Lecture 7

Functions

Chapter 3 Schaum
up to 3.8, 3.9

Function;

synonyms: map, mapping, transformation

associates elements of one set to elements of another

Local Time and Weather Around the World Cities Shown: Most Popular (143) Sort By: City °C

Accra	Mon 07:59	24 °C	Dublin *	Mon 08:59	11 °C	Nairobi	Mon 10:59	17 °C
Addis Ababa	Mon 10:59	15 °C	Edmonton *	Mon 01:59	13 °C	Nassau *	Mon 03:59	27 °C
Adelaide	Mon 17:29	11 °C	Frankfurt *	Mon 09:59	17 °C	New Delhi	Mon 13:29	30 °C
Algiers	Mon 08:59	28 °C	Guatemala City	Mon 01:59	19 °C	New Orleans *	Mon 02:59	28 °C
Almaty	Mon 13:59	33 °C	Halifax *	Mon 04:59	15 °C	New York *	Mon 03:59	22 °C
Amman *	Mon 10:59	28 °C	Hanoi	Mon 14:59	36 °C	Oslo *	Mon 09:59	17 °C
Amsterdam *	Mon 09:59	17 °C	Harare	Mon 09:59	16 °C	Ottawa *	Mon 03:59	16 °C
Anadyr	Mon 19:59	11 °C	Havana *	Mon 03:59	23 °C	Paris *	Mon 09:59	16 °C
Anchorage *	Sun 23:59	17 °C	Helsinki *	Mon 10:59	20 °C	Perth	Mon 15:59	25 °C
Ankara	Mon 10:59	24 °C	Hong Kong	Mon 15:59	34 °C	Philadelphia *	Mon 03:59	21 °C
Antananarivo	Mon 10:59	20 °C	Honolulu	Sun 21:59	28 °C	Phoenix	Mon 00:59	35 °C

tables…

Letter Grade	Number Grade
A+	97
A	95
A-	92
B+	87
B	85
B-	82
C+	77
C	75
C-	72
D+	67
D	65
D-	62
F	50

Function;

synonyms: map, mapping, transformation

associates elements of one set to elements of another

{ A, B, C }

0	0	0	$\to$	{ }
0	0	1	$\to$	{ C }
0	1	0	$\to$	{ B }
0	1	1	$\to$	{ B, C }
1	0	0	$\to$	{ A }
1	0	1	$\to$	{ A, C }
1	1	0	$\to$	{ A, B }
1	1	1	$\to$	{ A, B, C }

binary counting $\mathscr{P}(\{A, B, C\})$

Function;

synonyms: map, mapping, transformation

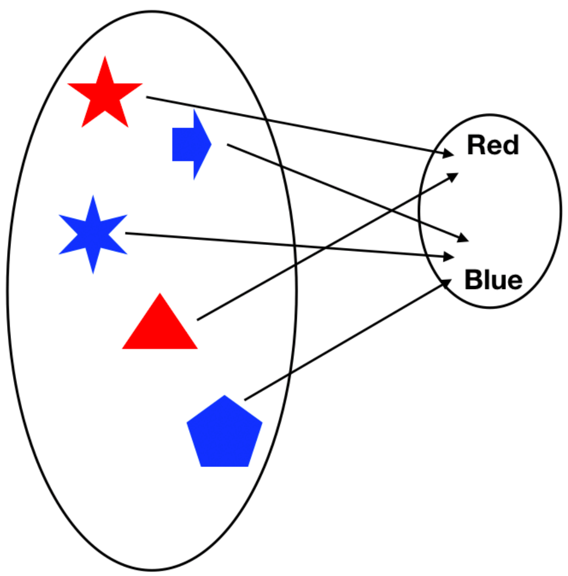
associates elements of one set to elements of another

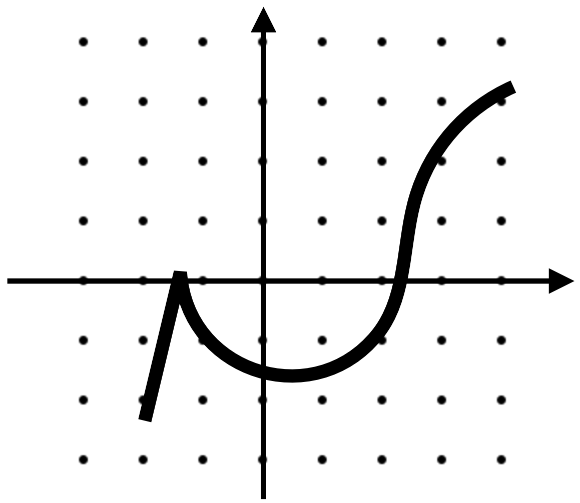
Function;

$$y = f(x)$$

$$x \xrightarrow{f} y$$

$$x \mapsto y$$





	1	2	3	4
1		x		
2			x	
3				x

Function;

Notation

$y = f(x)$

$x \xrightarrow{f} y$

$x \mapsto y$

Function;

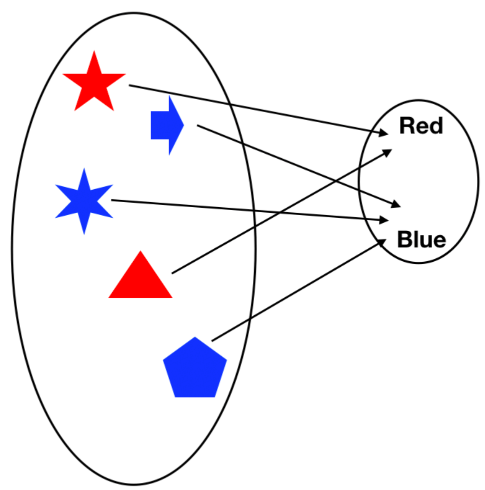
Notation:

$y = f(x)$

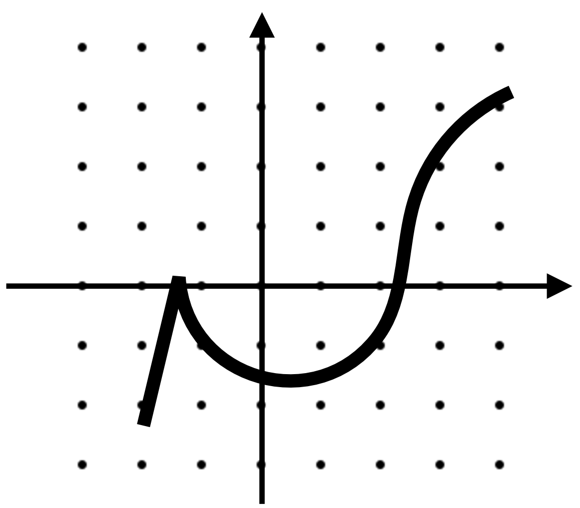
$x \xrightarrow{f} y$

$x \mapsto y$

Specification:



	1	2	3	4
1		x		
2			x	
3				x



Function;

Notation:

$$y = f(x)$$

$$x \xrightarrow{f} y$$

$$x \mapsto y$$

Specifications:

$$f(x) = 2(x + 1);$$
$$g(x) = 2x + 2;$$
$$h(x) = sin(x) + x;$$

Function;

Notation:

$y = f(x)$

$x \xrightarrow{f} y$

$x \mapsto y$

Specifications: ***graph (grafiek)***

$$\boxed{\begin{array}{l}\textit{given } f : A \to B\textit{, the graph of } f \textit{ is}\\ graph(f) = \{(x, f(x)) \mid x \in A\}\end{array}}$$

Note: $graph(f) \subseteq A \times B$

A graph is a binary relation.

Equality of functions:
Two functions f,g are equal if:
$f(x) = g(x)$ *for all* x *in* A

Function;

Notation:

$y = f(x)$

$x \xrightarrow{f} y$

$x \mapsto y$

Functions here:

from informal "mapping"

to formal relations.

