Lecture 5

Relations 2

Tuples & Cartesian products

$$A_1, A_2, A_3 \ldots, A_n$$

$$(a_1, a_2, a_3, \ldots, a_n)\ a_i \in A_i$$

$$A = A_1 \times A_2 \times A_3 \times \cdots \times A_n$$

Definition.

$$A = \{(a_1, a_2, \ldots, a_n) \,|\, a_i \in A_i\}$$

Notation: $B^n := B^{\times n} := B \times B \times B \times \cdots \times B$

$$(a_1, a_2, a_3, \ldots, a_n) = (b_1, b_2, b_3, \ldots, b_n)$$

if and only if

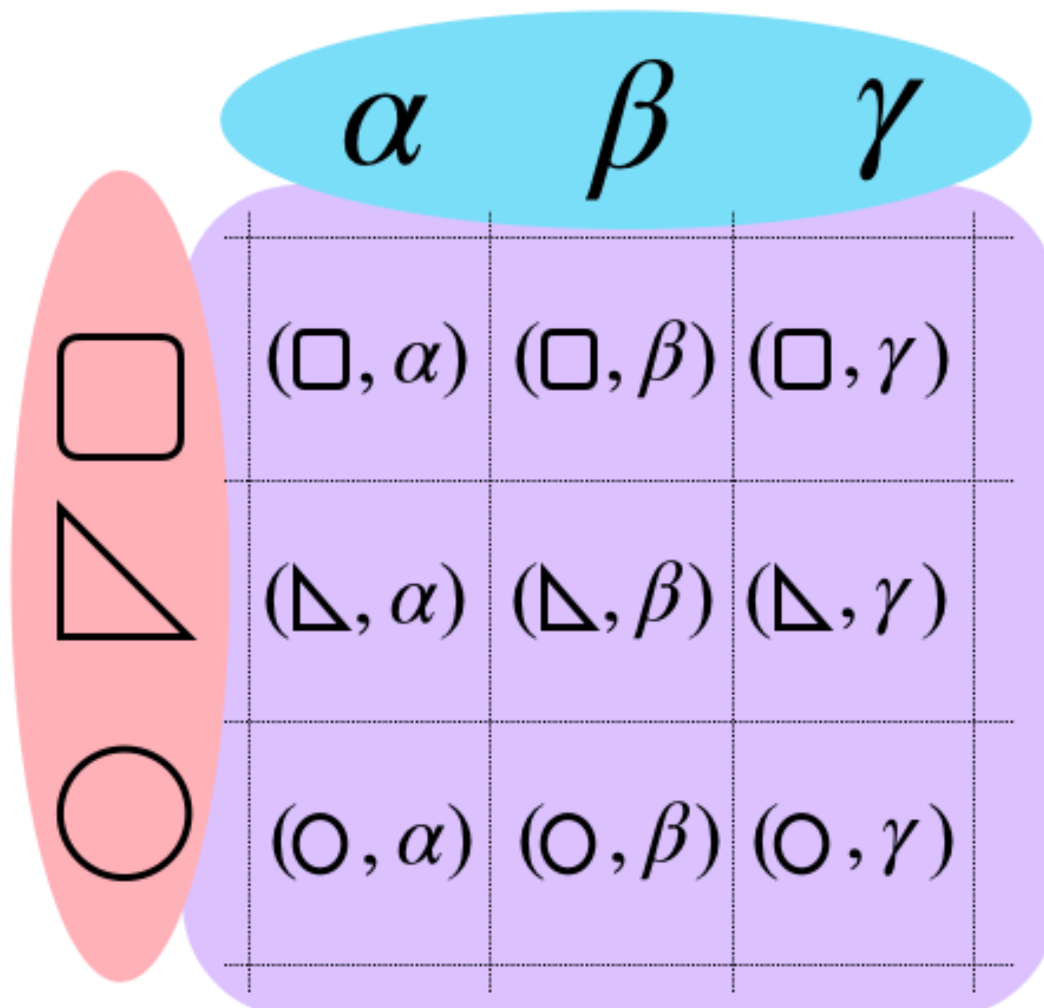
for all $i \in \{1, \ldots, n\}, a_i = b_i$.

Kuratowski?

Tuples & Cartesian products

$A = A_1 \times A_2$ **"ordered pairs"**

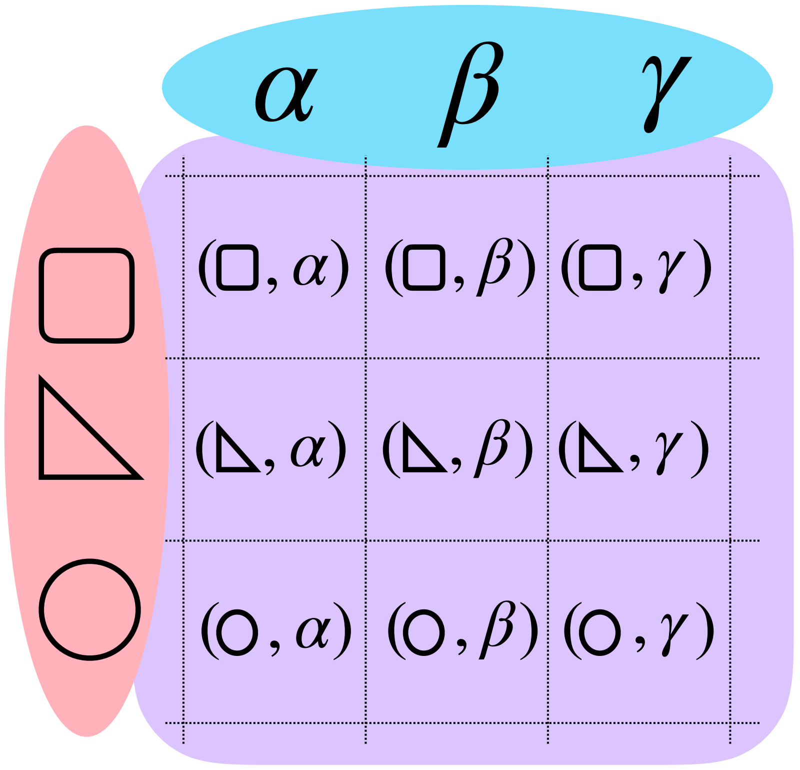
$$(a, b) = (c, d) \Leftrightarrow a = c \;\&\; b = d$$



Tuples & Cartesian products: counting
Counting reminder

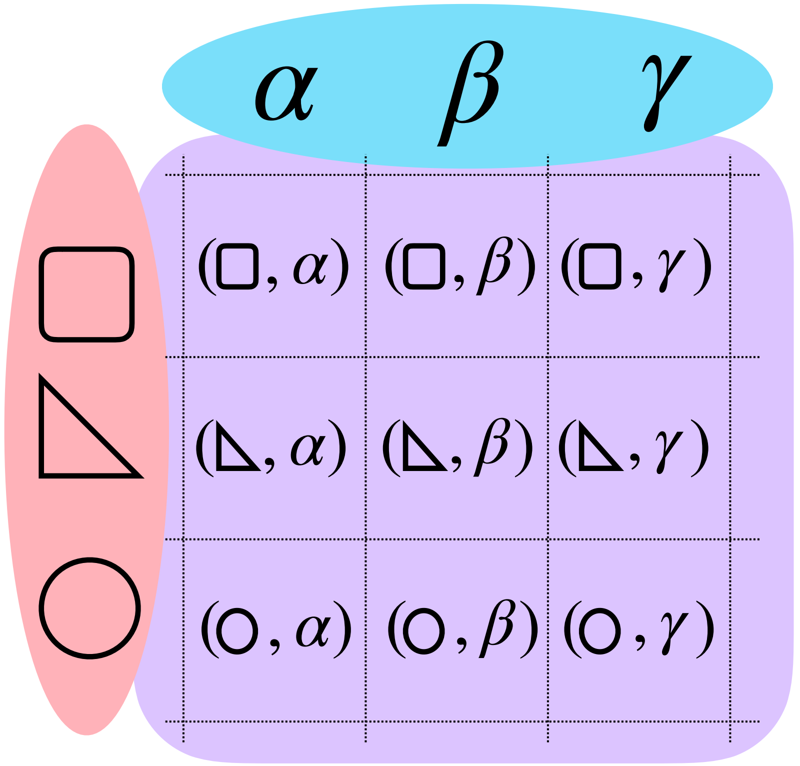
- $|A|$, $n(A)$, $\#(A)$, **or** $card(A)$
- **For sets A, B, C it holds that**

