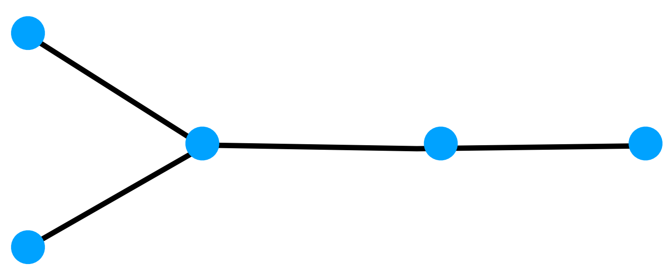
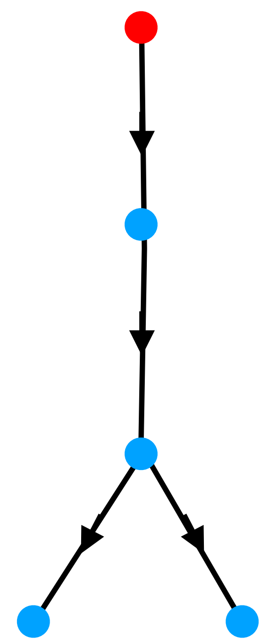
Lecture 14

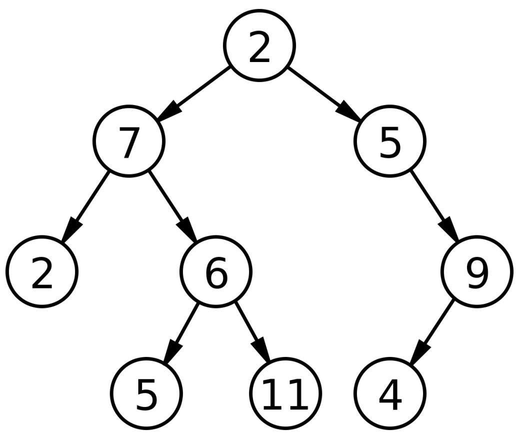
Trees

Types of trees (graphs)

- undirected (Ch. 8.8)
- directed (Ch. 9.4)
  - rooted (*arborescence*)
    - ordered rooted
- binary (Ch. 10)







Def. An undirected tree is an undirected acyclic connected graph.

Def. An directed tree is a directed connected graph without undirected cycles.

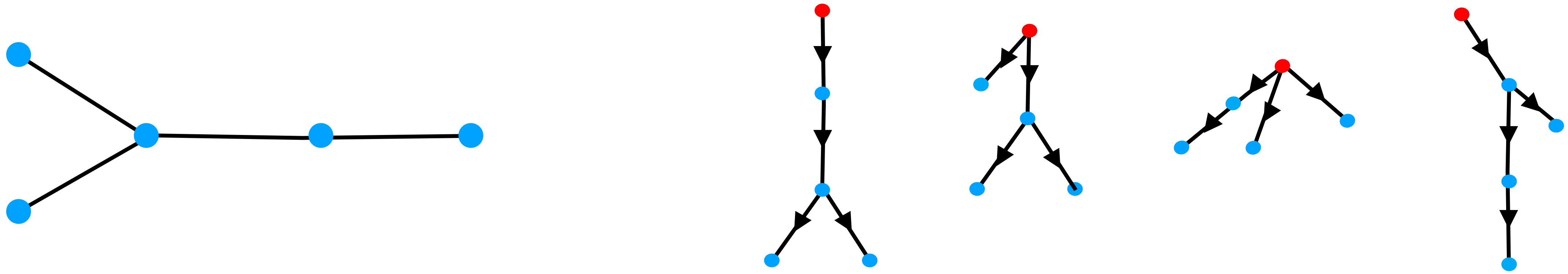
Def. A rooted tree is an undirected connected graph with a specal vertex called the root.

Def. DAG: directed acyclic graph: directed graph with no directed cycles.

Directed tree: DAG whose underlying undirected graph is a tree.