Function;

Notation:

$y = f(x)$

$x \xrightarrow{f} y$

$x \mapsto y$

Definition. A **function** from A to B is a binary relation $f \subseteq A \times B$ which is functional and total

What do "functional" and "total" mean?

Function;

Notation:

$y = f(x)$

$x \xrightarrow{f} y$

$x \mapsto y$

Definition. A **function** from A to B is a binary **relation** $f \subseteq A \times B$ which is functional and total.

What do "functional" and "total" mean?

$R \subseteq A \times B$

<u>Functional</u>: if aRb and aRc then $b = c$. [no 1-to-many!]

<u>Total</u>: if $a \in A$ then aRb for *some* $b \in B$. [domain is used up!]

<u>Injective</u>: if aRb and cRb then $a = c$. [no many-to-1]

<u>Surjective</u>: if $b \in B$ then aRb for some $a \in A$. [codomain is used up]

Function;

Notation:

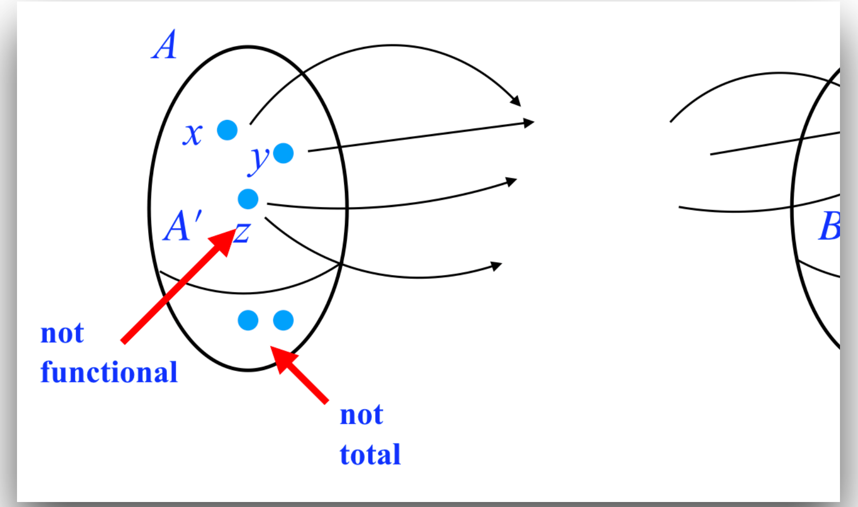
$y = f(x)$

$x \xrightarrow{f} y$

$x \mapsto y$

Definition. A **function** from A to B is a binary **relation** $f \subseteq A \times B$ which is functional.

What does "functional" mean?



Function;

Notation:

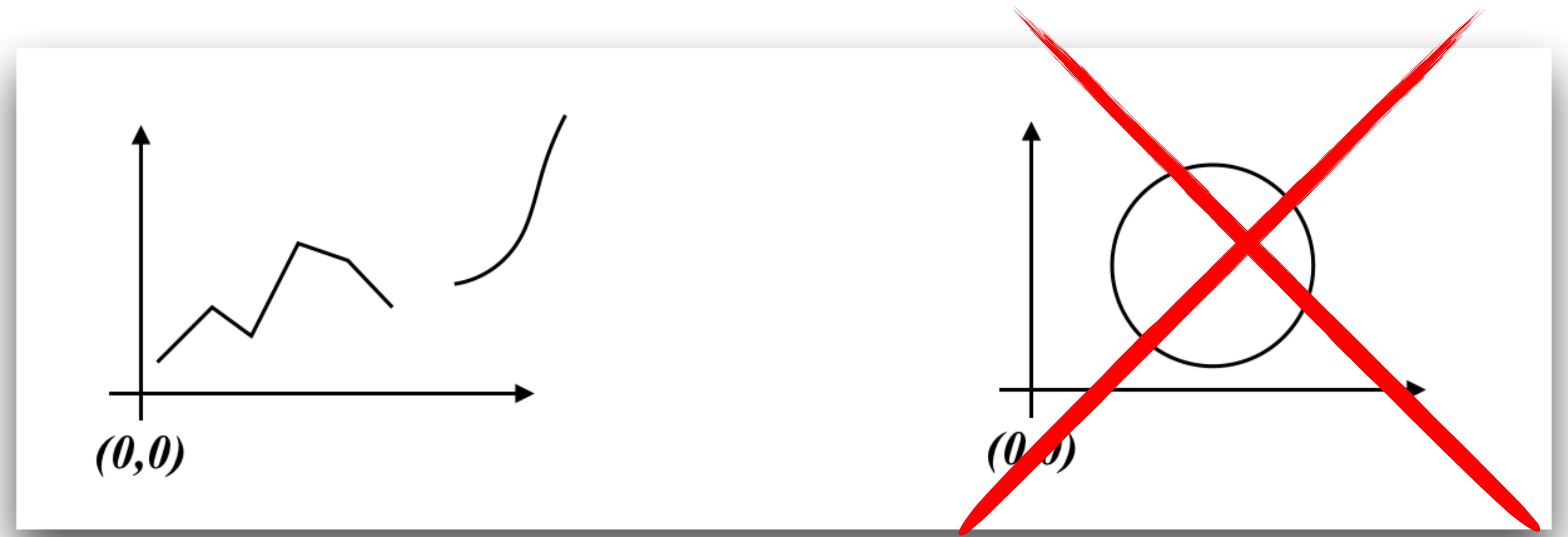
$y = f(x)$

$x \xrightarrow{f} y$

$x \mapsto y$

Definition. A **function** from A to B is a binary **relation** $f \subseteq A \times B$ which is functional.

What does "functional" mean?



Function;

Notation:

$y = f(x)$

$x \xrightarrow{f} y$

$x \mapsto y$

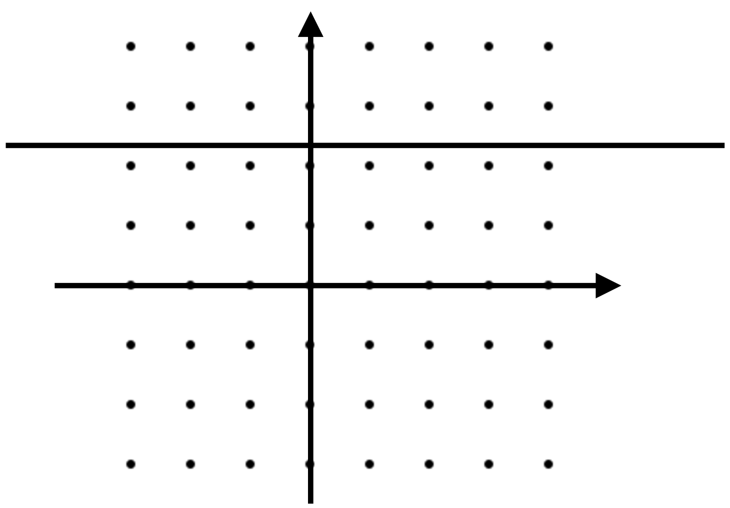
Definition. A **function** from A to B is a binary **relation** $f \subseteq A \times B$ which is functional.