  $|A \cup B| = |A| + |B| - |A \cap B|$



Counting reminder

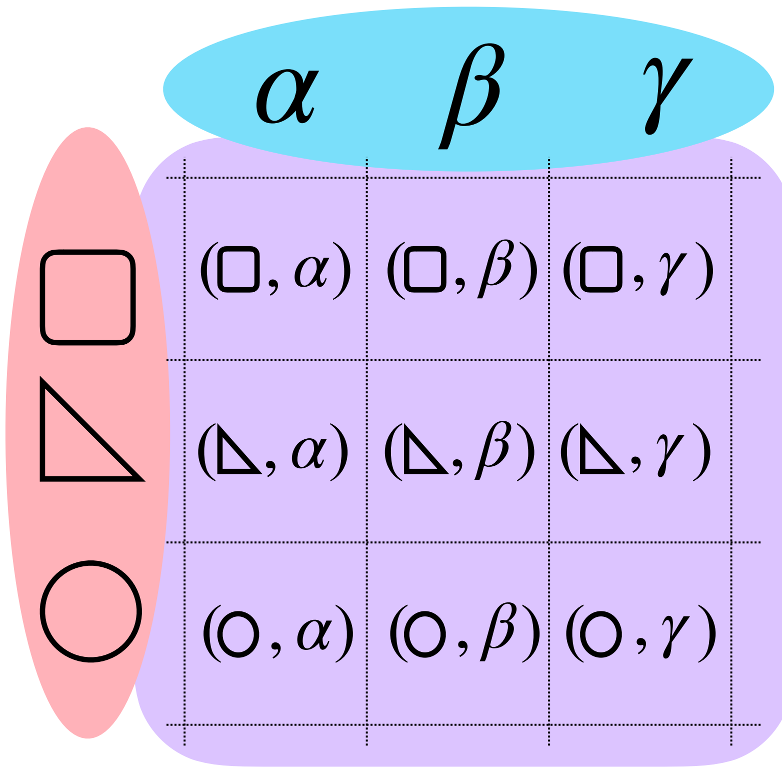
$$A = A_1 \times A_2 \times A_3 \times \cdots \times A_n$$



Tuples & Cartesian products: counting
Counting reminder

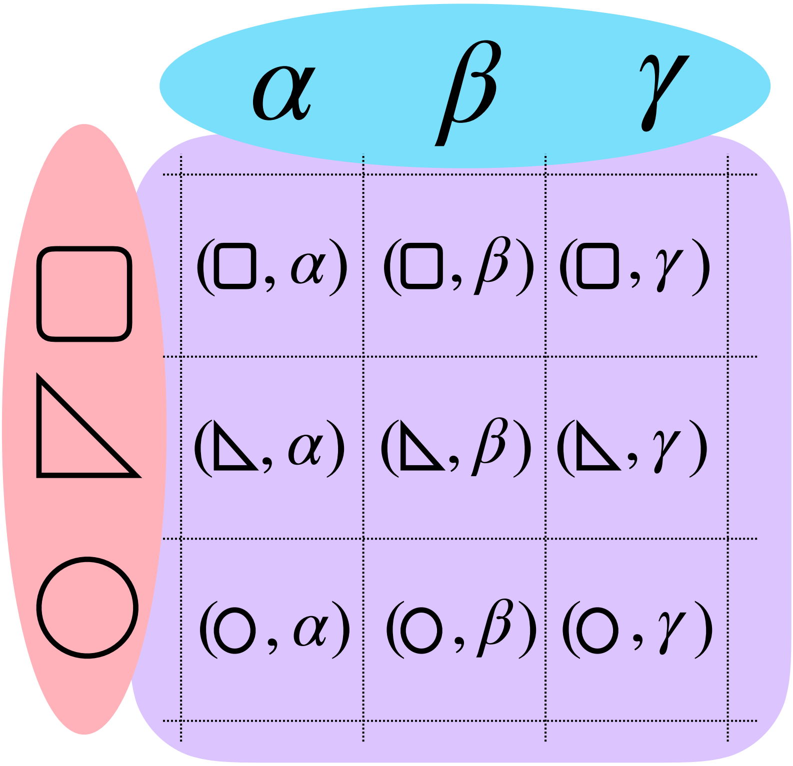
- $A = A_1 \times A_2 \times A_3 \times \cdots \times A_n$
- $|A|$ *?*

- $|A| = a; |B| = b; \quad |A \times B| = ?$



Tuples & Cartesian products: counting
Counting reminder

- $A = A_1 \times A_2 \times A_3 \times \cdots \times A_n$
- $|A|$ *?*
- $|A| = a; |B| = b; \quad |A \times B| = ?$
- ***notation helps:***
  $|A \times B| = |A| \times |B| = |A| \cdot |B|$



Tuples & Cartesian products: counting
Counting reminder

- $A = A_1 \times A_2 \times A_3 \times \cdots \times A_n$
- $|A|$ *?*

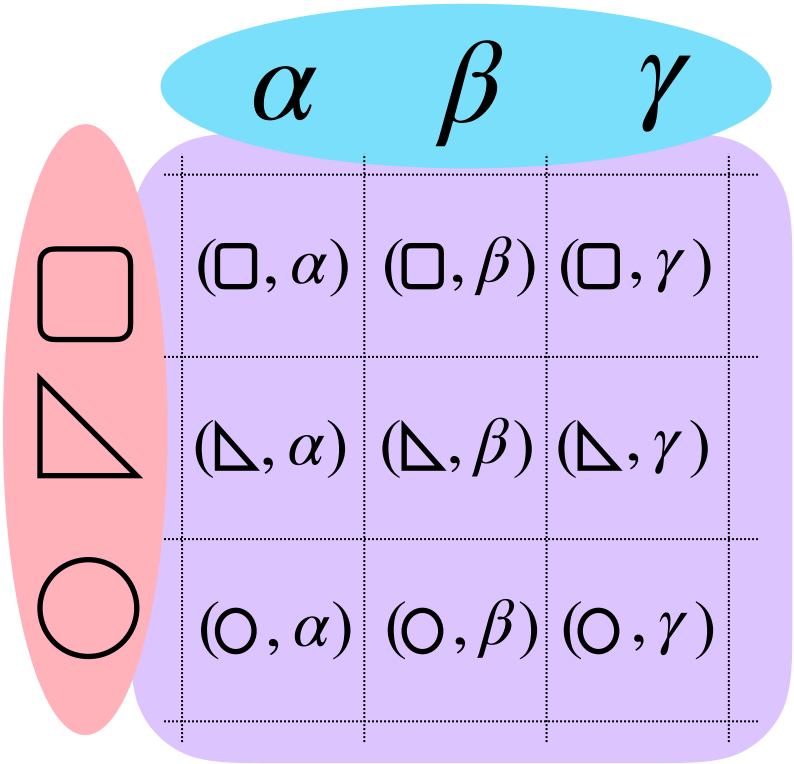
- $|A| = a; |B| = b; \quad |A \times B| = ?$

