FoCS

Lecture 6

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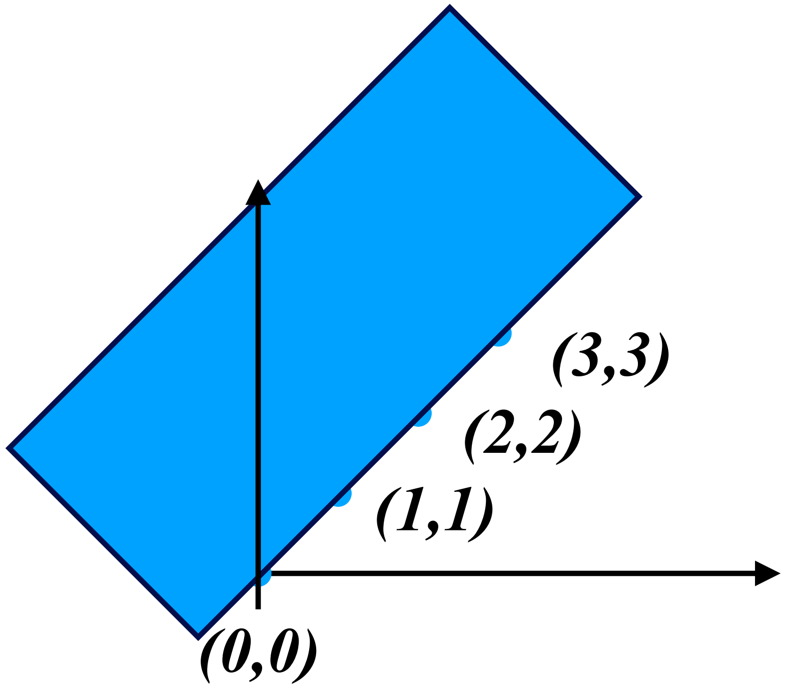
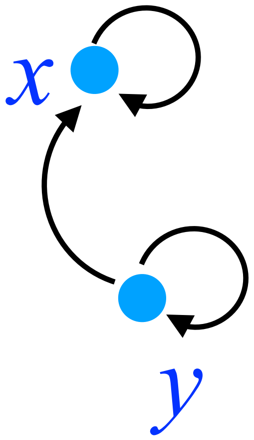
RELATIONS CONTINUED

Properties of binary relations

Definition. Relation $R \in A \times A$ is **reflexive** if
For all x, xRx.