Notation. To specify domain and range we write:

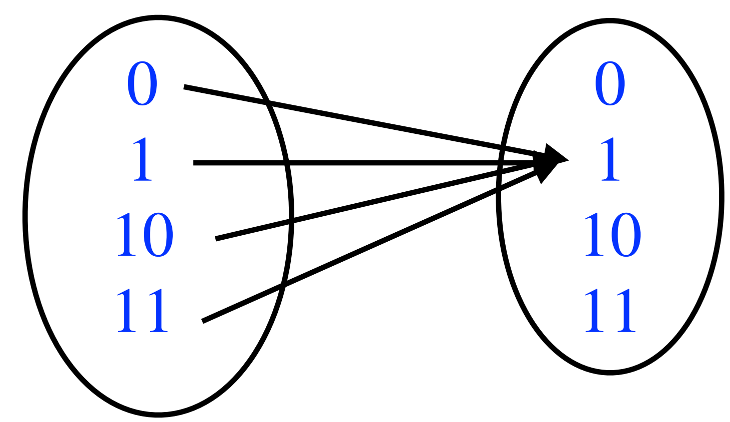
$f : A \to B$

Basic functions

Definition. $f : A \to B$ ***is constant if***
$f(x) = f(y)$ ***for all*** $x, y \in A$

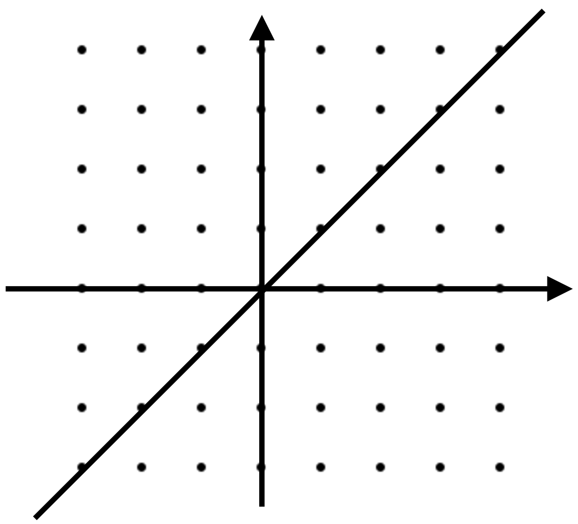


	1	2	3	4
1		x		
2		x		
3		x		

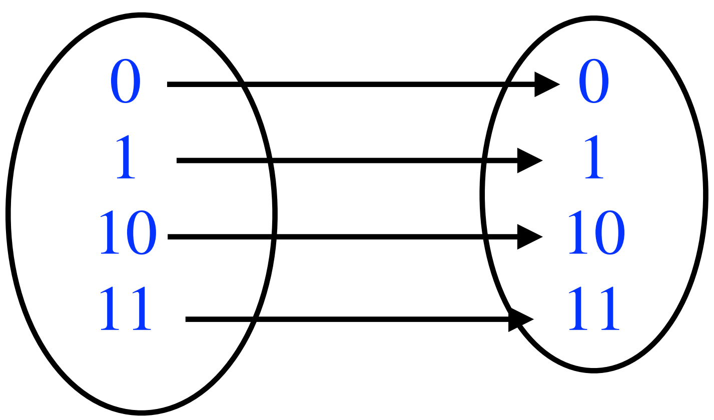


Basic functions

Definition. $f : A \to B$ ***is the identity (on*** A***) if***
$f(x) = x$ ***for all*** $x \in A$

