Lecture 6b

refresher & more details

Relations & tuples reminder

Tuples & Cartesian products

$A_1, A_2, A_3 \ldots, A_n$

$(a_1, a_2, a_3, \ldots, a_n)$ $a_i \in A_i$

$A = A_1 \times A_2 \times A_3 \times \cdots \times A_n$

Definition.

$A = \{(a_1, a_2, \ldots, a_n) \,|\, a_i \in A_i\}$

"ordered lists"

Relations:
subsets of Cartesian products:

$R \subseteq A_1 \times A_2 \times A_3 \times \cdots \times A_n$

Order matters:

$R \subseteq A \times B$

"relation from A to B"

"n-ary relation"
"binary relation"

Examples: $\leq \subseteq \mathbb{R} \times \mathbb{R}$; $\quad a \leq b \Leftrightarrow (a, b) \in \leq$.

Inverse relation

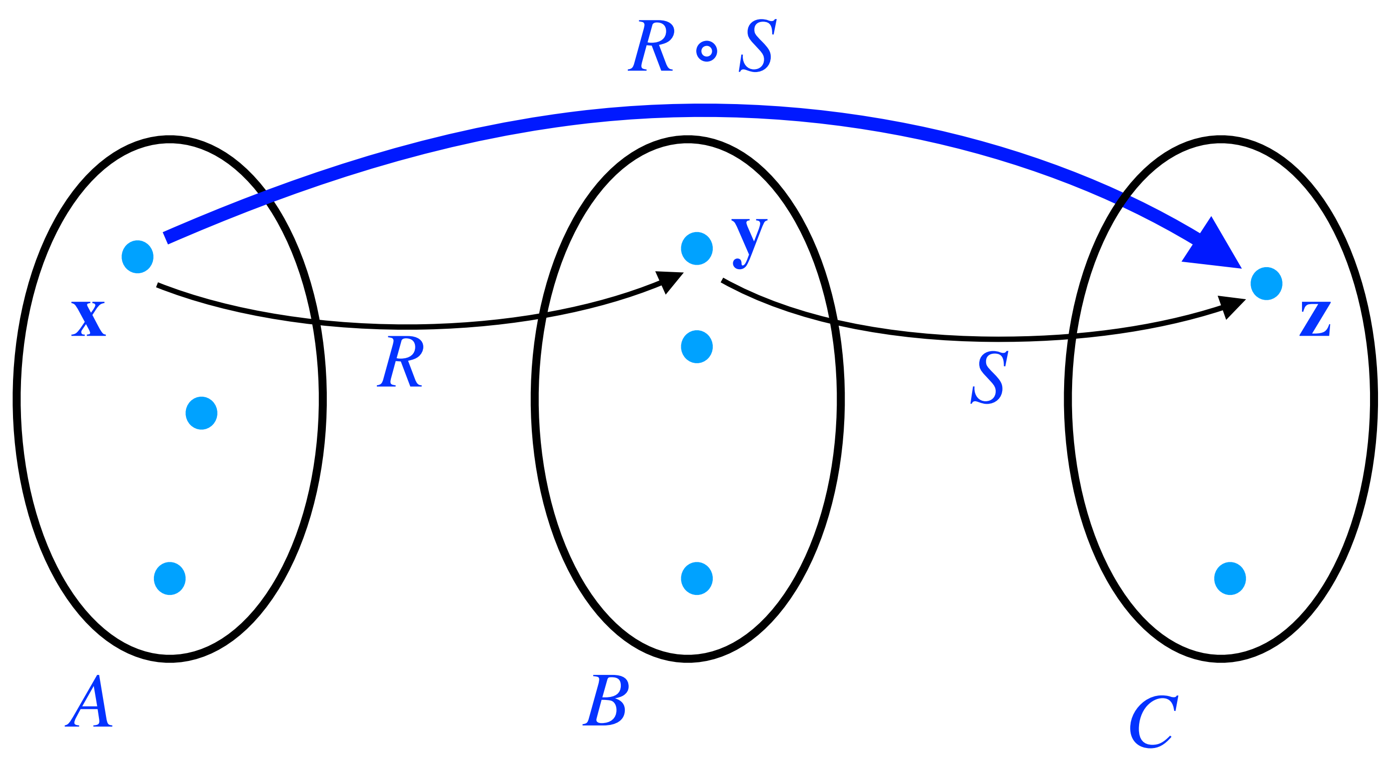
$R \subseteq A \times B$

$R^{-1} \subseteq B \times A,$ **defined with** $(b,a) \in R^{-1} \Leftrightarrow (a,b) \in R$

Composition of relations

$R \subseteq A \times B$ **and** $S \in B \times C$

$x \in A, y \in B, z \in C$ $\quad x(R \circ S)z \ \mathit{if}\ xRy \ \& \ yRz \ \mathit{for\ some}\ y \in B$



Relational properties (summary)

$R \subseteq A \times B$

Functional: if aRb and aRc then $b = c$. [no 1-to-many!]