If for all $x \in A$, $(x, x) \notin R$, then it is ***irreflexive***

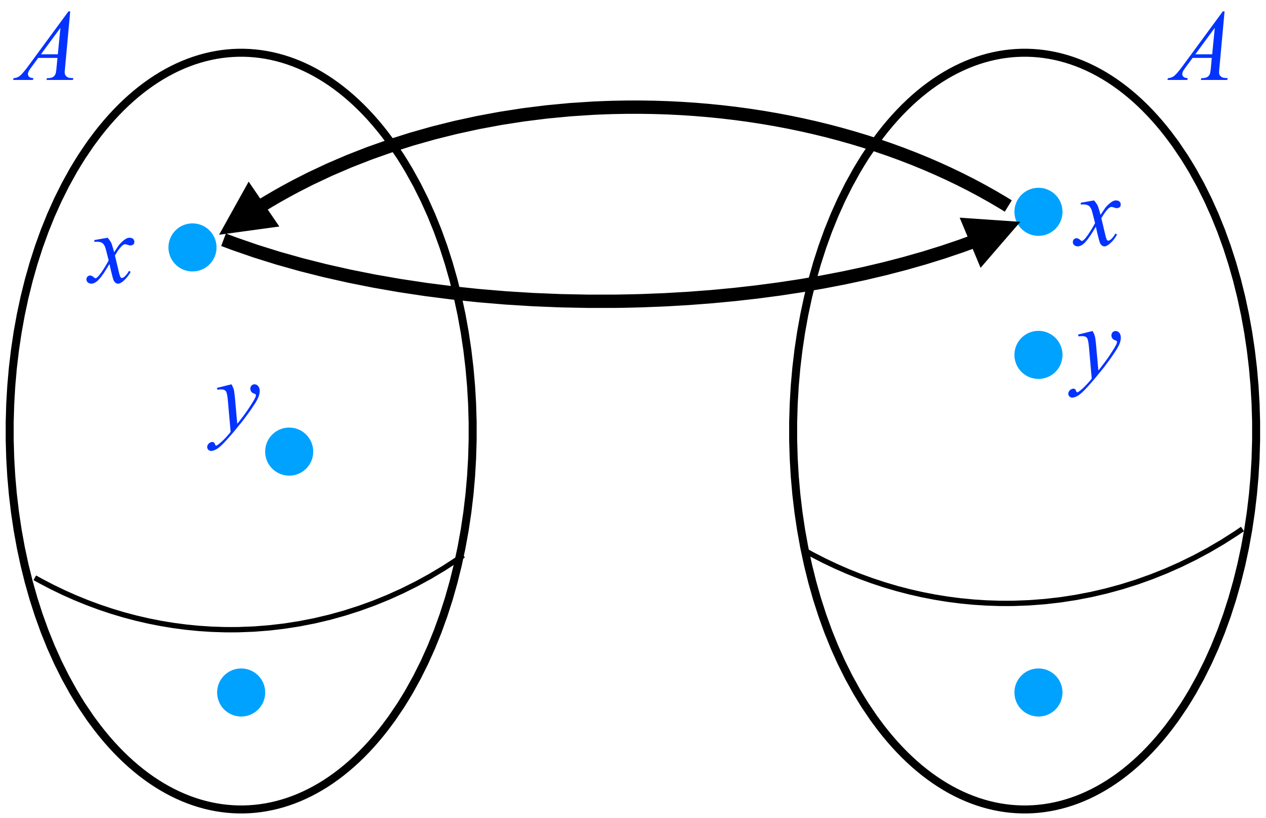




Properties of binary relations

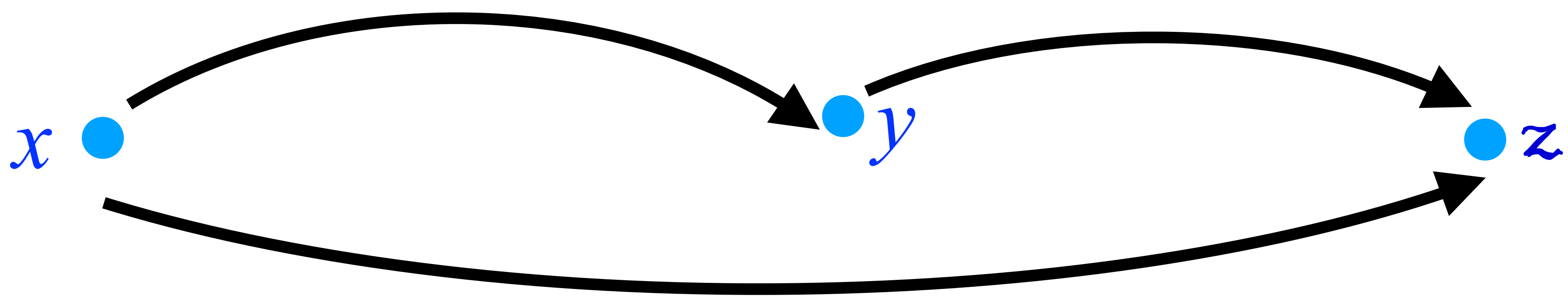
Definition. Relation $R \in A \times A$ is **symmetric** if
For all *x,y*, *if xRy then yRx.*

If xRy & yRx implies that x=y then it is ***antisymmetric***



Properties of binary relations

Definition. Relation $R \in A \times A$ is **transitive** if
For all x,y,z *if* xRy & yRz, *then* xRz



Extra clarification

not reflexive ≠ irreflexive

not symmetric ≠ antisymmetric

not reflexive: there exists x such that $(x, x) \notin R$
irreflexive: for all x $(x, x) \notin R$

Question: how would you express "not symmetric" formally, using "for all" etc…

Extra clarification

not reflexive ≠ irreflexive

not symmetric ≠ antisymmetric

not reflexive: there exists x such that $(x,x) \notin R$

irreflexive: for all x $(x,x) \notin R$

Question: how would you express "not symmetric" formally, using "for all" etc…

not symmetric: $\exists (x,y)$ s.t. $(x,y) \in R$ & $(y,x) \notin R$

N.B.: symmetricity does <u>not</u> mean <u>ALL</u> pairs-of-pairs $(x,y),(y,x)$ are in R.
It just means that <u>if</u> ONE of them is, <u>then</u> so is the other.

