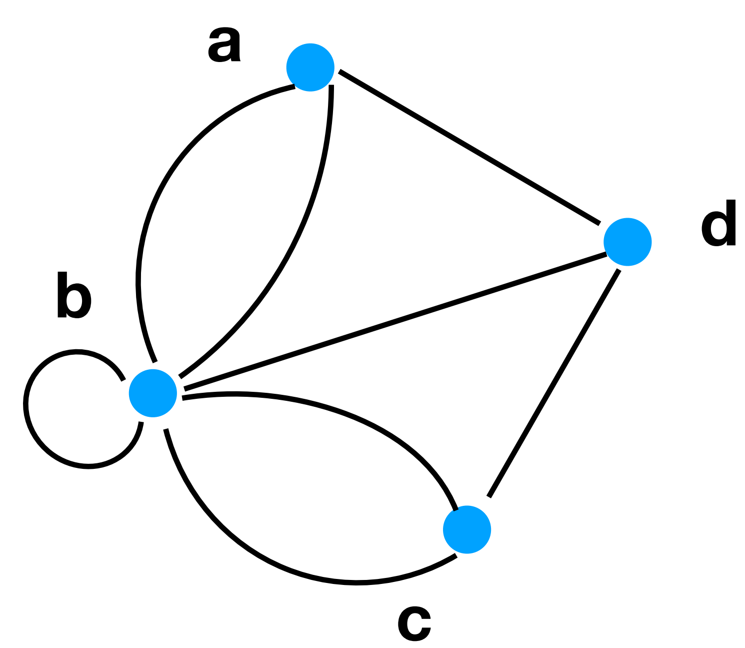

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$$
\begin{array}{c c}
 & \begin{array}{cccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \end{array} \\
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\end{array}
$$

Set has to be substituted with a multiset….

For now..
For now… we stick to intuition

A moment’s thought

concepts, "mathematical objects”, formalization, modelling, abstraction…

Bridges in reality, bridges on a map

v.s.

Graph — abstract concept

v.s.

graph (picture on paper),
set-theoretical notation for a graph,
adjacency matrix,
incidence matrix,

v.s.

graph as a binary relation
function (e.g. characteristic)

Walking the graph…

Paths

Path: a sequence $v_1, e_1, v_2, e_2 \ldots, v_n$, **with** $e_k = \{v_k, v_{k+1}\}$

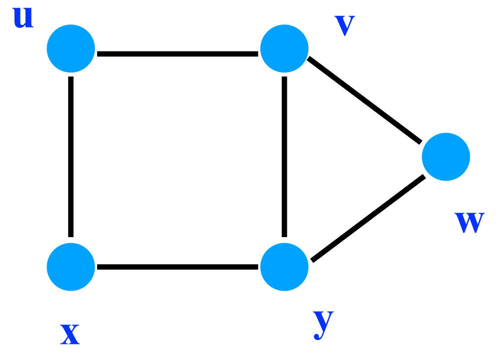
Lenght of path = number of edges in path (n)

We say: path from v_1 **to** v_2

Closed path: $v_1 = v_n$

In graphs (not multigraphs), vertices suffice:

$v_1, e_1, v_2, e_2 \ldots, v_n \rightarrow (v_1, v_2, \ldots, v_n)$



$u \rightarrow v \rightarrow w \rightarrow v \rightarrow u$

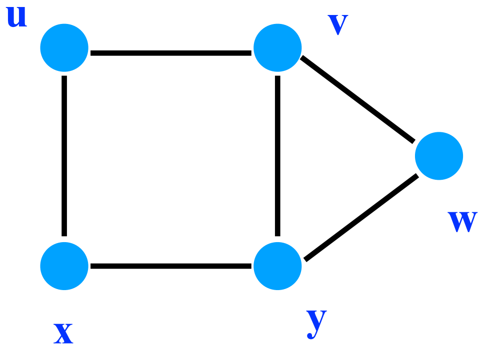
$v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_n$

$v_1 v_2 \ldots v_n$

Paths: more concepts

simple path (*walk*): distinct vertices

trail: path with distinct edges



Closed path: $v_1 = v_n$

cycle: closed path of length > 2, all distinct vertices except first/last (essentially, closed simple path)

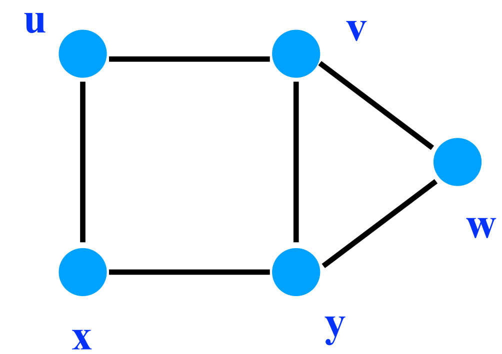
circuit: closed path, vertices may repeat, but edges cannot.

Same thing:

simple path:
Vertices may not repeat.
Edges may not repeat.

trail:
Vertices may repeat.
Edges cannot repeat. (a s.p. is a special trail.)

simple path:



$u \to v \to w \to y \to x$

cycle: closed path of length > 2
Vertices cannot repeat. Edges cannot repeat (Closed)

circuit:
(>2) Vertices may repeat. Edges cannot repeat (Closed)

Same thing:

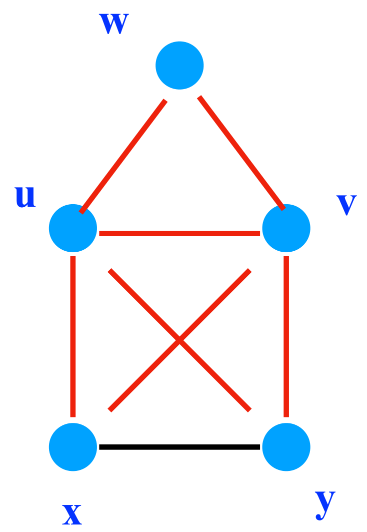
simple path:
Vertices may not repeat.
Edges may not repeat.