- ***notation helps:***

  $|A \times B| = |A| \cdot |B|$

  $|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_n|$

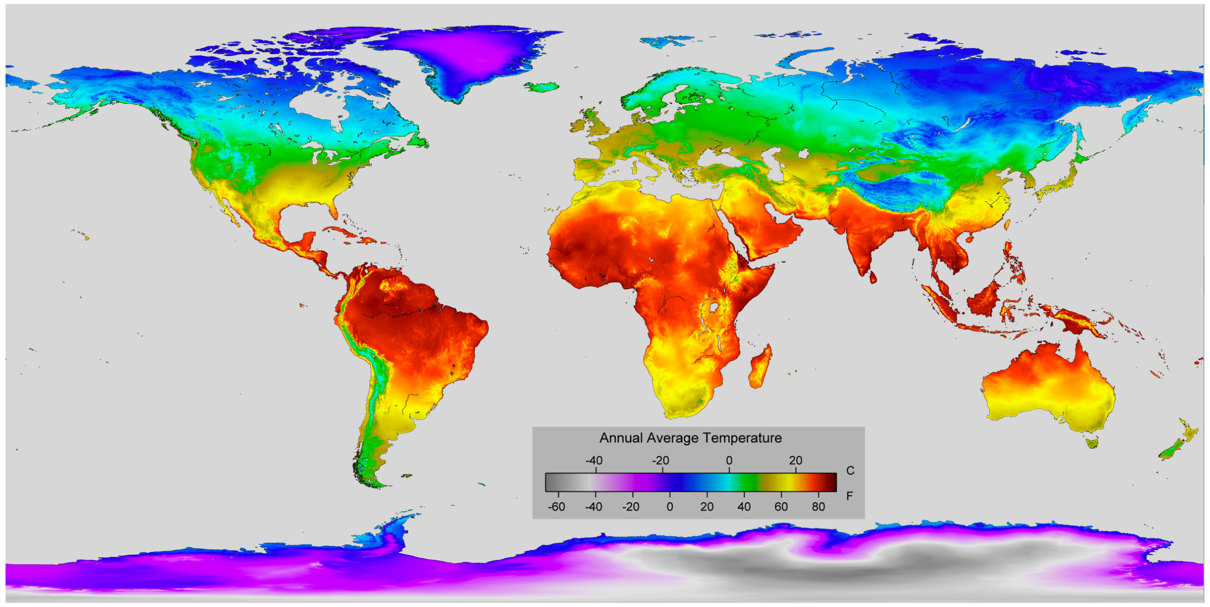
  $|A^n| = |A|^n$



Relations reminder

Subsets of Cartesian products:

$R \subseteq A_1 \times A_2 \times A_3 \times \cdots \times A_n$



(Latitude, longitude,temperature) $\in [0\ 2\pi] \times [0,2\pi] \times \mathbb{R}$

Relations reminder

Subsets of Cartesian products:

$R \subseteq A_1 \times A_2 \times A_3 \times \cdots \times A_n$

Ordering:

$R \subseteq A \times B$

"relation from A to B"

"n-ary relation"

"binary relation"

Relations reminder

Subsets of Cartesian products:

$R \subseteq A_1 \times A_2 \times A_3 \times \cdots \times A_n$

Ordering:

$R \subseteq A \times B$

"relation from A to B"

"n-ary relation"

"binary relation"

Notation: instead of $(a,b) \in R$ ***we write*** aRb

(or $(a,b) \in \star \Leftrightarrow a \star b$; $or (a,b) \in \square \Leftrightarrow a \square b;$***)***

Relations reminder

Subsets of Cartesian products:

$R \subseteq A_1 \times A_2 \times A_3 \times \cdots \times A_n$

Ordering:

$R \subseteq A \times B$

"relation from A to B"

"n-ary relation"

"binary relation"

Notation: instead of $(a, b) \in R$ ***we write*** aRb

Instead of $(42{,}256) \in <$ ***we write*** $42 < 256$.

Relations and counting intermezzo

Set of all subsets of S, $|S|=n$ has …. elements?

Given $|A|=a$, $|B|=b$, a relation from A to B is *any subset of* $A \times B$ *(card?)*

How many relations from A to B are there?

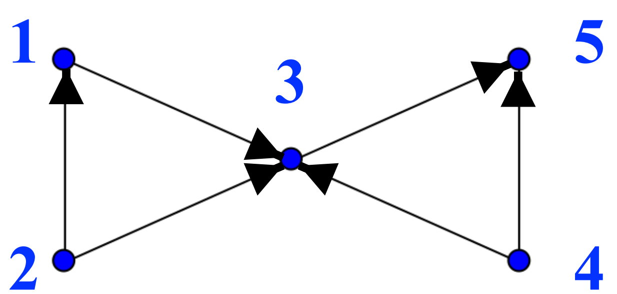
Lets work this out:

Special binary relations

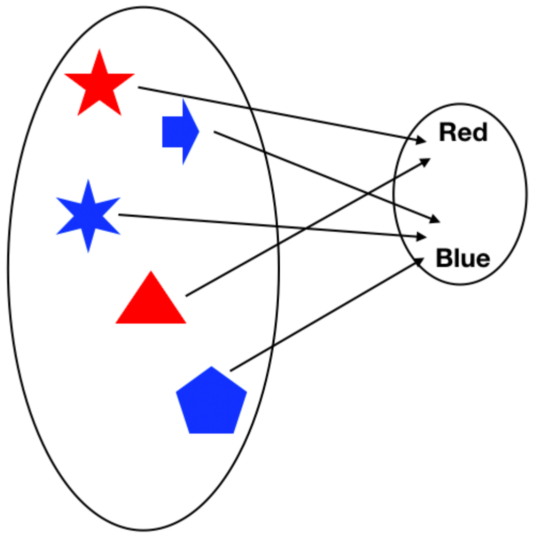
- **empty relation**
- **relations in** A: $R \subseteq A^2$
- ***inverse relation*** $R^{-1} = \{(b,a) \mid (a,b) \in R\}$