Extra clarification

not reflexive $\neq$ irreflexive

not symmetric $\neq$ antisymmetric

not symmetric: there exist x,y such that $(x, y) \in R$ & $(y, x) \notin R$

antisymmetric: for all x $(x, y) \in R$ & $(y, x) \in R$ **then** $x = y$

Not symmetric/reflexive: violates definition of …

Antisymmetric: only symmetric pairs are reflexive ones

Extra clarification

Examples:

reflexive: $\leq$
not reflexive: "product of x an y is even" on integers

irreflexive: $<$, $\subset$

Question?

not reflexive $\Rightarrow$ $\exists\, x$ s.t. $(x,x) \notin R$.

xRy if $x \cdot y$ is even ; xRx IF x^2 is even. but $3^2 = 9$ ...

IRREFLEXIVE : xRx never,

"Product of x, y is even" is neither reflexive nor irreflexive on integers

However, on evens it is reflexive (product of evens is even)

Extra clarification

Examples:

symmetric: $=$
not symmetric: $<$

antisymmetric: $\leq$
not antisymmetric: "is sibling of"; x, y such that $2x \geq y$ on $\mathbb{N}$ (see below!)

IS $<$ ON $\mathbb{Z}$ ANTISYMMETRIC?

Question?

YES, BY VACUOUS REASONS. NOTE, IMPLICATION "IF A THEN B" IS ONLY FALSE IF A & NOT-B. IF A IS ALWAYS FALSE A→B IS TRUE.

$(x<y \ \& \ y<x) \Rightarrow x=y$ IS TRUE BECAUSE FOR NO x, y IS $(x<y) \& (y<x)$ TRUE. THIS IS FALSE, SO $(x<y \ \& \ y<x) \Rightarrow (x=y)$ IS TRUE.

Extra clarification

Examples:

symmetric: =
not symmetric: <

antisymmetric: $\leq$
not antisymmetric: "is sibling of"; x, y such that $2x \geq y$ on $\mathbb{N}$ (see below!)

IS $<$ ON $\mathbb{Z}$ ANTISYMMETRIC?

Question?

$xRy \iff 2x \geq y.$

not antysimmetric $\Rightarrow$ [IT IS NOT THE CASE THAT IF xRy & yRx THEN $x=y$]

$\iff \exists\, x, y$ st xRy & yRx AND $x \neq y$.

NOTE $2 \cdot 3 \geq 4$ & $2 \cdot 4 \geq 3$ SO $3R4$ & $4R3$ BUT $4 \neq 3$.

Extra example

Transitivity: xRy **&** $yRz \Rightarrow xRz$

example (trivial): <, =

example (non-trivial) : empty relation (vacuous truth), IMPLICATION ITSELF.

counterexample: orthogonality

counterexample: two transitive relations $R = \{(1,2),(3,4)\}$ $S = \{(2,3),(4,5)\}$

their composition is not transitive

WHY TRANSITIVE

Question?

$R \circ S = \begin{array}{ccc} 1 & 2 & 3 \\ 3 & 4 & 5 \end{array} = \{(1,3),(3,5)\}$ $(1,5)$ missing

COMPOSITION OF TRANSITIVE RELATIONS MAY NOT BE TRANSITIVE

NOTE: ALL RELATIONS WHERE $(xRy$ & $yRz)$ NEVER HAPPENS <u>ARE</u> TRANSITIVE

Equivalence relations

Definition. Binary relation $R \subseteq A^2$ is an ***equivalence relation*** *if it is*
- *reflexive*
- *symmetric*
- *transitive*

Example: "="

Slightly more complicated: *aRb if $|a| = |b|$ [absolute value equality]*

More complicated: "is congruent to (mod n)" $[a \equiv b \pmod{n} \Leftrightarrow (n \mid (b-a))]$

Counterexample: <

REMAINDER $a \bmod n = a - \lfloor \frac{a}{n} \rfloor$

"Floor"

FIRST INTEGER LESS OR EQUAL TO a/k

$a \equiv b \pmod{n}$ iF

$a \bmod n = b \bmod n$

EQUIV: $a \equiv b \pmod{n}$ iF $(b-a) = k \times n$ for an integer k

Equivalence relations

Definition. Binary relation $R \subseteq A^2$ is an ***equivalence relation*** *if it is*

- *reflexive*
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- *transitive*

Example: "="

Slightly more complicated: *aRb if $|a| = |b|$ [absolute value equality]*

More complicated: "is congruent to (mod n)" $[a \equiv b \pmod{n} \Leftrightarrow (n \mid (b-a))]$

Counterexample: <

$aRb \Leftrightarrow a \equiv b \pmod{n}$

$aRb \Leftrightarrow$ remainder of division of a & b with n is the same.