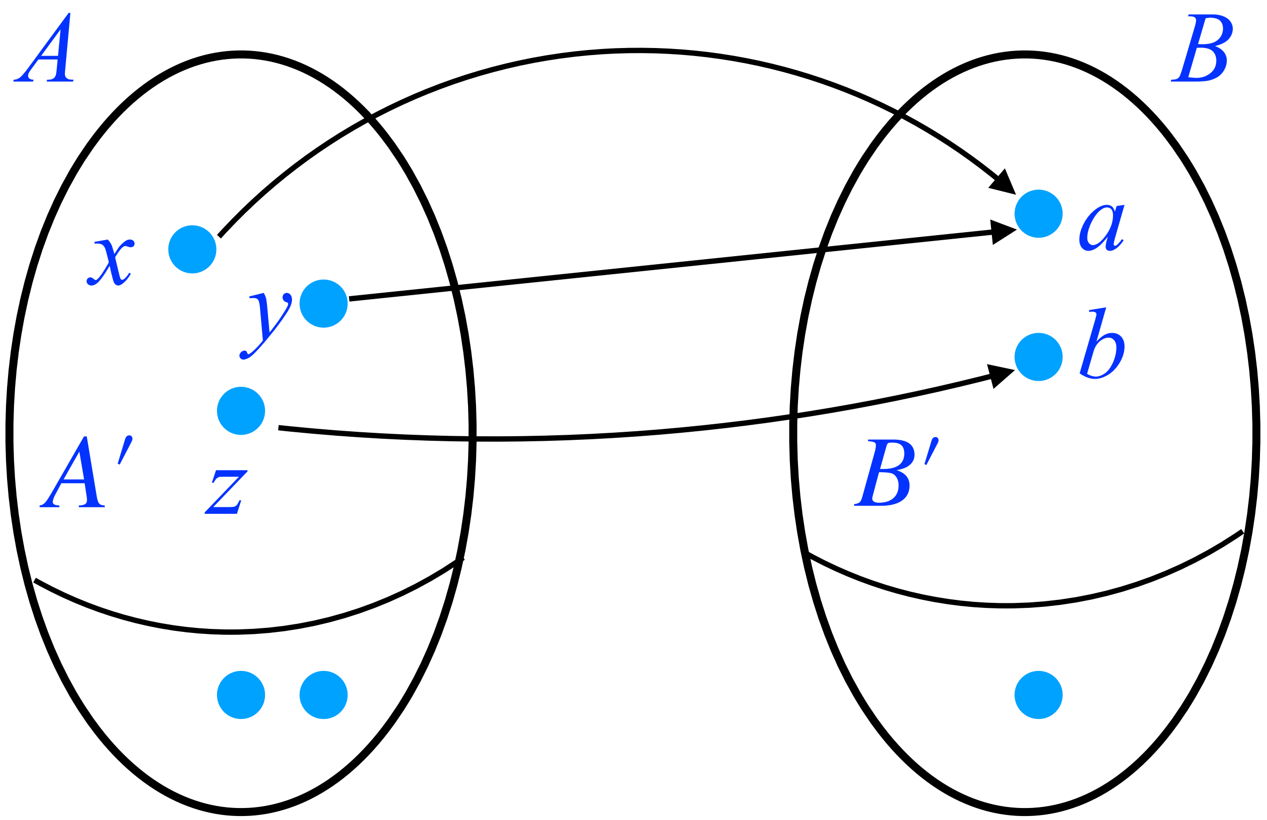


	1	2	3	4
1	x			
2		x		
3			x	



identity relation…Inherited notation $id_A, \mathbf{1}_A$

Domain, range…



$R \subseteq A \times B$

Domain

$dom(R) = A'$

Range or image

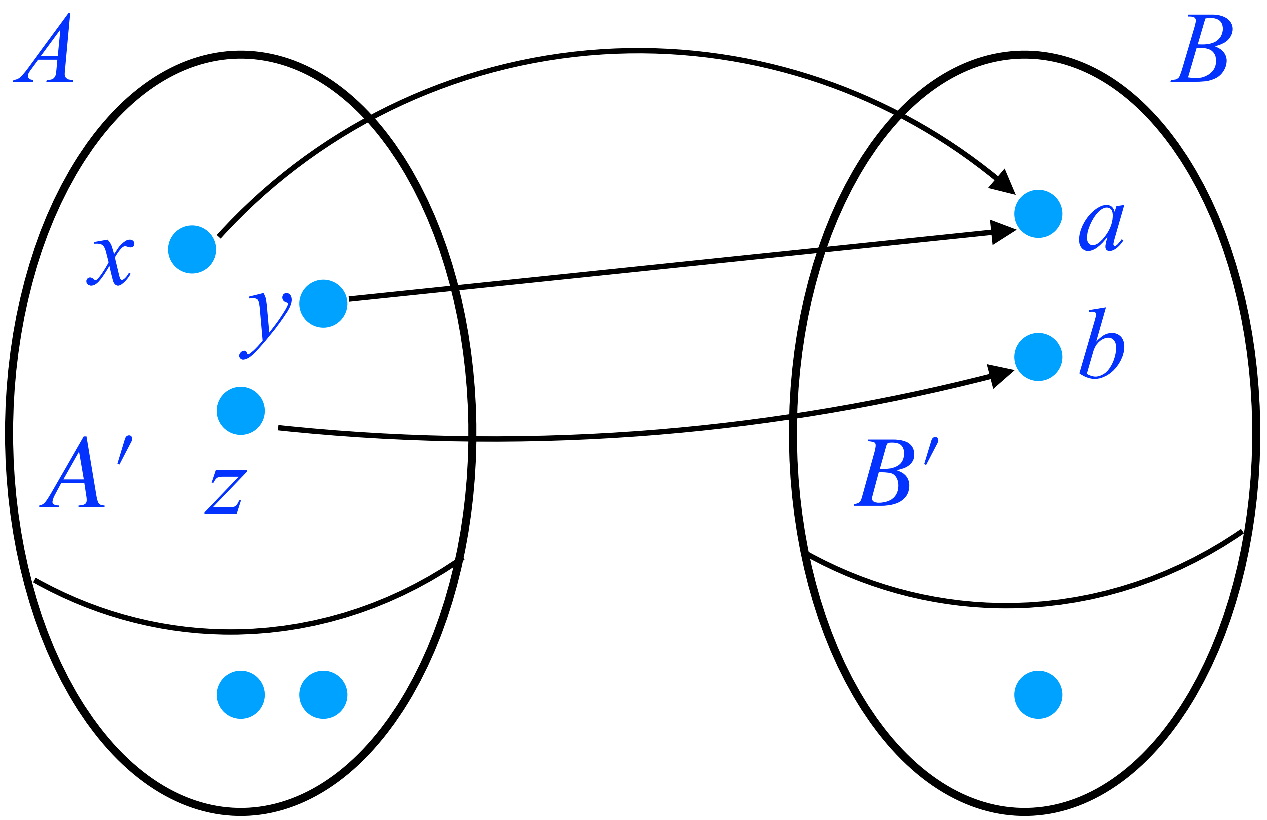
$range(R) = B'$

Codomain

Preimage

Image of $V \subseteq A$ **(under f)** : $f(V) = \{f(x) \mid x \in V\}$

Domain, range…



$R \subseteq A \times B$

Domain

$dom(R) = A'$

Range or image

$range(R) = B'$

Codomain

Preimage

Preimage of $W \subseteq B$ **(under f)** : $f^{-1}(W) = \{x \mid f(x) \in W\}$

Domain, range…

Image of $V \subseteq A$ **(under f)** : $f(V) = \{f(x) \mid x \in V\}$

Preimage of $W \subseteq B$ **(under f)** : $f^{-1}(W) = \{x \mid f(x) \in W\}$

Some highlights:

- V **is in** A **(***domain***),** $f(V)$ **in** B **(***codomain***) [image in codomain/range]**
- W **is in** B **(***codomain***),** $f^{-1}(W)$ **in** A **(***domain***) [preimage in domain]**

- $f^{-1}(W)$ **is** ***notation. In general,*** f^{-1} ***is not a function (not functional)***

Can all be made fully formal as (proper) functions on powersets…

Domain, range…

Image of $V \subseteq A$ **(under f)** : $f(V) = \{f(x) \,|\, x \in V\}$

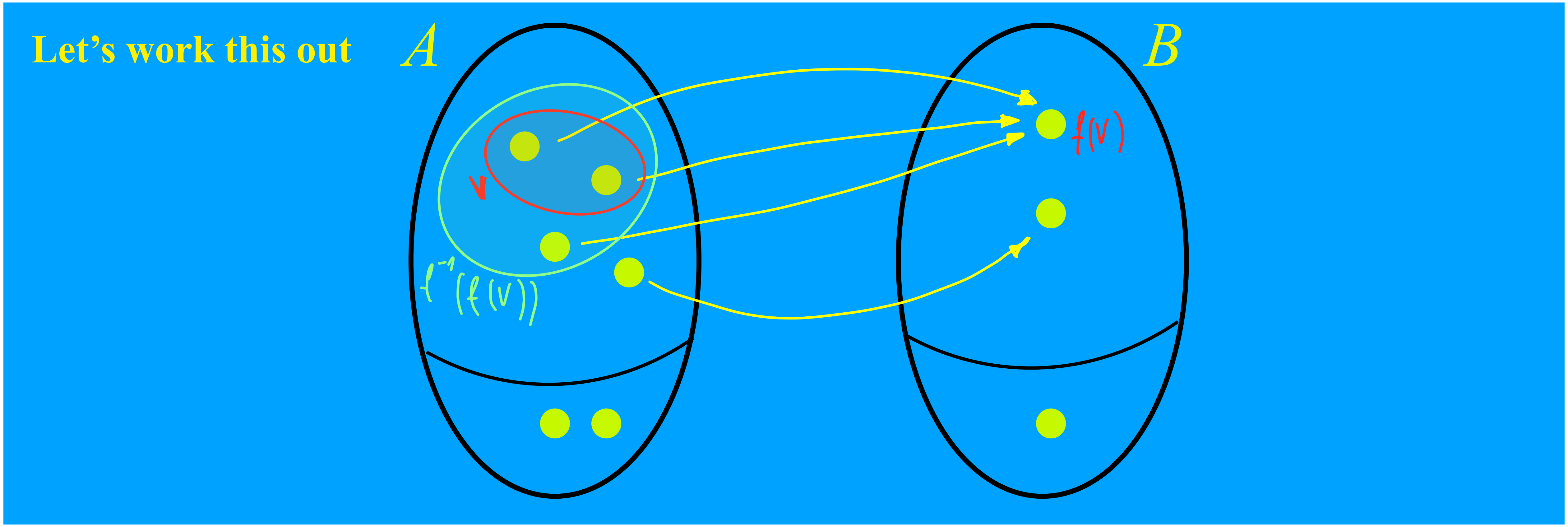
Preimage of $W \subseteq B$ **(under f)** : $f^{-1}(V) = \{x \,|\, f(x) \in W\}$

Properties; $V \subseteq A$; $W \subseteq B$;

What is the relationship between

V **and** $f^{-1}(f(V))$? $V \subseteq f^{-1}(f(V))$

W **and** $f(f^{-1}(W))$?



Domain, range…

Image of $V \subseteq A$ **(under f)** : $f(V) = \{f(x) \,|\, x \in V\}$

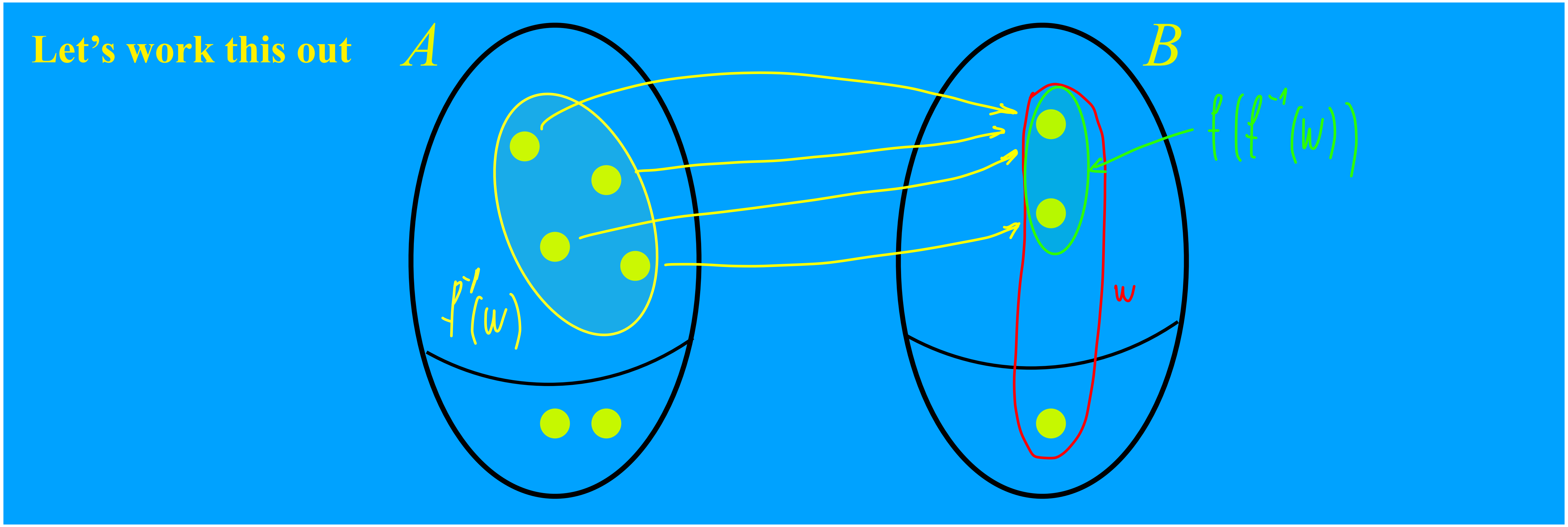
Preimage of $W \subseteq B$ **(under f)** : $f^{-1}(V) = \{x \,|\, f(x) \in W\}$

Properties; $V \subseteq A$; $W \subseteq B$;

What is the relationship between

V **and** $f^{-1}(f(V))$?