One or two examples… IF NEEDED

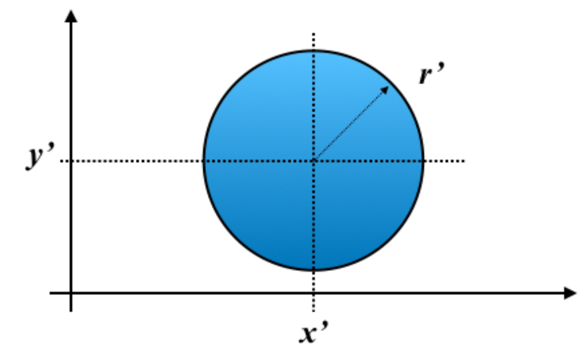
Representing relations



Directed graphs



Arrow diagrams



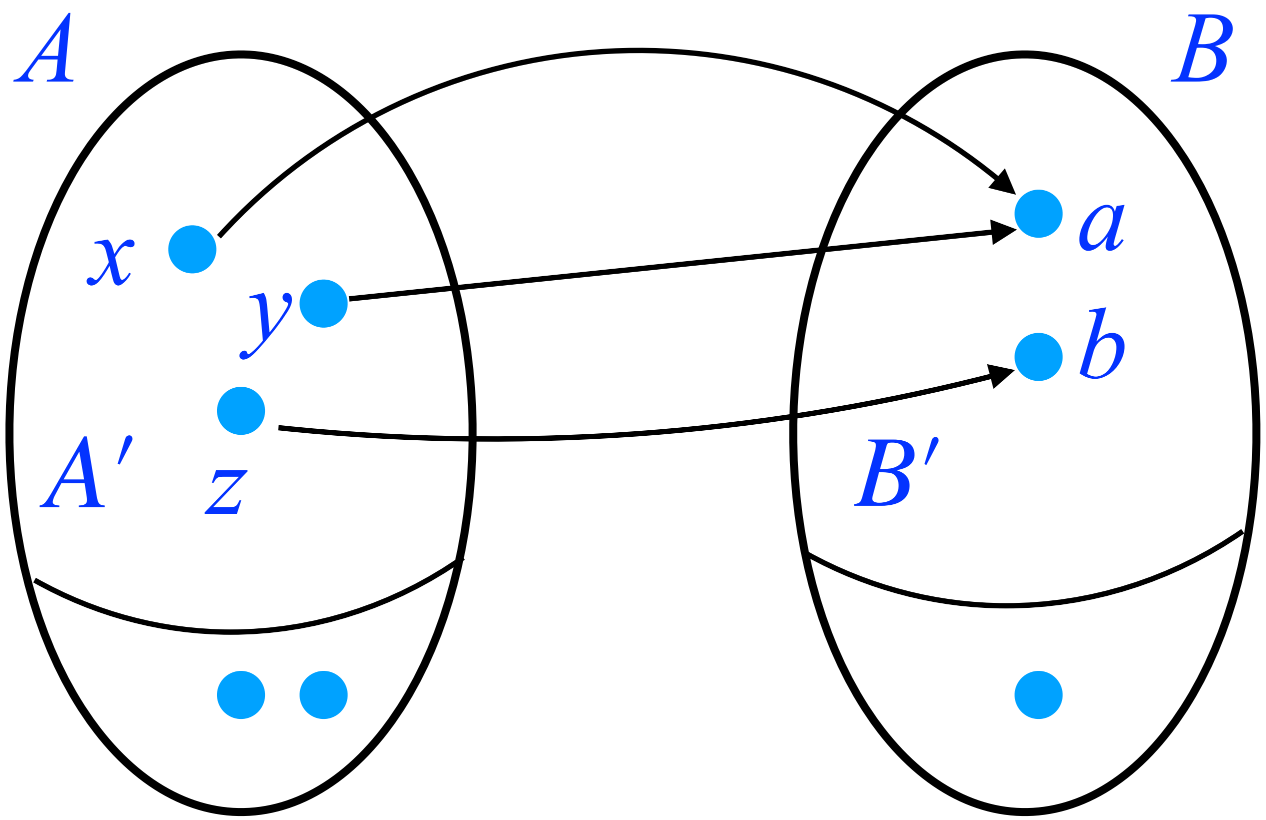
Graphs (plot)

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Matrix

(adjacency matrix of the graph)

Domain, range, image, preimage



$R \subseteq A \times B$

Domain

$dom(R) = A'$

Range or image

$range(R) = B'$

Codomain

Preimage

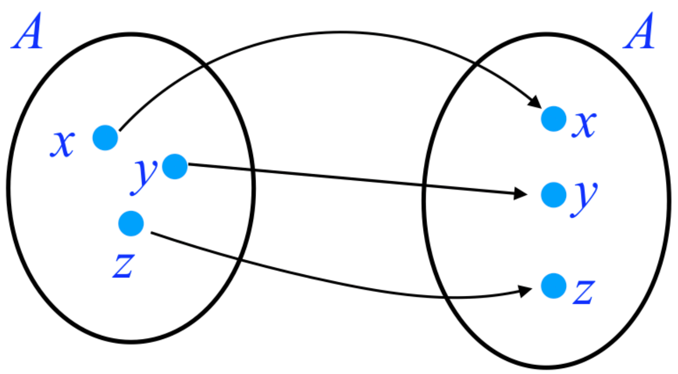
More details: $\forall$'s & $\exists$'s

Special relations continued: identity

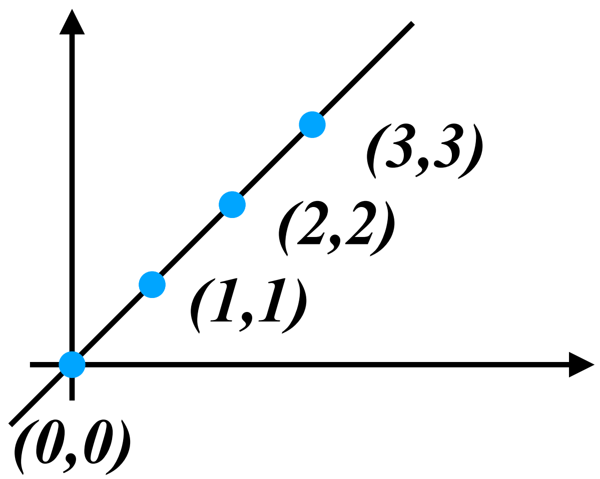
$R \subseteq A^2$

$R = \{(x,x) \mid x \in A\}$

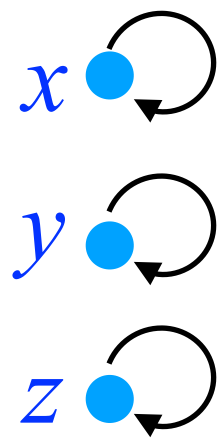
notation: $id_A, \mathbf{1}_A, \Delta_A$



Arrow diagram



Graph (plot)



Graph with self-loops

Main binary relation types

$R \subseteq A \times B$

Functional: if aRb and aRc then $b = c$. [no 1-to-many!]

Total: if $a \in A$ then aRb for *some* $b \in B$. [domain is used up!]

Injective: if aRb and cRb then $a = c$. [no many-to-1]

Surjective: if $b \in B$ then aRb for some $a \in A$. [codomain is used up]

Once more…

Main binary relation types

$R \subseteq A \times B$

Functional: if aRb and aRc then $b = c$. [no 1-to-many!]

Total: if $a \in A$ then aRb for *some* $b \in B$. [domain is used up!]

Injective: if aRb and cRb then $a = c$. [no many-to-1]

Surjective: if $b \in B$ then aRb for some $a \in A$. [codomain is used up]

Instead of "for some"