trail:
Vertices may repeat.
Edges cannot repeat.

cycle: closed path of length > 2
Vertices cannot repeat. Edges cannot repeat (Closed)

circuit:
(>2) Vertices may repeat. Edges cannot repeat (Closed)

trail (& not simp. path)



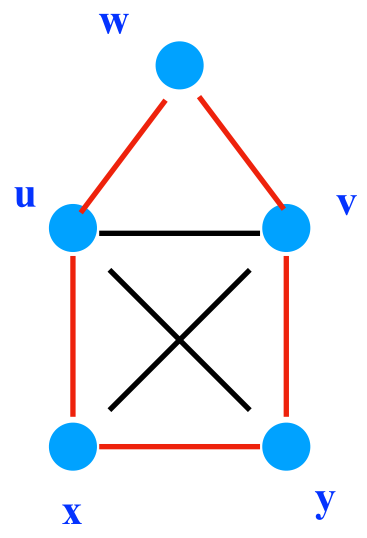
$$y \to v \to w \to u \to x \to v \to u \to y$$

Same thing:

simple path:
Vertices may not repeat.
Edges may not repeat.

trail:
Vertices may repeat.
Edges cannot repeat.

cycle:



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Same thing:

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Vertices may repeat.
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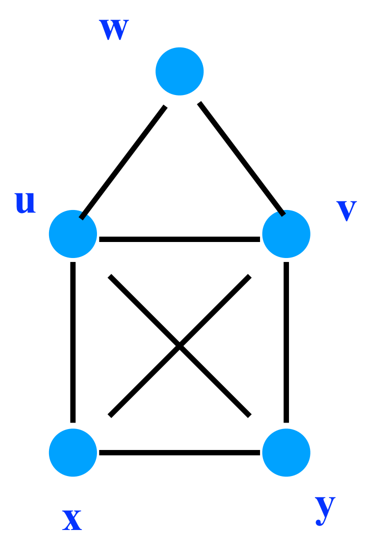
circuit (& not a cycle)



$$y \to v \to w \to u \to x \to v \to u \to y \to x$$

cycle: closed path of length > 2
Vertices cannot repeat. Edges cannot repeat (Closed)

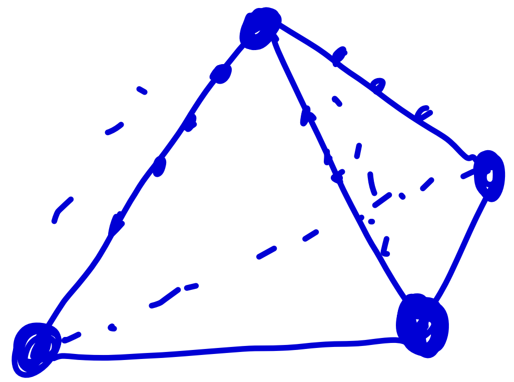
circuit:
(>2) Vertices may repeat. Edges cannot repeat (Closed)

Paths

Cycle and circuit not mutually exclusive.

Terminology *not* fully consistent between bodies of work. *Must be consistent within one work*
Check (and *give*) definitions

A small exercise

T

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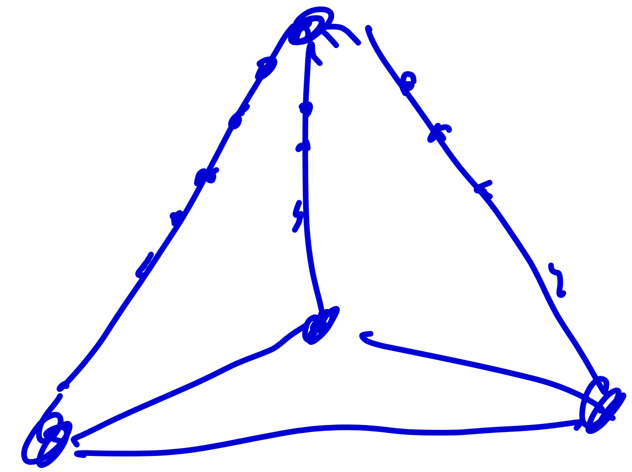
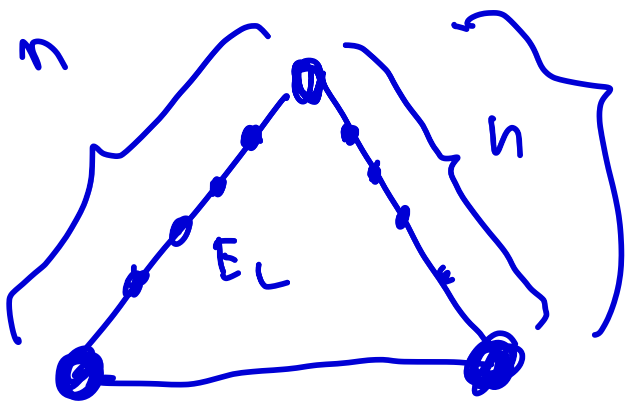


"tetrahedron"

Face F =


How many vertices?

$$|F| = |E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

$$= 2n - 1$$

$$|T| = |F_1| + |F_2| + |F_3| - |F_1 \cap F_2| - |F_2 \cap F_3| - |F_1 \cap F_3|$$

$$+ |F_1 \cap F_2 \cap F_3| =$$

$$= 3(2n-1) - 3n + 1$$

$$= 6n - 3 - 3n + 1 = 3n - 2.$$