$a \equiv b \pmod{2}$ ["evenness"]

$10 \equiv 13 \pmod{3}$

Equivalence relations

Definition. Binary relation $R \subseteq A^2$ *is an* ***equivalence relation*** *if it is*

- *reflexive*
- *symmetric*
- *transitive*

Equivalence relations = always capture some notion of equality
“the same up to ‘irrelevant’ properties”

Partial orders

Definition. Binary relation $R \subseteq A^2$ is a ***partial order*** *if it is*

- *reflexive*
- *antisymmetric*
- *transitive*

Example: $\leq$

More complicated: "divisibility" $b \mid a$, ***if*** $\frac{a}{b} \in \mathbb{Z}$ (b divides a)

Counterexample: $<$

$\underline{aRb \Leftrightarrow b \mid a.}$

(1) $a \mid a$ ✓

(2) $a \mid b$ & $b \mid a \Rightarrow a = b$ ✓

(3) aRb, $bRc \Rightarrow aRc$

$b \mid a$, $c \mid b \Rightarrow c \mid a$

$30 \mid 60$ $15 \mid 30 \Rightarrow 15 \mid 60$ ✓

Characterizations (1)

UNDERSTANDING MAIN PROPERTIES VIA COMPOSITION

(1) **R is reflexive** $\Leftrightarrow id \subseteq R$
(2) **R is symmetric** $\Leftrightarrow R^{-1} \subseteq R$
(3) **Transitive** $\Leftrightarrow R \circ R \subseteq R$
(then also $R^{\circ n} \subseteq R$**)**

recall $id_A = \{ (a,a) \mid a \in A \}$
= set of all pairs (a,a)

Let's prove these

$id \subseteq R$ means
$\rightarrow (x,y) \in id \Rightarrow (x,y) \in R$
$\rightarrow (x,y) \in id \Rightarrow (x,x) \in R$ ✓

(1) Reflexive $\Rightarrow \forall x\ (x,x) \in R \Rightarrow id \subseteq R$

$id \subseteq R \Rightarrow \forall x\ (x,x) \in R \Rightarrow R$ is reflexive

Characterizations (1)

(1) **R is reflexive** $\Leftrightarrow id \subseteq R$
(2) **R is symmetric** $\Leftrightarrow R^{-1} \subseteq R$
(3) **Transitive** $\Leftrightarrow R \circ R \subseteq R$
(then also $R^{\circ n} \subseteq R$**)**

Let's prove these

(2) $\Rightarrow$

R is symm $\Rightarrow$ $[\underbrace{xRy \Rightarrow yRx}_{(c1)}]$; need $(x,y) \in R^{-1} \Rightarrow (x,y) \in R$.

(2) $\Leftarrow$ $R^{-1} \subseteq R$

$\Leftrightarrow$ $\underbrace{(y,x) \in R \Rightarrow (x,y) \in R}_{(c1)\ \checkmark}$

$\Rightarrow$ if $(x,y) \in R^{-1} \Rightarrow (x,y) \in R$ &
$(y,x) \in R \Rightarrow (x,y) \in R$

Characterizations (1)

(1) **R is reflexive** $\Leftrightarrow id \subseteq R$

(2) **R is symmetric** $\Leftrightarrow R^{-1} \subseteq R$

(3) **Transitive** $\Leftrightarrow R \circ R \subseteq R$
(then also $R^{\circ n} \subseteq R$**)**

Let's prove these

(3) $\Rightarrow$ $\underbrace{[xRy \ \& \ yRz \Rightarrow xRz]}_{(C_1)}$;

need $\underbrace{(a,c) \in R \circ R}$ $\Rightarrow (a,c) \in R$

$\exists y \ (a,y) \in R \ \& \ (y,c) \in R$

Let a & c are such that that $\exists$ y FOR which (a,y) & (y,c) are in R

But then by (C_1) $(a,c) \in R$. ✓

$\Leftarrow$ ASSUME THAT $[(a,c) \in R \circ R$

$\Rightarrow \exists y \ (a,y) \in R, (y,c) \in R] \Rightarrow (a,c) \in R$

Need: $\underbrace{(a,y), (y,c) \in R}$ $\Rightarrow (a,c) \in R$

But $(a,c) \in R \circ R$ so $\Rightarrow (a,c) \in R$!!