Total: if $a \in A$ then aRb for *some* $b \in B$. [domain is used up!]

Injective: if aRb and cRb then $a = c$. [no many-to-1]

Surjective: if $b \in B$ then aRb for some $a \in A$. [codomain is used up]

$R \subseteq A \times A$

Reflexive: $\forall a \in A, aRa$

Symmetric: $aRb \Leftrightarrow bRa$

Antisymmetric: $aRb \;\&\; bRa \Rightarrow a = b$

Transitive: $aRb \;\&\; bRc \Rightarrow aRc$

Equivalence relation, partial orders
(just further types of relations)

R **is reflexive:** $\forall a \in A, aRa$

R is **symmetric:** $aRb \Leftrightarrow bRa$

R is **antisymmetric:** $aRb \;\&\; bRa \Rightarrow a = b$

R *is* **transitive:** $aRb \;\&\; bRc \Rightarrow aRc$

R *is* **an equivalence relation: reflexive & symmetric & transitive**

R *is* **a partial order: reflexive & antisymmetric & transitive**

BIOINFORMATICA TOY EXAMPLE

DNA := SEQUENCE OF NUCLEOTIDES; $N = \{A, C, G, T\}$

Gene g of length ℓ : $g \in N^{\times \ell}$; e.g. $g = (A, T, G, \ldots, T, A, G)$

Genome G : set of genes : $G \subseteq \bigcup_{\ell=1}^{\infty} N^{\times \ell}$; $G_{tot} = \bigcup_{\ell=1}^{\infty} N^{\times \ell}$

Each person has a genome : $R \subseteq \text{Persons} \times \{G \mid G \text{ is a genome}\}$

R is functional. Is it injective?

GENE SIMILARITY. $S \subseteq G_{tot} \times G_{tot}$

$g_1 \, S \, g_2$ if $|g_1| = |g_2|$ (equally long) & differ in less than 1%.

1) Is gene similarity an equivalence relation?

Gene is not transcribed unless it starts with ATG.

$S' \subseteq G_{TOT} \times G_{TOT}$ "equitranscribable" $g_1 \, S' \, g_2$

if the same on first three letters.

is this an equivalence relation?

I & E MOCK EXAMPLE

Business = (name, investment, profit p/a) $\in$ Names $\times \mathbb{R}^+ \times \mathbb{R}^+ = \mathcal{B}$

$\preceq \in \mathcal{B} \times \mathcal{B}$ $\quad (b_1 \preceq b_2)$ "more valuable"

if $\quad b_1 = (n_1, i_1, p_1)$, $b_2 = (n_2, i_2, p_2)$ &

$i_1 \leq i_2$ & $p_1 \leq p_2$.

$\Rightarrow$ <u>PARTIAL ORDER.</u> (Alpha, 10, 1000) incomparable to (Beta, 100, 10)

Name doesn't matter.

Equivalence relation: more examples

-for numbers x,y , xRy iff x and y have the same parity (both even or both odd)

-for numbers x,y , xRy iff $x^2 = y^2$

-for strings: sRr iff s and r are equally long

-for fractions (rational numbers) x,y , xRy iff x-y is an integer

Need to check that these relations are reflexive & symmetric & transitive.

“equal in some sense”

math jargon: iff means “if and only if”

Partial orders: more examples

a) Take any set and its powerset. The relation ⊆ on the elements of the powerset is a partial order

b) The set of natural numbers and the relation of divisibility *(aRb if b|a)*

c) Strings ordered by length

(not alphabetically, so one is in relation with the other if one is shorter or of equal lenght; if you add alphabetic ordering, then it is both partially and totally ordered, which is a special case)

d) so for strings s, r, *sRr* if s is shorter than r.