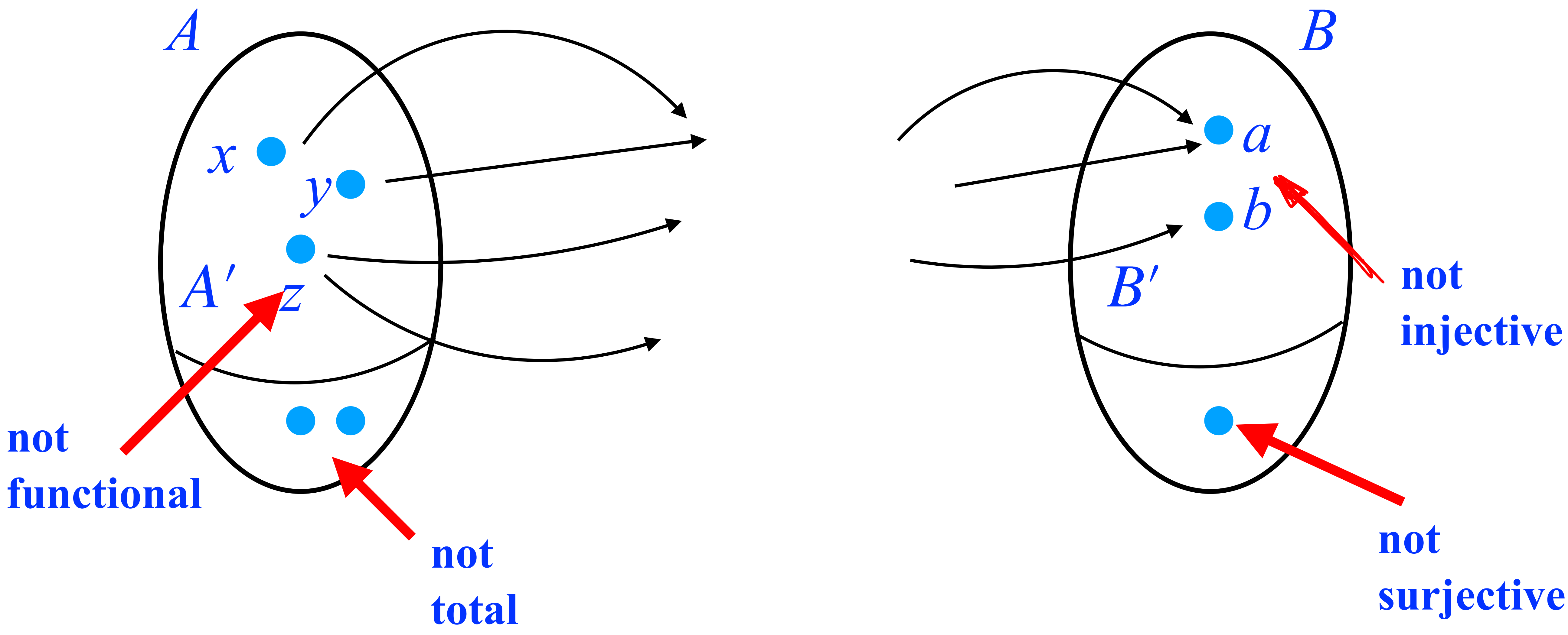
$a \in A$ then aRb "for some" b

$\forall a \in A \;\; \exists b \in B$ s.t. $(a,b) \in R$

$\cap$

Main binary relation types

$R \subseteq A \times B$



Main binary relation types

$$R \subseteq A \times B$$

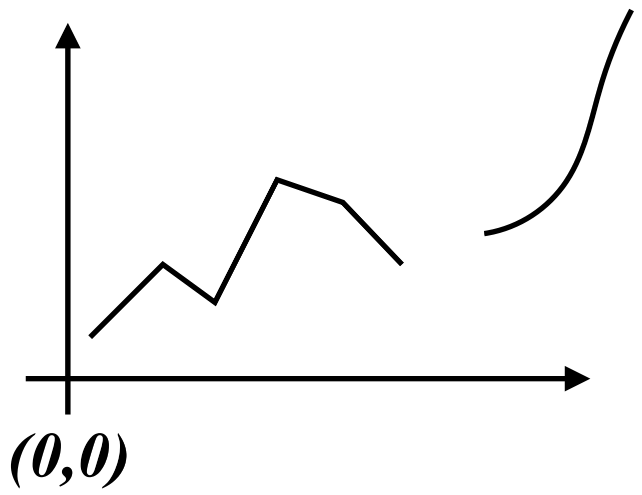
$$A = \{1,2,3\}; B = \{a, b, c, d\}$$

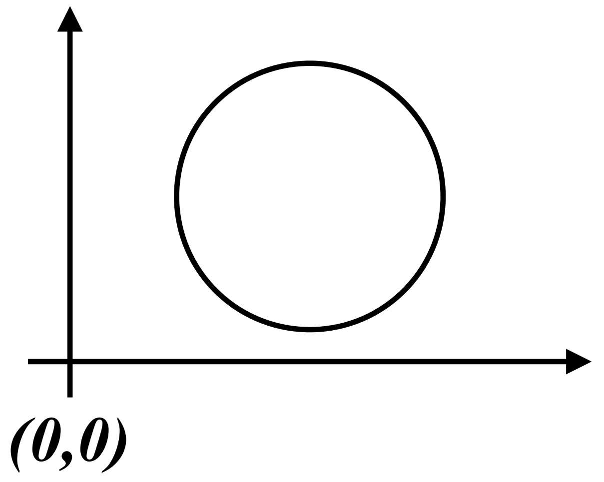
Examples: functional, injective, surjective, total

Functions!

$R \subseteq A \times B$

Functional relations. $R : A \to B$





$R \subseteq \mathbb{R} \times \mathbb{R}$

More on inverse relations

$R \subseteq A \times B$

$R^{-1} \subseteq B \times A$, **defined with** $(b, a) \in R^{-1} \Leftrightarrow (a, b) \in R$

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Representations are **intuitive**

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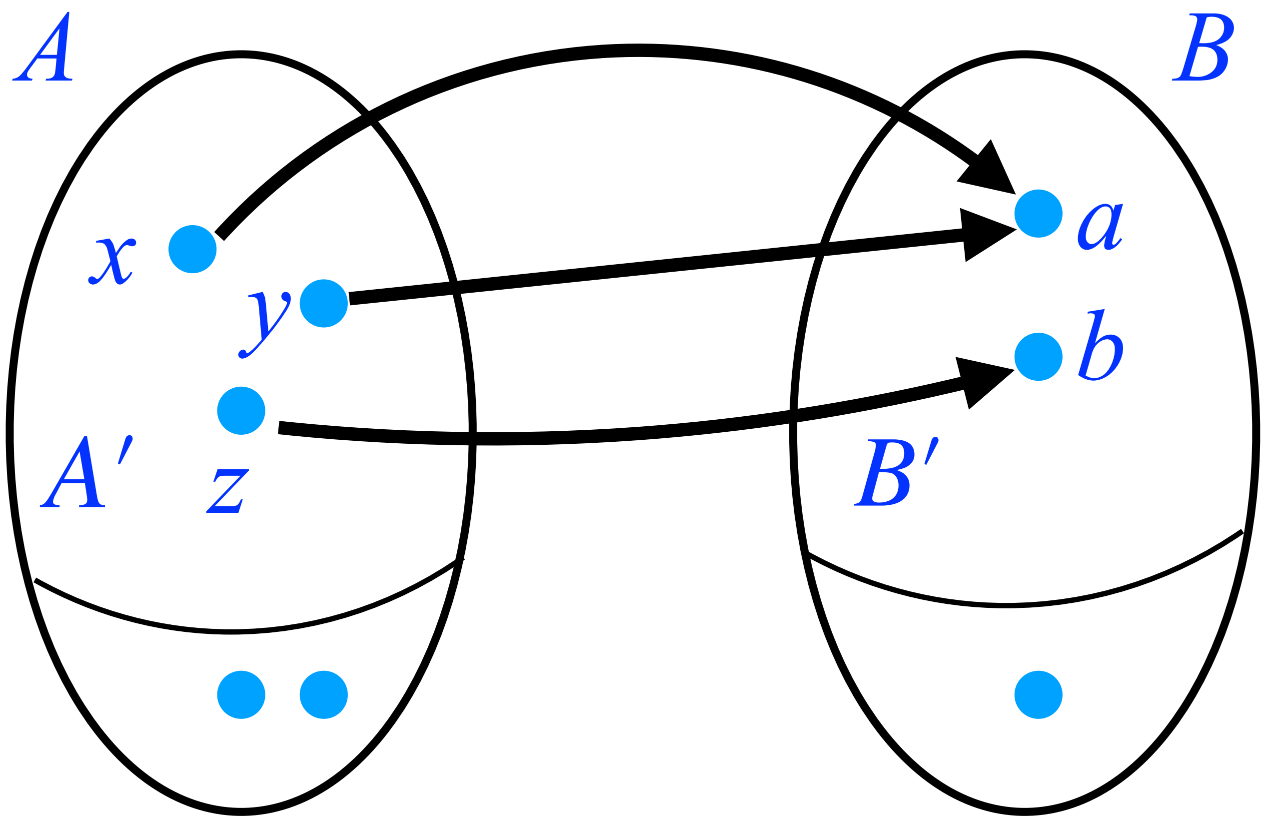