Characterizations (2)

(1) **R is irreflexive** $\Leftrightarrow id \cap R = \varnothing$

(2) **R is antisymmetric** $\Leftrightarrow R^{-1} \cap R \subseteq id$

Let's prove these

(1) $\Rightarrow$ $\underbrace{\forall x \quad (x,x) \notin R}_{(C1)}$.

$(a,b) \in id \cap R \Rightarrow$ (C1)

$(a,b) \in id \Rightarrow (a=b) \Rightarrow (a,b) \notin R$

$\Rightarrow (a,b) \notin id \cap R \quad \forall (a,b)$

$\Rightarrow \varnothing = id \cap R$.

$\Leftarrow$ $\underbrace{id \cap R = \varnothing}_{(C1)}$, $\forall (a,a)$, $(a,a) \in id \overset{(C1)}{\Rightarrow} \underbrace{(a,a) \notin R}$

$\Leftrightarrow$ irreflexive

Characterizations (2)

(1) **R is irreflexive** $\Leftrightarrow id \cap R = \varnothing$

(2) **R is antisymmetric** $\Leftrightarrow R^{-1} \cap R \subseteq id$

Let's prove these

(2) $\Leftarrow$ IF $(x,y) \in R$ & $(y,x) \in R \Rightarrow (x,y) \in id \Rightarrow x = y$. ✓

$\Rightarrow$ yRx & $xRy \Rightarrow x = y$; $(x,y) \in R$ & $(y,x) \in R \Rightarrow x = y \Rightarrow (x,y) = (x,x) \in id$.

Characterizations (3)

(1) $R \subseteq A \times B$ **is functional** $\Leftrightarrow R^{-1} \circ R \subseteq id_B$
(2) $R \subseteq A \times B$ **is surjective** $\Leftrightarrow id_B \subseteq R^{-1} \circ R$

(3) $R \subseteq A \times B$ **is injective** $\Leftrightarrow R \circ R^{-1} \subseteq id_A$
(4) $R \subseteq A \times B$ **is total** $\Leftrightarrow id_A \subseteq R \circ R^{-1}$

} ***via inverse***

Maybe one

(1) $\Rightarrow$ $[xRy \ \& \ xRz \Rightarrow y = z]$ (C1)

$(a,b) \in R^{-1} \circ R$, $\exists y$ $\underbrace{(a,y) \in R^{-1}}$ & $(y,b) \in R$

$\Rightarrow \exists y$ $(y,a) \in R$ & $(y,b) \in R$ (by C1)

$\Rightarrow a = b \Rightarrow (a,b) = (b,b) \in \underline{id_B}$

$\Leftarrow$ if (a,b) st. (y,a) & $(y,b) \in R \Rightarrow a = b$. THIS IS FUNCTIONALITY !

Characterizations (3)

(1) $R \subseteq A \times B$ **is functional** $\Leftrightarrow R^{-1} \circ R \subseteq id_B$

(2) $R \subseteq A \times B$ **is surjective** $\Leftrightarrow id_B \subseteq R^{-1} \circ R$

(3) $R \subseteq A \times B$ **is injective** $\Leftrightarrow R \circ R^{-1} \subseteq id_A$

(4) $R \subseteq A \times B$ **is total** $\Leftrightarrow id_A \subseteq R \circ R^{-1}$

$\Big\}$ ***via inverse***

Maybe one

(3) via inverse : R is injective $\Leftarrow ?$ R^{-1} is functional

$aR^{-1}b$ & $aR^{-1}c \Rightarrow b=c$

bRa & $cRa \Rightarrow b=c$ } injectivity

(3) $\Rightarrow$ $\boxed{R^{-1} \subseteq B \times A \text{ is functional iff } R \circ R^{-1} \subseteq id_A}$

↳ Relabel ! R exchange with R^{-1}, B with A, get (1). ALREADY PROVEN TRUE.

Closure

For property P (reflexivity, symmetricity, transitivity)…

P-closure of relation R = "smallest" relation with property P

But what does smallest mean??