Need to check that these relations are reflexive & antisymmetric & transitive.

they are "ordered" in some sense

Main properties can be expressed in different ways:

R **is reflexive** $\Leftrightarrow id \subseteq R$

R **is symmetric** $\Leftrightarrow R^{-1} \subseteq R$

R *is* **transitive** $\Leftrightarrow R \circ R \subseteq R$

(then also $R^{\circ n} \subseteq R$**)**

R **is irreflexive** $\Leftrightarrow id \cap R = \emptyset$

R **is antisymmetric** $\Leftrightarrow R^{-1} \cap R \subseteq id$

$R \subseteq A \times B$ **is functional** $\Leftrightarrow R^{-1} \circ R \subseteq id_B$

$R \subseteq A \times B$ **is surjective** $\Leftrightarrow id_B \subseteq R^{-1} \circ R$

$R \subseteq A \times B$ **is injective** $\Leftrightarrow R \circ R^{-1} \subseteq id_A$

$R \subseteq A \times B$ **is total** $\Leftrightarrow id_A \subseteq R \circ R^{-1}$

$$id = \{(x,x) \mid x \in dom(R)\}$$

$$id_A = \{(x,x) \mid x \in A\}$$

$$id_B = \{(x,x) \mid x \in B\}$$

End of recap, on to new stuff

Closure

For property P (reflexivity, symmetricity, transitivity)…

P-closure of relation R = "smallest" relation, containing R with property P

But what does smallest mean??
When is one relation "smaller" than another?
What is the order?

Closure

For property P (reflexivity, symmetricity, transitivity)…

P-closure of relation R = "smallest" relation, containing R with property P

Lemma. If R and S are reflexive (symmetric) over set A, then $R \cap S$ *and* $R \cup S$ are reflexive (symmetric) .

Work it!

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EXAMPLE: SYMMETRIC, UNION.

R & S ARE SYMMETRIC, $S^{-1} \subseteq S$ & $R^{-1} \subseteq R$. SUFFICES: $(R \cup S)^{-1} \subseteq R \cup S$

NOTE: $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$, ALSO, $V \subseteq W$, $P \subseteq Q \Rightarrow (V \cup P) \subseteq W \cup Q$.

$\Rightarrow (R \cup S)^{-1} = R^{-1} \cup S^{-1} \subseteq R \cup S$. DONE.

Lemma: intersection and union of transitive relations is transitive.

Proofs! direct and by contradiction

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Proofs! direct and by contradiction

EXAMPLE: INTERSECTION, BY CONTRADICTION.

ASSUME R, S ARE TRANSITIVE BUT $\underbrace{R \cap S \text{ IS NOT}}_{(*)}$.

THEN $\exists\, a,b,c$ such that $\underbrace{(a,b) \in R \cap S,\ (b,c) \in R \cap S}_{(1)}$ & $(a,c) \notin R \cap S$

(1) $(a,b),(b,c) \in R$ AND $(a,b),(b,c) \in S$,

BUT THEN SINCE R & S ARE TRANSITIVE, (a,c) is in R AND S

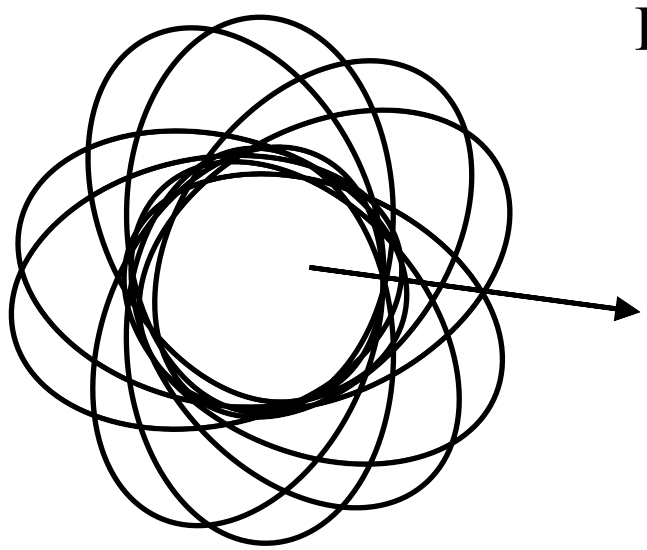
SO IT IS IN $R \cap S$. CONTRADICTION WITH $(*)$

Closure

For property *P* (reflexivity, symmetricity, transitivity)…

P*-closure of relation *R* = "smallest" relation, containing *R* with property *P

Formally, the order we care about is ⊆.
To construct the smallest relation with property
P* containing *R* *we take the intersection of all relations with property P containing R.