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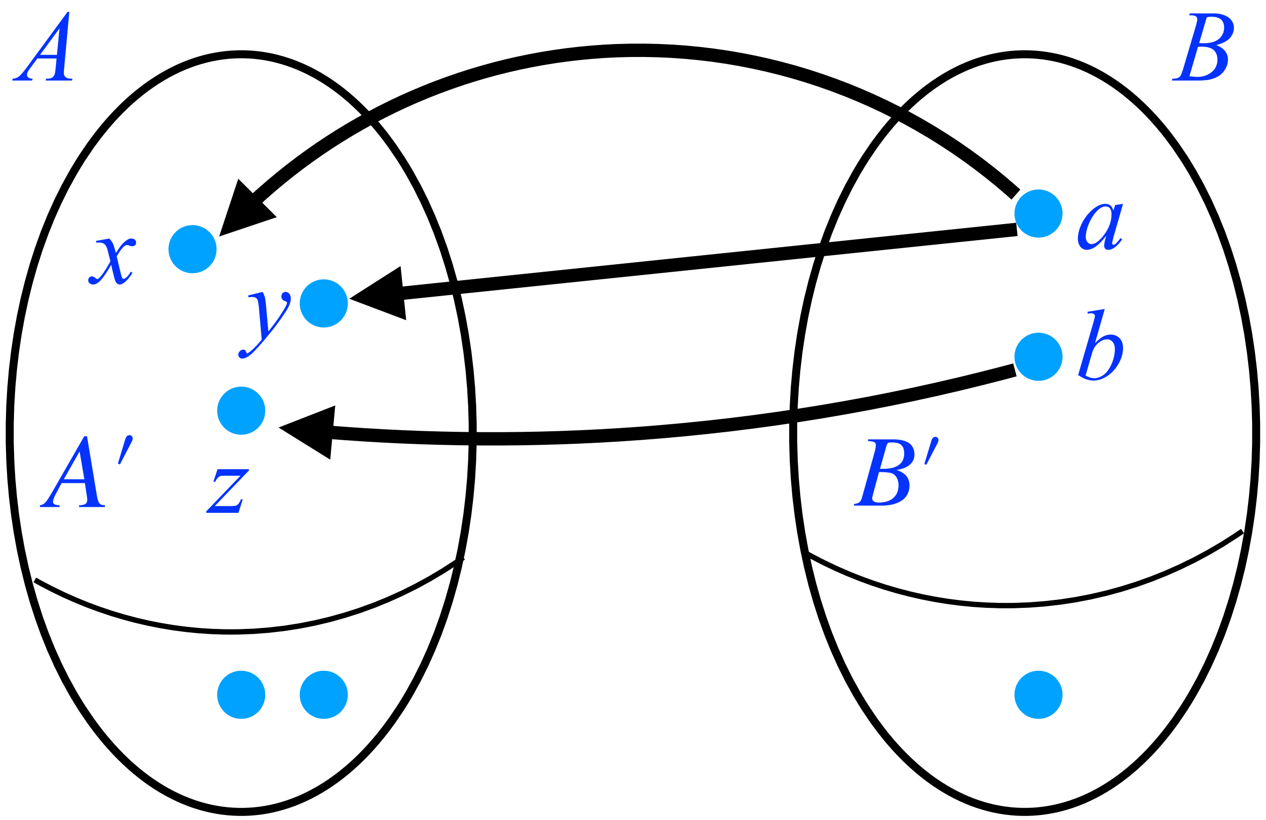

More on inverse relations

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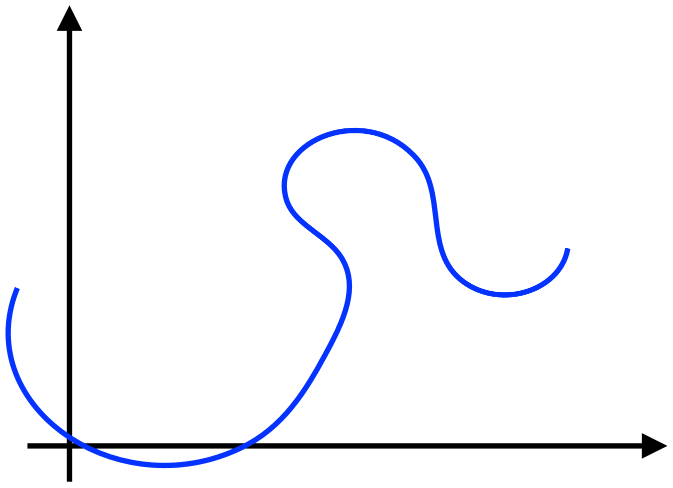
Representations are intuitive

$R \subseteq \mathbb{R} \times \mathbb{R}$



(0,0)

More on inverse relations

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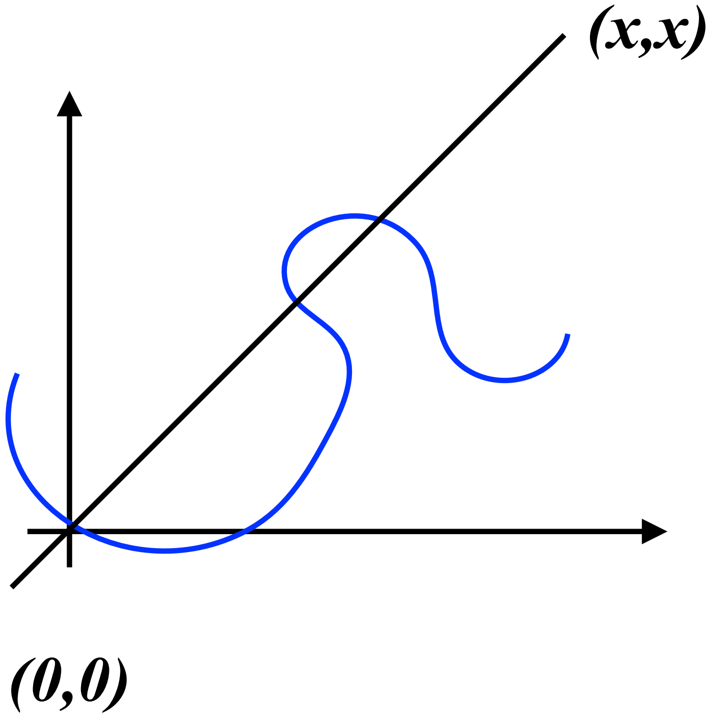

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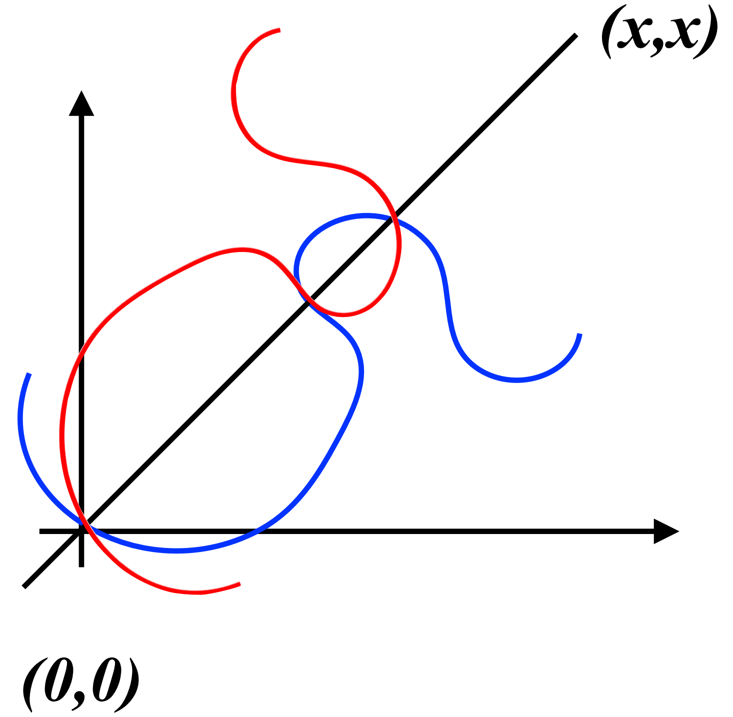