Closure

For property *P* (reflexivity, symmetricity, transitivity)…

P*-closure of relation *R* = "smallest" relation with property *P

But what does smallest mean??

Intuition… start adding pairs that are missing… but is this process always ending with the same relation…

is "P-closure" well-defined?

Closure

Easy ones: symmetric and reflexive closure

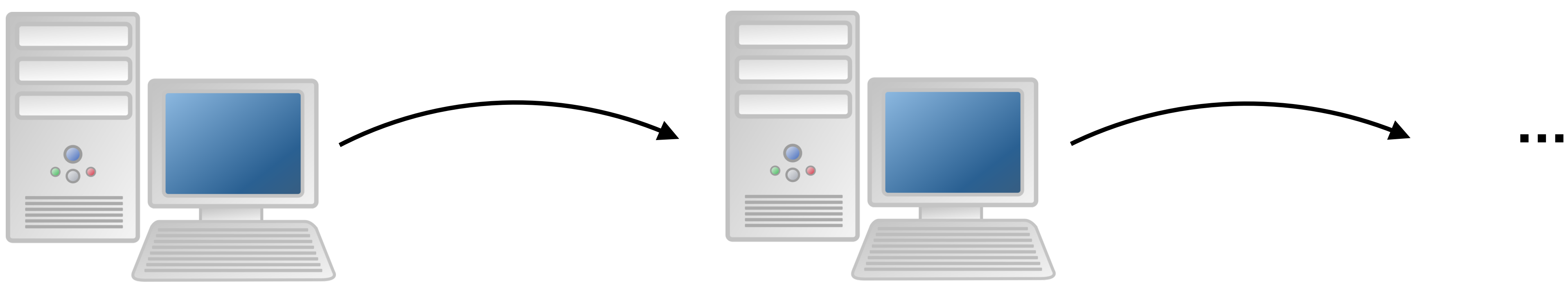
(1) R **is reflexive** $\Leftrightarrow id \subseteq R$
(2) R **is symmetric** $\Leftrightarrow R^{-1} \subseteq R$

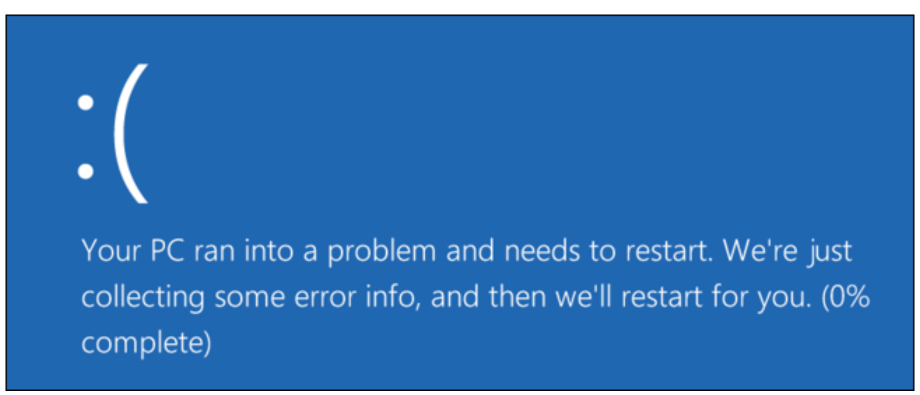
Given R, its symmetric closure is $S = R \cup id_A$

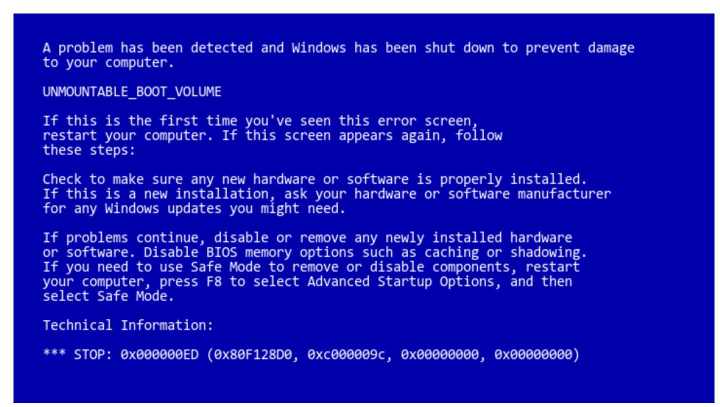
Given R, its reflexive closure is $S = R \cup R^{-1}$