Each contains *R*

Intersection preserves *P*

Smallest because contained in all!

Could there be two smallest ones?
No. It is unique!

Closure

For property *P* (reflexivity, symmetricity, transitivity)…

P*-closure of relation *R* = "smallest" relation, containing *R* with property *P

More intuitively… start adding pairs that are missing, and add only those you must add.

Closure

Easy ones: symmetric and reflexive closure

(1) R **is reflexive** $\Leftrightarrow id \subseteq R$
(2) R **is symmetric** $\Leftrightarrow R^{-1} \subseteq R$

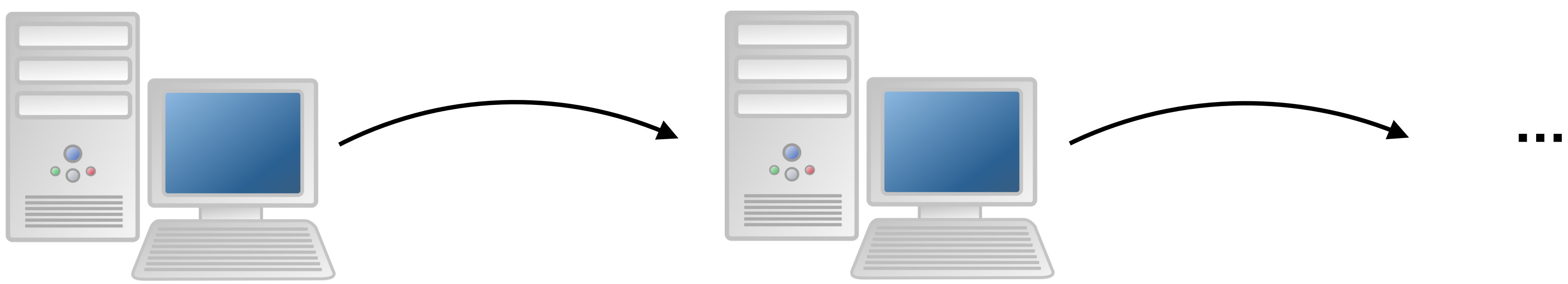
Given R, its reflexive closure is $S = R \cup id_A$

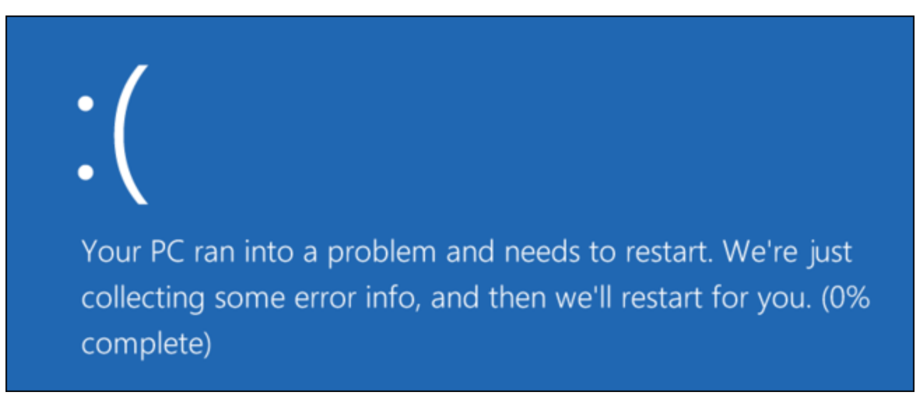
Given R, its symmetric *closure is* $S = R \cup R^{-1}$

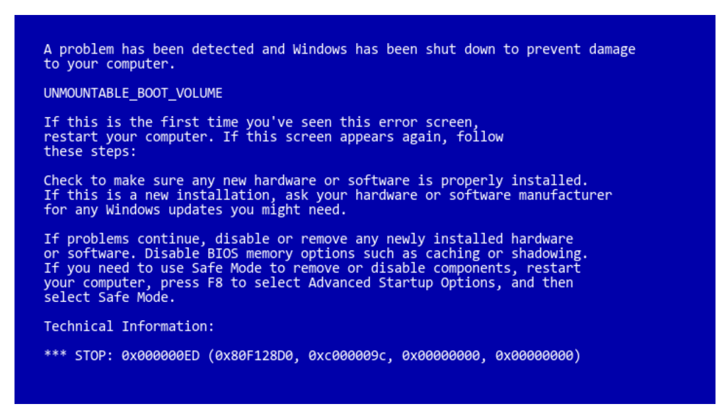
Without proof; the above are the intersections of all relations containing *R* *satisfying (reflexive, symmetric)*