W **and** $f(f^{-1}(W))$? $f(f^{-1}(W)) = W \cap f(A)$



Surjective, injective

Definition. $f : A \to B$ **is surjective if** $f(A) = B$ **.**

Definition. $f : A \to B$ **is injective if for all** x, y **if f(x)=f(y) then x=y.**

In other words…

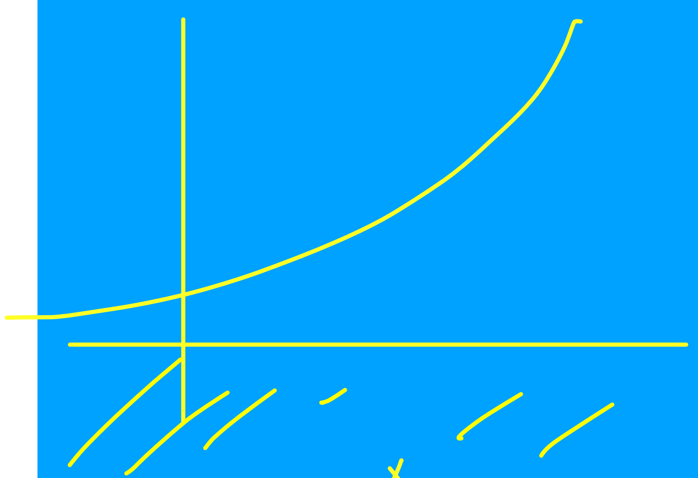
... surjective (injective) as a relation...

Surjective, injective

Examples: $f : \mathbb{R} \to \mathbb{R}$

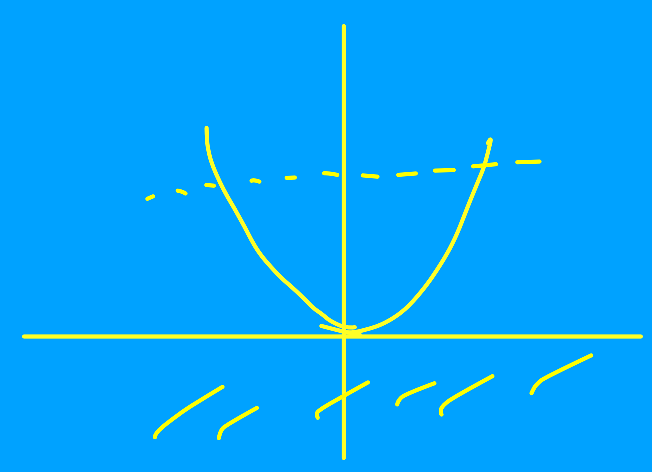
$$f(x) = 2^x$$
$$f(x) = x^2$$
$$f(x) = x^3$$

Plotting a plot to plot



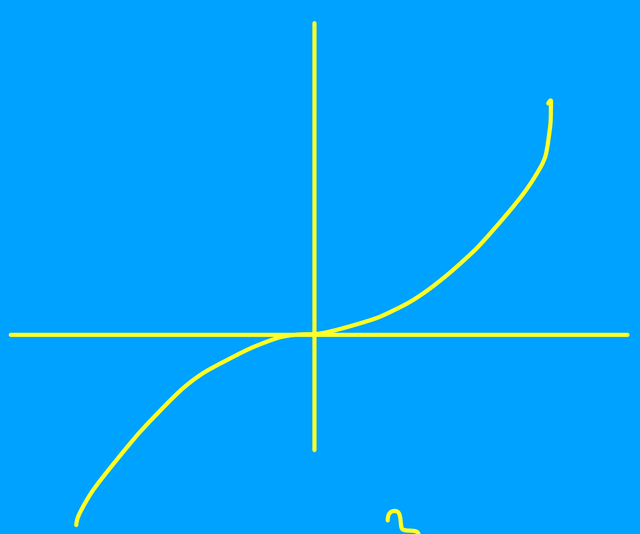
$f(x) = 2^x$

inj, not sur



$f(x) = x^2$

not inj., not sur.



$f(x) = x^3$

inj. & sur.

Bijection (1-to-1)

Definition. $f : A \to B$ **is bijective if it is both surjective and injective.**

$V \subseteq A; W \subseteq B;$ $\quad V = f^{-1}(f(V))$ & $V = f(f^{-1}(W))$

Bijection (1-to-1)

Definition. $f : A \to B$ **is bijective if it is both surjective and injective.**

$V \subseteq A; W \subseteq B;$

$V = f^{-1}(f(V))$ & $W = f(f^{-1}(W))$

These are two properties. One of them is equivalent to injectivity, the other surjectivity. Which is which?

Solution.

a) $\forall\ V \subseteq A,\ V = f^{-1}(f(V)) \iff f$ is injective

b) $\forall\ W \subseteq B,\ W = f(f^{-1}(W)) \iff f$ is surjective

Proof (b)

$\Rightarrow$ by contradiction. assume $y \in B$ & $y \notin f(A)$ and take any $z \in f(A)$. set $W = \{z, y\}$ then $f(f^{-1}(W)) \neq W$ as $y \in W$ but $y \notin f \circ f^{-1}(W)$ as $y \notin f(A)$.

$\Leftarrow$ by contradiction. assume f surjective & $W \supsetneq f \circ f^{-1}(W)$ [only option since $f(f^{-1}(W)) = W \cap f(A)$]. $\Rightarrow \exists\, z \in W,\ z \notin f \circ f^{-1}(W)$ $\Rightarrow z \notin f(f^{-1}(\{z\})) \Rightarrow f^{-1}(\{z\}) = \emptyset \Rightarrow \exists\, z$ st $\forall\, a \in A\ f(a) \neq z$. Not surjective

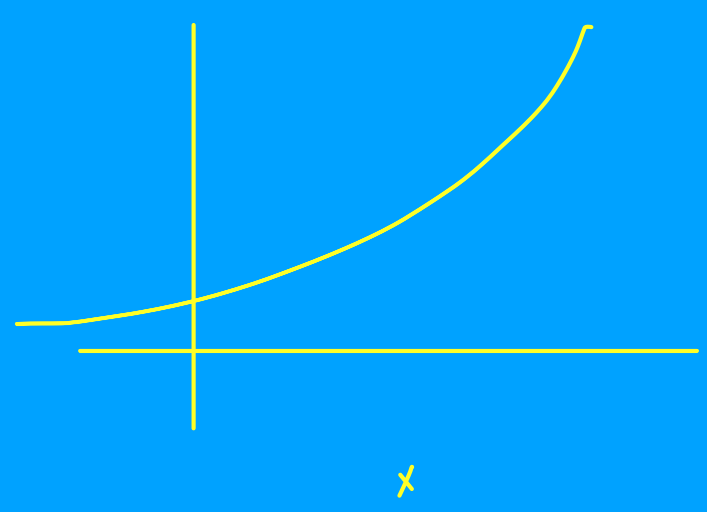
Bijection (1-to-1)