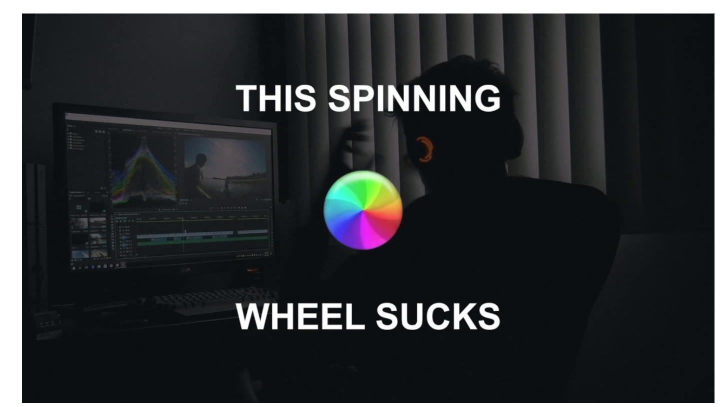
Closure

Transitive closure: super important









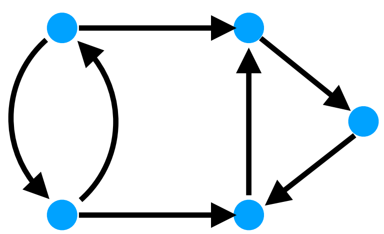
Closure: transitive

(1) Transitive $\Leftrightarrow R \circ R \subseteq R$
(then also) $R^{\circ n} \subseteq R$

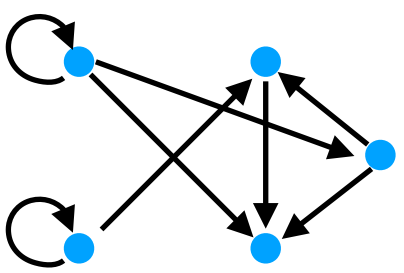
Suppose $R \circ R \nsubseteq R$

How about $R' = R \circ R \cup R$. Are we done?

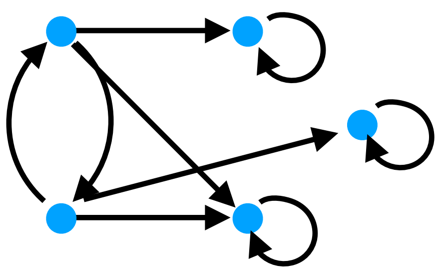
R



$R \circ R$



$R^{\circ 3}$



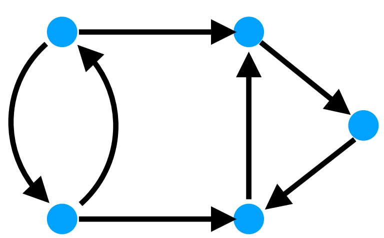
Closure: transitive

(1) Transitive $\Leftrightarrow R \circ R \subseteq R$
(then also) $R^{\circ n} \subseteq R$

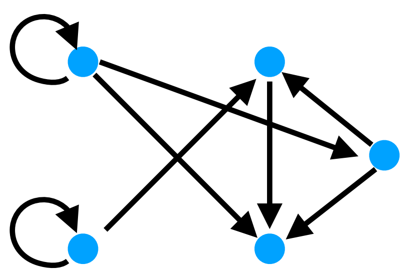
Suppose $R \circ R \not\subseteq R$

How about $R' = R \circ R \cup R$. Are we done?