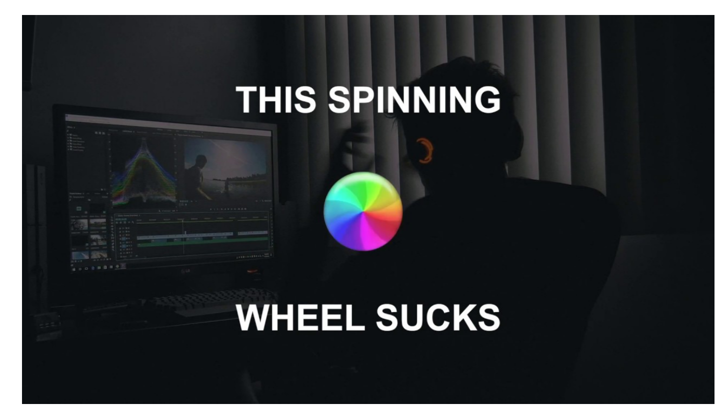
Closure

Transitive closure: super important







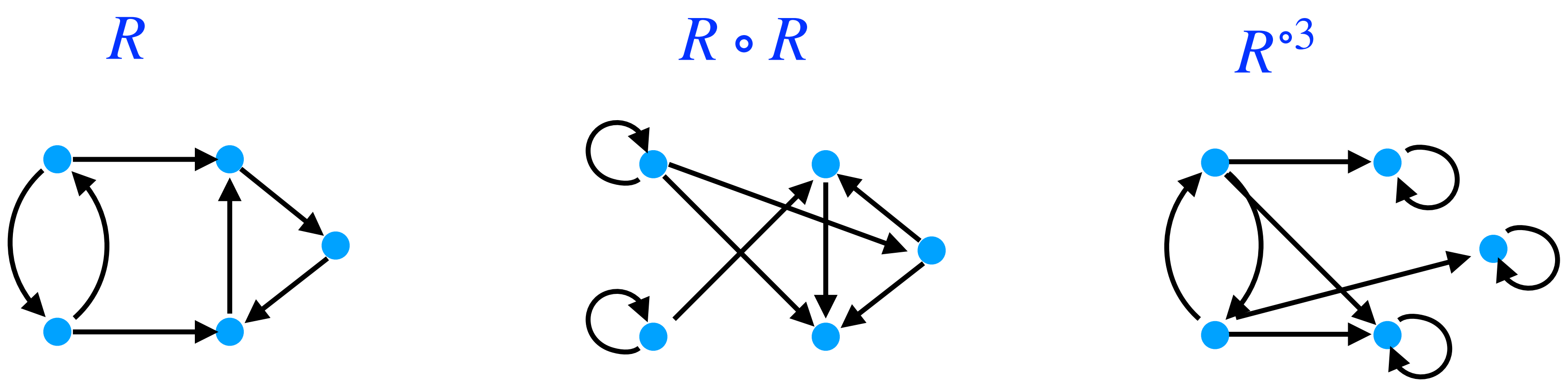


Closure: transitive

(1) Transitive $\Leftrightarrow R \circ R \subseteq R$
(then also) $R^{\circ n} \subseteq R$

Suppose $R \circ R \nsubseteq R$

How about $R' = R \circ R \cup R$. Are we done?



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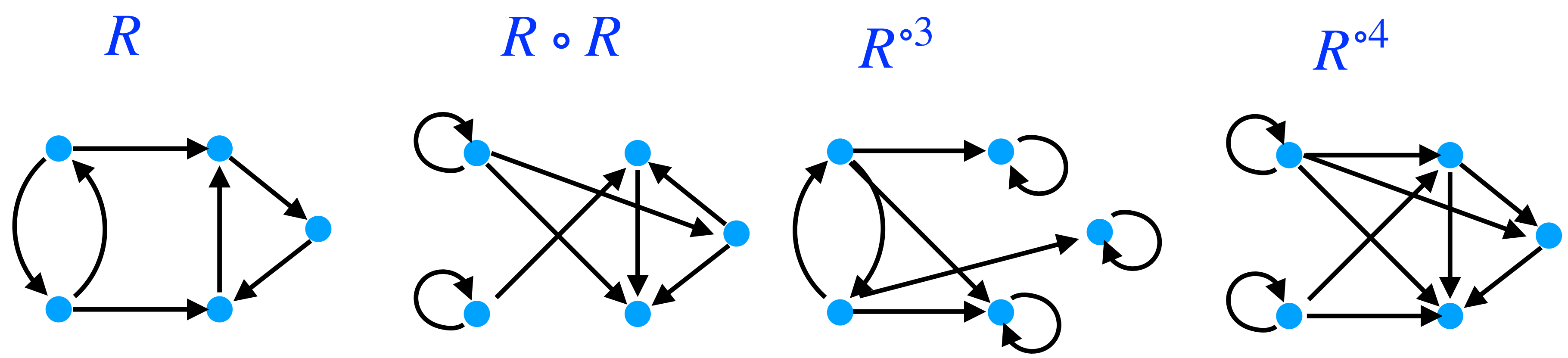