Definition. $f : A \to B$ **is bijective if it is both surjective and injective.**

Examples: $f : \mathbb{R} \to \mathbb{R}$ $\qquad$ $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$

$f(x) = 2^x$ $\qquad$ $f(x) = 2^x$

Plotting a plot to plot



Bijection (1-to-1)

Definition. $f : A \to B$ **is bijective if it is both surjective and injective.**

Examples: $f : \mathbb{R} \to \mathbb{R}$

$f(x) = tan(x)$

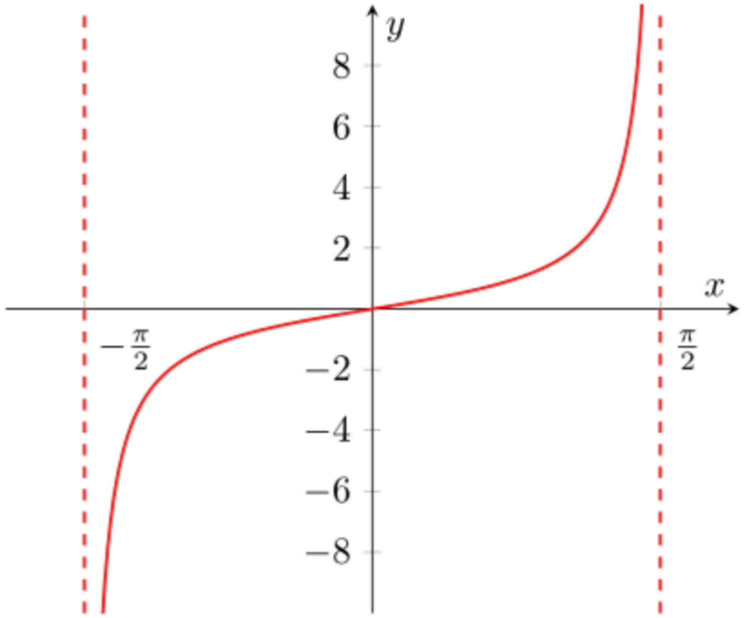
$f : [-\pi/2, \pi/2] \to \mathbb{R}$

$f(x) = tan(x)$

Restriction on domain (and codomain) **can *always* yield a 1-1 function…**

To make surjective, restrict codomain to range.

To make injective, restrict domain..



Bijection (1-to-1)

Definition. $f : A \to B$ **is bijective if it is both surjective and injective.**

Examples: $f : \mathbb{R} \to \mathbb{R}$

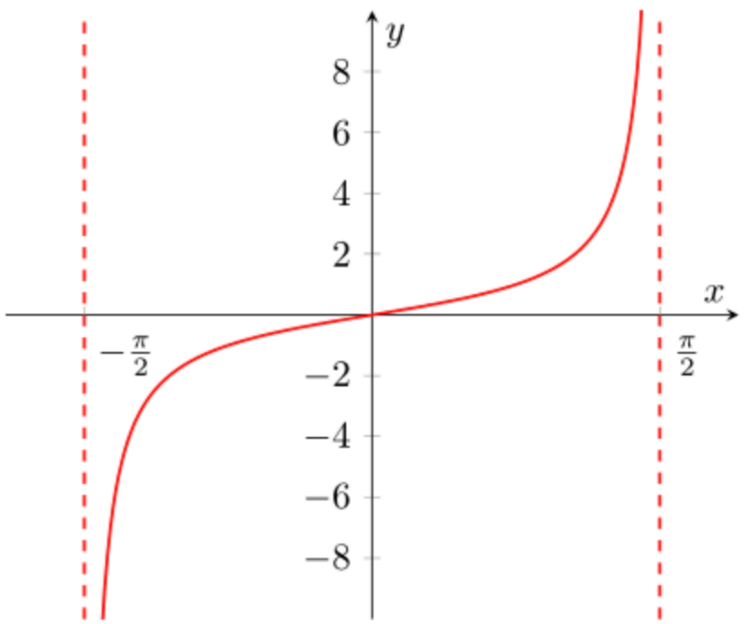
$f(x) = tan(x)$

$f : [-\pi/2, \pi/2] \to \mathbb{R}$

$f(x) = tan(x)$

Restriction on domain
can *often* yield a 1-1 function…

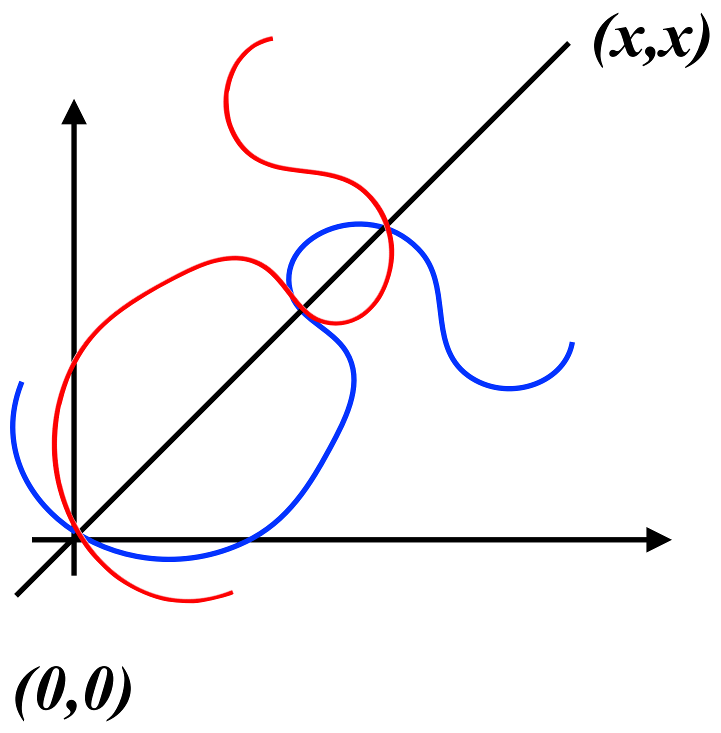
intuitively.. for each $f^{-1}(\{b\}), b \in B$**,**
choose *one*…
[deep waters]



Inverse function

$R \subseteq A \times B$

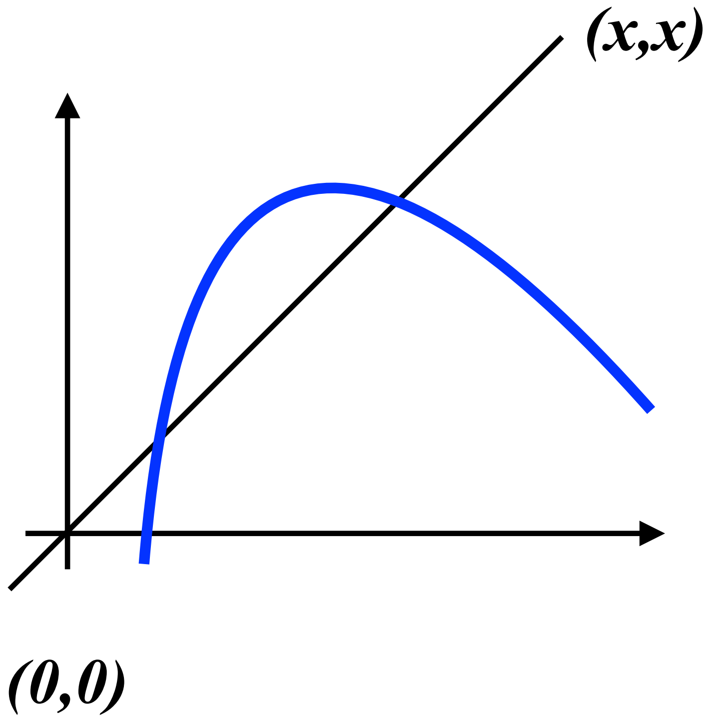
$R^{-1} \subseteq B \times A$, **defined with** $(b,a) \in R^{-1} \Leftrightarrow (a,b) \in R$