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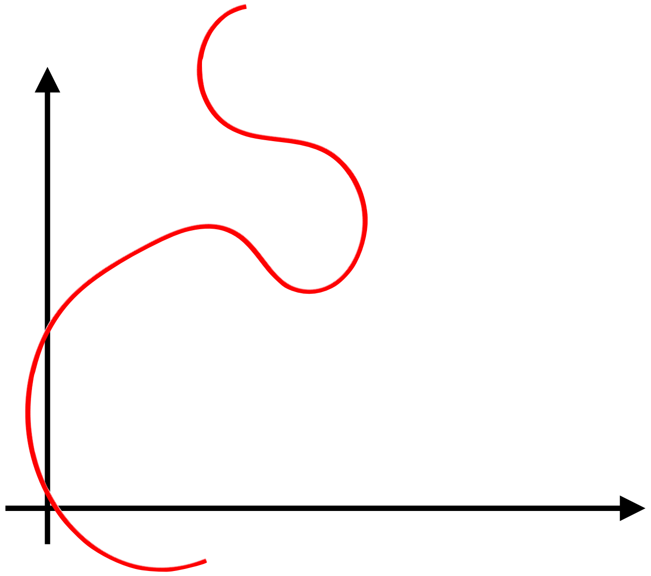


(0,0)

More on inverse relations

$R \subseteq A \times B$

$R^{-1} \subseteq B \times A$, **defined with** $(b,a) \in R^{-1} \Leftrightarrow (a,b) \in R$

Properties:

- $(R^{-1})^{-1} = R$
- $dom(R^{-1}) = range(R)$ & $range(R^{-1}) = dom(R)$
- *if R is injective,* R^{-1} *is functional*
- *if R is total,* R^{-1} *is surjective*

One or two examples…

Composition of relations

Afk	Vak	studiegidsnummer
CW1	Continue wiskunde 1	4031CW103
FDSD	Fundamentals of Digital Systems Design	4031FDSD6
(w)FI1	Fundamentele Informatica 1	4031FINF1
Intro Inf	Introductie studie Informatica	
LAfCS	Linear Algebra for Computer Scientists 1	4031LACS1
(w)PM	Programmeermethoden	4031PRGR6
vPM	Vragenuur Programmeermethoden	4031PRGR6
St&Pr	Studying and Presenting	4031STPEC
OI	Orientatie Informatica	4031ORINF

studiegidsnummer	Docent
4031CW103	Dr. J.-H. Evertse
4031FDSD6	Dr. T.P.Stefanov
4031FINF1	Dr. J.M. de Graaf
	Dr. J.M. de Graaf/M. Derogee
4031LACS1	Dr. D. Holmes
4031PRGR6	Dr. W.A. Kosters
4031PRGR6	Dr. W.A. Kosters
4031STPEC	Dr. ir. F.F.J. Hermans
4031ORINF	Prof.dr. M.E.H. van Reisen

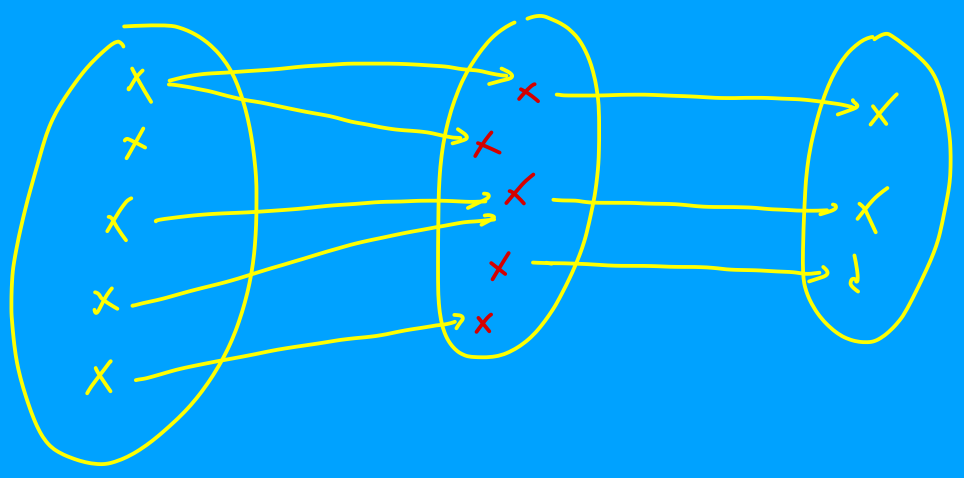
Composition of relations

$R \subseteq A \times B$ **and** $S \in B \times C$

$x \in A, y \in B, z \in C$

$x(R \circ S)z \;\; if \;\; xRy \;\&\; yRz \;\; for \;\; some \;\; y \in B$

Example if needed:



Composition of relations

$R \circ R^{-1}$?

R= {(1,1), (1,2),(2,3),(3,2),(3,4)}

Composition of relations

$R \circ R^{-1}$?

always *symmetric*

R= {(1,1), (1,2),(2,3),(3,2),(3,4)}

$R = \{ \underline{(a,b)}, \ldots \} \qquad R^{-1} = \{ (b,a) \ldots \}$

$\underline{(a,a)} \in R \circ R^{-1}$

$\exists y$ s.t. $(a,y) \in R$ & $(y,a) \in R^{-1}$ ✓

Composition of relations

SUPPLEMENTARY

Connection to matrix multiplication

See Schaum 2.5

$R= \{(1,1), (1,2),(2,3),(3,2),(3,4)\}$

Composition of relations

Theorem 2.1. Relation composition is associative

Proof

$S \quad T \quad U \quad V$

$\rightarrow P \subseteq S \times T$

$\rightarrow R \subseteq T \times U$

$\rightarrow Q \subseteq U \times V$

$$\underbrace{(P \circ R)}_{} \circ Q \stackrel{?}{=} P \circ \underbrace{(R \circ Q)}_{}$$

Composition of relations

Note on notation ("direction") of compostions for functions and relations…they are opposite…

$$x\overrightarrow{(R \circ S)}y \quad y = \overleftarrow{(g \circ f)}(x)$$

Why?

Properties of binary relations

Properties of binary relations

- reflexivity
- (anti) symmetricity
- transitivity

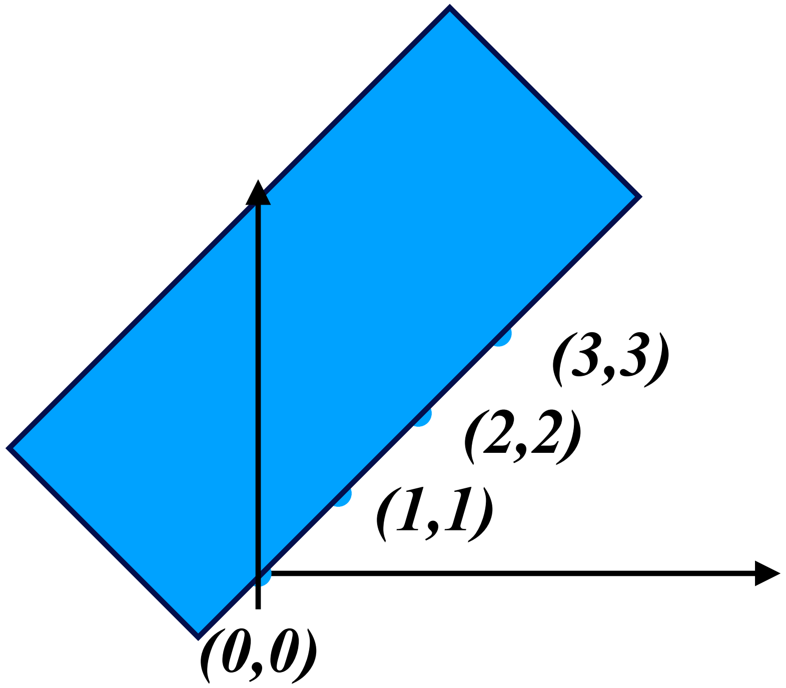
- partial orders

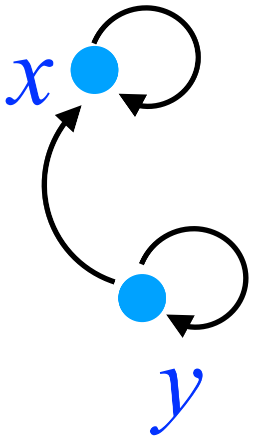
- equivalence

Properties of binary relations

Definition. Relation $R \in A \times A$ is **reflexive** if
For all x, xRx.