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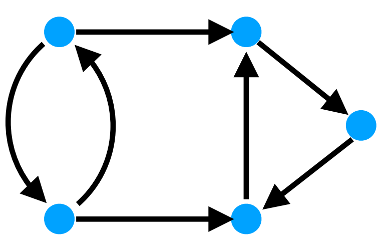
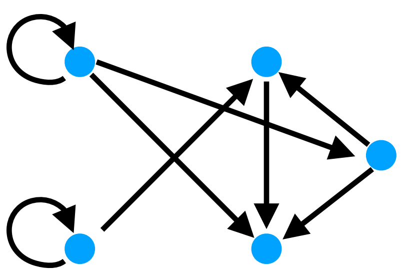
How about $R' = R \circ R \cup R$**. Are we done?**

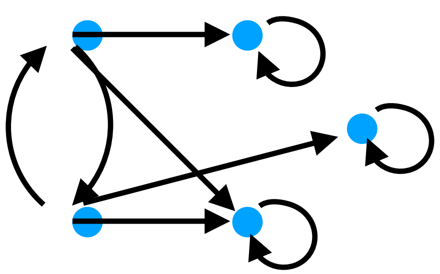
R



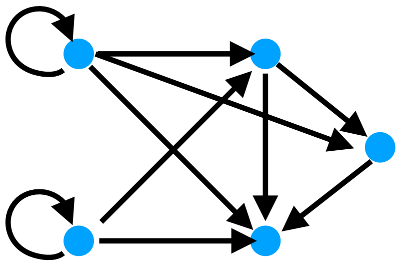
$R \circ R$



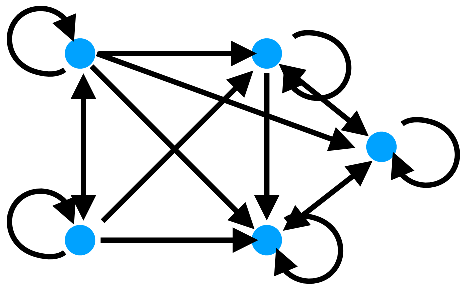
$R^{\circ 3}$



$R^{\circ 4}$



R^{+}



Closure: transitive

(1) **Transitive** $\Leftrightarrow R \circ R \subseteq R$
(then also) $R^{\circ n} \subseteq R$

The transitive closure R^+ **of** R **is given with** $R^+ = \bigcup_{k=1}^{\infty} R^{\circ k}$

Domain can be infinite
so union can be infinite

Without proof; the above is the intersecitons of all relations containing R
which is transitive; it is the "smallest"

Closure: transitive

The transitive closure R^+ of R is given with $R^+ = \bigcup_{k=1}^{\infty} R^{\circ k}$

Example: $R \in \mathbb{Z} \times \mathbb{Z}; aRb$ iff $b = a + 1$

What is R^+?

Work it!

Closure: transitive

The transitive closure R^+ of R is given with $R^+ = \bigcup_{k=1}^{\infty} R^{\circ k}$

Example: $R \in \mathbb{Z} \times \mathbb{Z}; aRb$ **iff** $b = a + 1$

math jargon; iff means "if and only if"

Work it!

aRb iff $b = a+1$

$a\, R \circ R\, b = a R^{\circ 2} b$ iff $b = a+2$

⋮

$a \underbrace{R \circ R \circ R \circ \cdots R}_{k\text{-times}} b = a R^{\circ k} b$ iff $b = a+k$

aR^+b if...

$aR^{\circ k}b$ for some k..

⋮

R^+ is $<$ (strictly less than)