Inverse function

$R \subseteq A \times B$

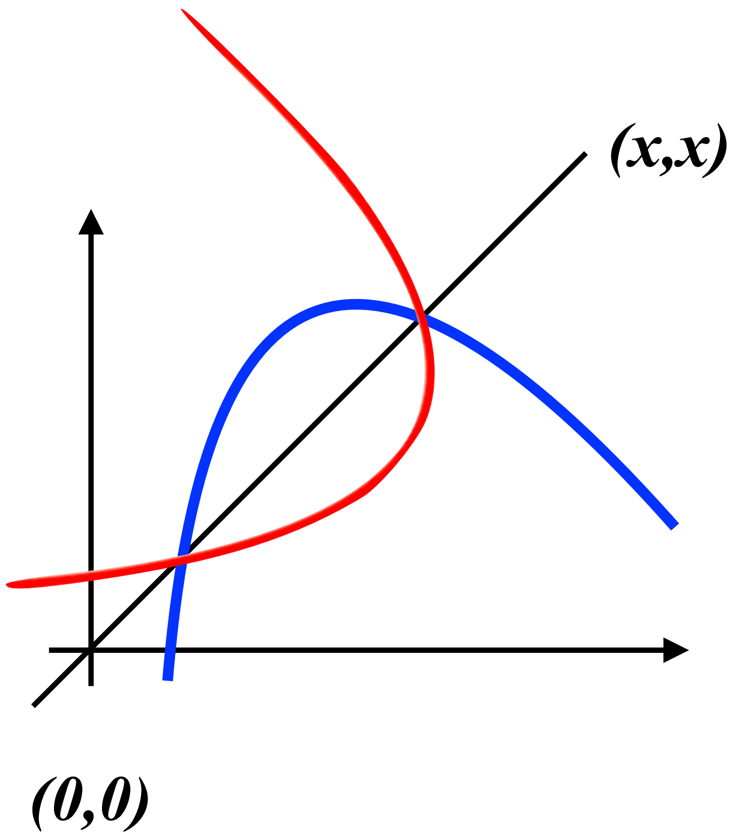
$R^{-1} \subseteq B \times A,$ **defined with** $(b,a) \in R^{-1} \Leftrightarrow (a,b) \in R$



Inverse function

$R \subseteq A \times B$

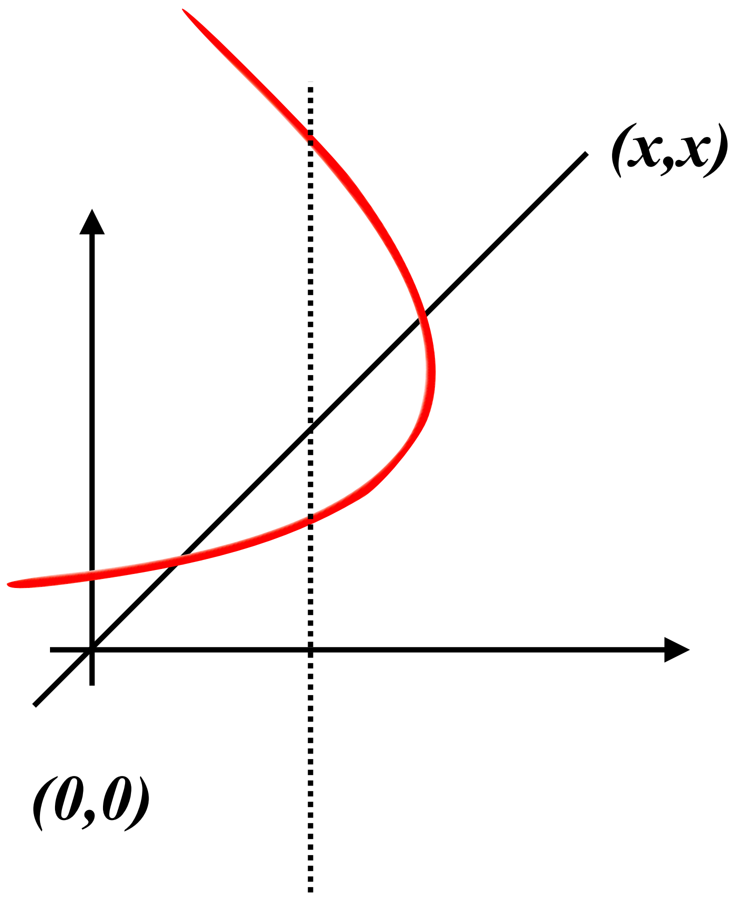
$R^{-1} \subseteq B \times A,$ **defined with** $(b,a) \in R^{-1} \Leftrightarrow (a,b) \in R$



Inverse function

$R \subseteq A \times B$

$R^{-1} \subseteq B \times A,$ **defined with** $(b,a) \in R^{-1} \Leftrightarrow (a,b) \in R$



Inverse function

$R \subseteq A \times B$

$R^{-1} \subseteq B \times A$, **defined with** $(b, a) \in R^{-1} \Leftrightarrow (a, b) \in R$

For functions not triv. since $R^{-1} \subseteq B \times A$, **is not functional, unless...?**

Inverse function

$R \subseteq A \times B$

$R^{-1} \subseteq B \times A,$ **defined with** $(b, a) \in R^{-1} \Leftrightarrow (a, b) \in R$

For functions not triv. since $R^{-1} \subseteq B \times A,$ **is not functional, unless** R **is injective**

Inverse function

$R \subseteq A \times B$

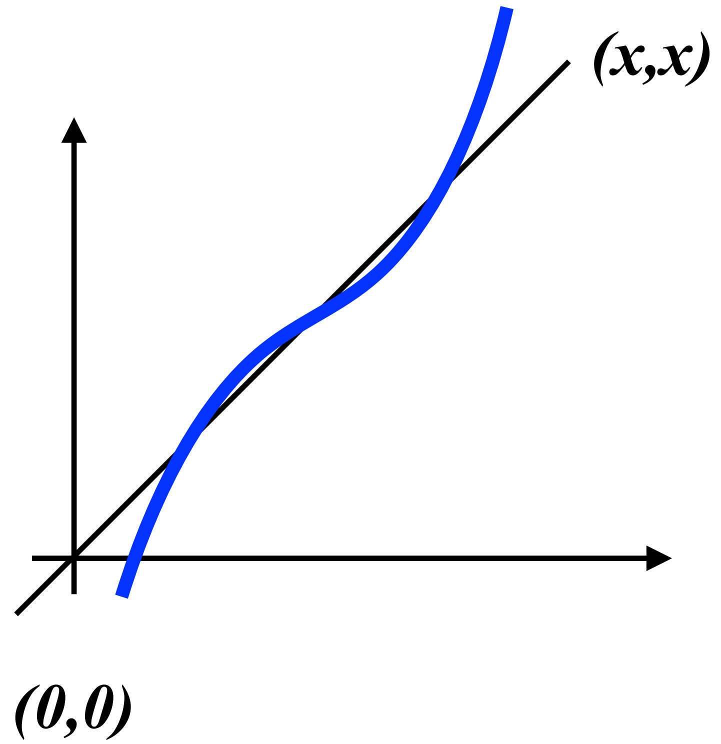
$R^{-1} \subseteq B \times A,$ **defined with** $(b, a) \in R^{-1} \Leftrightarrow (a, b) \in R$

For functions not triv. since $R^{-1} \subseteq B \times A,$ **is not functional, unless** R **is injective**