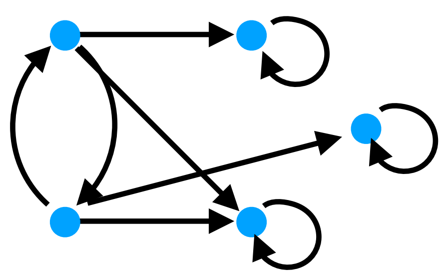
R



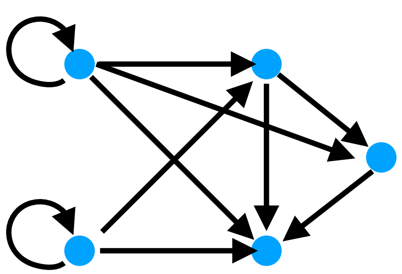
$R \circ R$



$R^{\circ 3}$



$R^{\circ 4}$

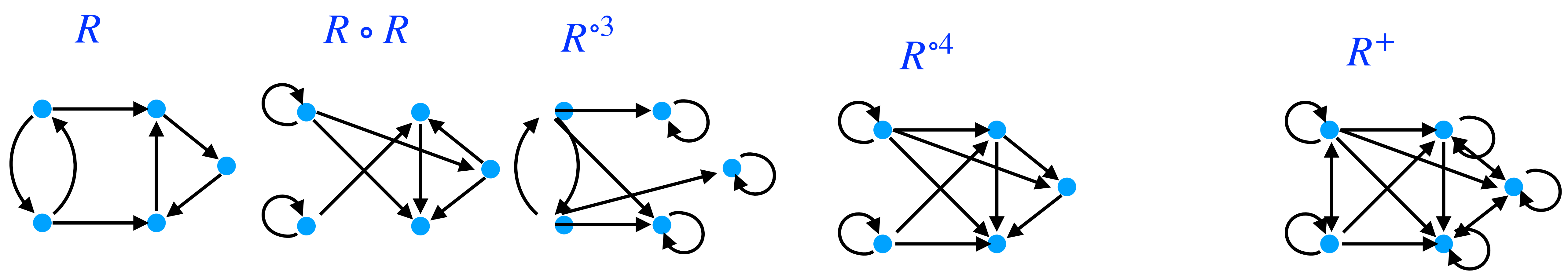


Closure: transitive

(1) Transitive $\Leftrightarrow R \circ R \subseteq R$
(then also) $R^{\circ n} \subseteq R$

Suppose $R \circ R \nsubseteq R$

How about $R' = R \circ R \cup R$. Are we done?



Closure: transitive

(1) **Transitive** $\Leftrightarrow R \circ R \subseteq R$
(then also) $R^{\circ n} \subseteq R$

The transitive closure R^{+} **of** R **is given with** $R^{+} = \bigcup_{k=1}^{\infty} R^{\circ k}$

Domain can be infinite…

Closure: transitive

The transitive closure R^+ of R is given with $R^+ = \bigcup_{k=1}^{\infty} R^{\circ k}$

Example: $R \in \mathbb{Z} \times \mathbb{Z}; aRb$ **iff** $b = a + 1$

What is R^+**?**

Work it!

$R^+ = \bigcup R^{\circ k}$

Closure: transitive

The transitive closure R^+ **of** R **is given with** $R^+ = \bigcup_{k=1}^{\infty} R^{\circ k}$

Example: $R \in \mathbb{Z} \times \mathbb{Z}; aRb$ **iff** $b = a + 1$

math jargon; iff means "if and only if"

Work it!

$R^+ = \bigcup R^{\circ k}$

$R^{\circ k} \Leftrightarrow a R^{\circ k} b \Leftrightarrow b = a + k$

aRb ... if $\exists k$ st $b = a + k$ ....

$\Leftrightarrow a < b$. $R^+ = <$

Closure: transitive

Property: intersection and union of transitive relations is transitive.

Proofs! direct and by contradiction

Intersection

R st $R \circ R \subseteq R$ (1)

S st $S \circ S \subseteq S$ (2)

$R \cap S = (x,y)$ st xRy & xSy.

Assume

$(x,y) \in R \cap S$ & $(x,z) \in R \cap S$.

$\Rightarrow (x,z) \overset{(1)}{\in} R$ & $(x,z) \overset{(2)}{\in} S$

$\Rightarrow (x,z) \in R \cap S$. done.

Closure: transitive

Property: intersection and union of transitive relations is transitive.

Proofs! direct and by contradiction

BY CONTRADICTION: (intersection)

Assume R, S transitive & $R \cap S$ is not.

$\Rightarrow \exists \underbrace{(x,y), (y,z) \in R \cap S}_{(*)}$ st $(x,z) \notin R \cap S$.

$\Rightarrow$ Either $(x,z) \notin R$ or $(x,z) \notin S$. ASSUME $(x,z) \notin R$.

But then $(x,y) \in R$ & $(y,z) \in R$ (by $(*)$) & $(x,z) \notin R$

$\Rightarrow$ R is NOT TRANSITIVE. CONTRADICTION.

IF $(x,z) \notin S$ WE GET $[(x,y) \in S$ & $(y,z) \in S$ & $(x,y) \notin S] \Rightarrow$ Contradiction ✓ $\square$