*If for all $x \in A$, $(x, x) \notin R$, then it is **irreflexive***

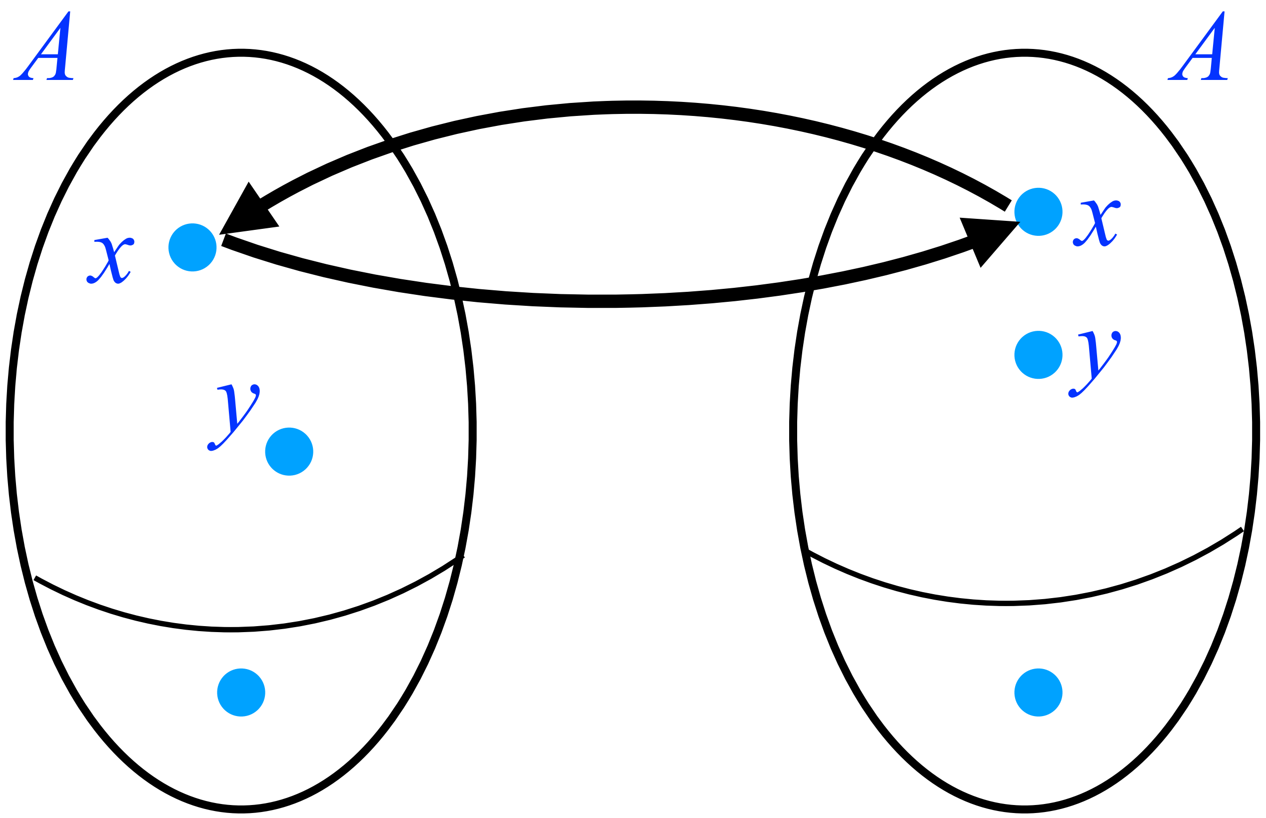




Properties of binary relations

Definition. Relation $R \in A \times A$ is **symmetric** if
For all x,y, *if* xRy *then* yRx.

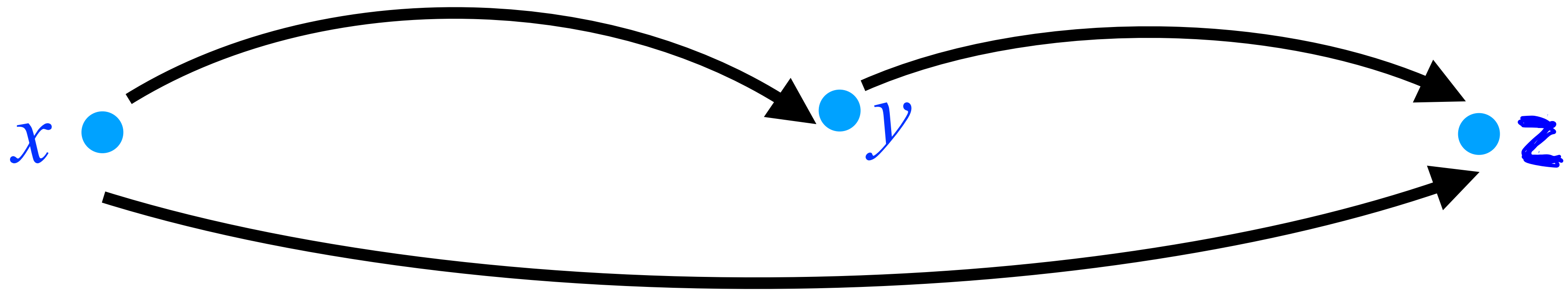
If xRy *&* yRx *implies that* $x=y$ *then it is* ***antisymmetric***



Properties of binary relations

Definition. Relation $R \in A \times A$ is **transitive** if

For all x,y,z *if* xRy *&* yRz*, then* xRz



Putting relations in a relation…
(I'm so meta even this acronym)

reflexive

irreflexive

symmetric

antisymmetric

transitive

$(x \leq y)$ $[\mathbb{Z}, \leq \subseteq \mathbb{Z}^2]$

$(x < y)[\mathbb{Z}, < \subseteq \mathbb{Z}^2]$

[**to be disjoint**]

$\mathscr{P}(U), R \subseteq \mathscr{P}(U)^2, xRy \textit{ if } x \cap y = \varnothing$

$(x \subseteq_U y)$ $[\mathscr{P}(U), \subseteq_U \subseteq \mathscr{P}(U)^2]$

$(x \subsetneq_U y)$ $[\mathscr{P}(U), \subsetneq_U \subseteq \mathscr{P}(U)^2]$

$x = y$

Question?

Putting relations in a relation…
(I'm so meta even this acronym)

reflexive

irreflexive

symmetric

antisymmetric

transitive

$(x \leq y)$ $[\mathbb{Z}, \leq\, \subseteq \mathbb{Z}^2]$ R AS T

$(x < y)[\mathbb{Z}, <\, \subseteq \mathbb{Z}^2\,]$ IR AS T

[to be disjoint]

$\mathscr{P}(U)$, $R \subseteq \mathscr{P}(U)^2, xRy \text{ if } x \cap y = \varnothing$ S

$(x \subseteq_U y)$ $[\mathscr{P}(U),\ \subseteq_U\, \subseteq \mathscr{P}(U)^2]$ R AS T

$(x \subsetneq_U y)$ $[\mathscr{P}(U),\ \subsetneq_U\, \subseteq \mathscr{P}(U)^2]$ IR AS T

$x = y$ R S T

Question?