Inverse function

$R \subseteq A \times B$

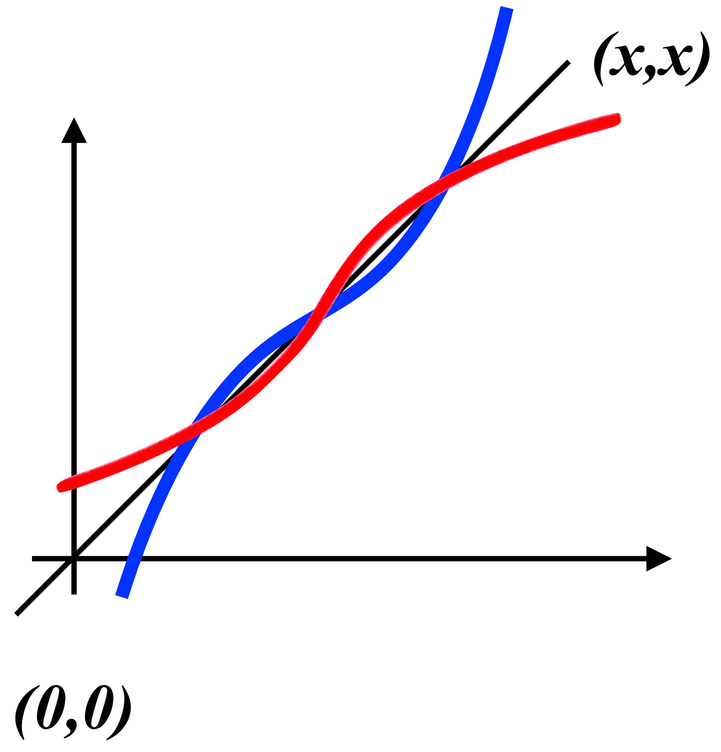
$R^{-1} \subseteq B \times A,$ **defined with** $(b,a) \in R^{-1} \Leftrightarrow (a,b) \in R$



Inverse function

$R \subseteq A \times B$

$R^{-1} \subseteq B \times A,$ **defined with** $(b,a) \in R^{-1} \Leftrightarrow (a,b) \in R$



Inverse function

Theorem 3.1. A function *f is invertible (has an inverse) if and only if f is bijective.*

Function composition

Definition. Let $f : A \to B$, $g : B \to C$ **be functions.** The **composition of *f* and *g***, *denoted* $g \circ f$ *("g composed with f", "g after f") is a function from A to C, defined with:*

$(g \circ f)(x) = g(f(x))$, *for all* $x \in A$

Function composition

Definition. Let $f : A \to B$, $g : B \to C$ **be functions.** The **composition of *f* and g**, *denoted* $g \circ f$ *("g composed with f", "g after f") is a function from A to C, defined with:*

$(g \circ f)(x) = g(f(x))$, *for all* $x \in A$ (note : Range(f) $\subseteq$ Dom(g))!

Careful: for relations we had $xR \circ Sy$, but for functions $y = g \circ f(x)$

Language: *relations* : $R \circ S =$ 'R then S'

functions : $g \circ f =$ 'g after f'

ambiguity… be careful, context matters.

Function composition: *brackets not necessary*

Property. Function composition is associative:
$f \circ g \circ h = (f \circ g) \circ h = f \circ (g \circ h)$

Note: for function composition to be defined domains & ranges must match:

$\mathrm{Range}(h) \subseteq \mathrm{Dom}(g)$ $\quad \mathrm{Range}(g) \subseteq \mathrm{Dom}(f)$

Let's work this out

$(f \circ g) \circ h\,(x) = (f \circ g)(h(x)) = f(g(h(x)))$

$=$

$f \circ (g \circ h)(x) = f((g \circ h)(x)) = f(g(h(x)))$

Much simpler than for relations.

Function composition: *brackets not necessary*

Property. Let $f : A \to B$, $g : B \to C$.
Then if f *and* g *are injective (surjective) then* $g \circ f$ *is injective (surjective)*

Let's work this out

EXAMPLE: INJECTIVE.

f, g injective $\Big[\overbrace{f(x) = f(y) \Rightarrow x = y}^{(1)} ; \overbrace{g(x) = g(y) \Rightarrow x = y}^{(2)} \Big]$

$f \circ g(x) = f \circ g(y) \iff f(g(x)) = f(g(y)) \overset{(1)}{\Longrightarrow} g(x) = g(y) \overset{(2)}{\Longrightarrow} x = y$

Function composition

A property: Let $f: A \to B$ be a 1-1 function and f^{-1} be its inverse.

Then $f \circ f^{-1} = id_B$ and $f^{-1} \circ f = id_A$

identity on B

identity on A.

Let's work this out

Trivial to see if we think of f and f^{-1} as relations.

(a,b) & (b,a) $\Rightarrow$ (a,a)

f $\quad f^{-1} \quad \underbrace{f^{-1} \circ f}$

using order for function composition!

Some excercises

Property. A = {a,b,c}; B = {x,y,z}, C = {r,s,t,u}

$f = \{(a,y),(b,x), (c,y)\}$ $[f : A \to B]$

$g = \{(a,b),(b,c), (c,a)\}$ $[g : A \to A]$

$h = \{(r,c),(s,b), (t,b),(u,a)\}$ $[h : C \to A]$

Comment: f, g, h are defined by their graphs, $\{(a, f(a)) \mid a \in A\}$

a) $g \circ f$?; b) determine $f \circ g \circ h$. c) what is it's (b)) range?

d) determine f^{-1}, g^{-1}, h^{-1} if they exist.

Let's work this out

Homework.

Some excercises

Property. A = {a,b,c}; B = {x,y,z}, C = {r,s,t,u}

$f = \{(a,y),(b,x), (c,y)\}$ [$f : A \to B$]

$g = \{(a,b),(b,c), (c,a)\}$ [$g : A \to A$]

$h = \{(r,c),(s,b), (t,b),(u,a)\}$ [$h : C \to A$]

Comment: f, g, h are defined by their graphs, $\{(a, f(a)) \mid a \in A\}$

a) $g \circ f$?; b) determine $f \circ g \circ h$. c) what is it's (b)) range?
d) determine f^{-1}, g^{-1}, h^{-1} if they exist.

Let's work this out

Homework.

Mathematical formalism…

Function

$$(\forall x \in A)[$$
$$(\exists y \in B)(y = f(x)) \wedge$$
$$\neg(\exists y \in B)(\exists z \in B)(y \neq z \wedge y = f(x) \wedge z = f(x))]$$

$$(\forall/\exists y \in B) = (shorthand) = (\forall/\exists y)_B$$

Injective

$$(\forall x)_A(\forall y)_B(f(x) = f(y) \Rightarrow x = y)$$

?

$$(\exists y)_B(\forall x)_A(y = f(x))$$

?

$$(\exists y)_B(\forall x)_A(y \neq f(x))$$

Mathematical formalism…

$\forall$ -“for all”… inverted “A”

$\exists$ -exists… flipped “E”

Some excercises

Property. A = {a,b,c}; B = {x,y,z}, C = {r,s,t,u}

$f = \{(a,y),(b,x),(c,y)\}$ $[f : A \to B]$

$g = \{(a,b),(b,c),(c,a)\}$ $[g : A \to A]$

$h = \{(r,c),(s,b),(t,b),(u,a)\}$ $[h : C \to A]$

Comment: f, g, h are defined by their graphs, $\{(a, f(a)) \mid a \in A\}$

a) $g \circ f$?; b) determine $f \circ g \circ h$. c) what is it's (b)) range?
d) determine f^{-1}, g^{-1}, h^{-1} if they exist.

Let's work this out

Solution

a) $g \circ f$ is not defined, because $f(A)$ is not in $\mathrm{Dom}(g)$

b) $g \circ h = \{(r,a),(s,c),(t,c),(u,b)\}$ [e.g. $h(r) = c$ so $g(h(r)) = g(c) = a$]

$f \circ g \circ h = f \circ (g \circ h) = \{(r,y),(s,y),(t,y),(u,x)\}$

c) $\{x,y\}$

d) f & g are not injective $\Rightarrow$ no inverse by Theorem 3.1.

$g^{-1} = \{(b,a),(c,b),(a,c)\}$