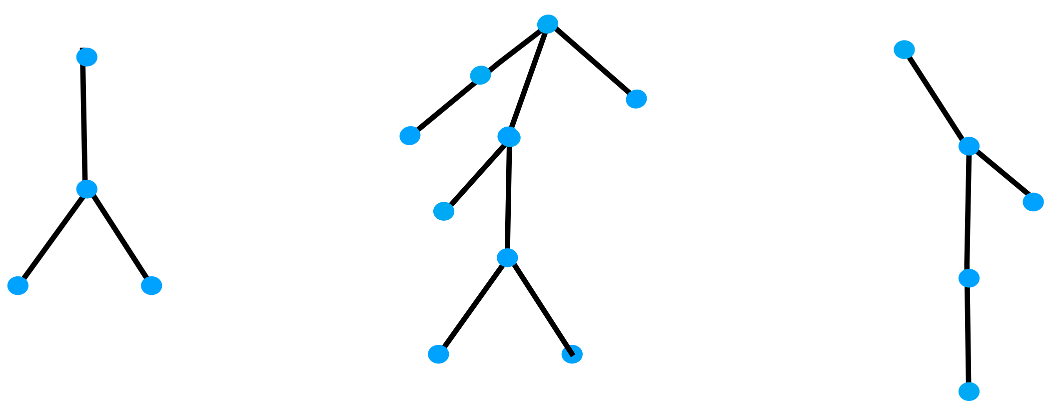
Def. A rooted tree is an undirected connected graph with a specal vertex called the root.

The root naturally induces a directionality in the graph, from the root to the leaves.



Def. A *forest* is an undirected graph without cycles (acyclic).

Every forest is a collection of trees.



Recall: graph is connected if there exists a simple path between any two vertices

Theorem. If a graph is acyclic and connected then there exists a unique simple path between any two vertices. Converse holds as well.

Proof: suppose there are two different simple paths from u to v, then the graph has a cycle. Contradiction.

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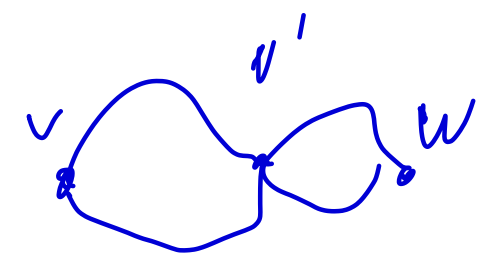
DETAILS: LET P_1 & P_2 BE TWO DISTINCT PATHS

U to V, $P_1 = U\, v_1 v_2 \ldots v_n V$ $\quad P_2 = U\, v_1' \ldots v_n' V$

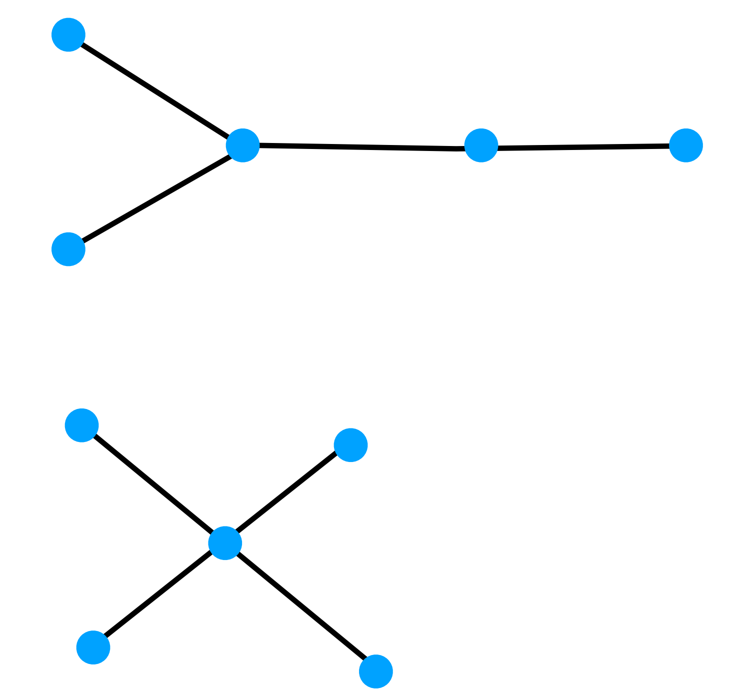
$\rightarrow$ $C = U V_1 \ldots V_n V V_n' V_{n-1}' \ldots v_1' U$ is a circuit.



G has circuit $\Rightarrow$ has cycle. Find repeating vertex, remove sequence between.

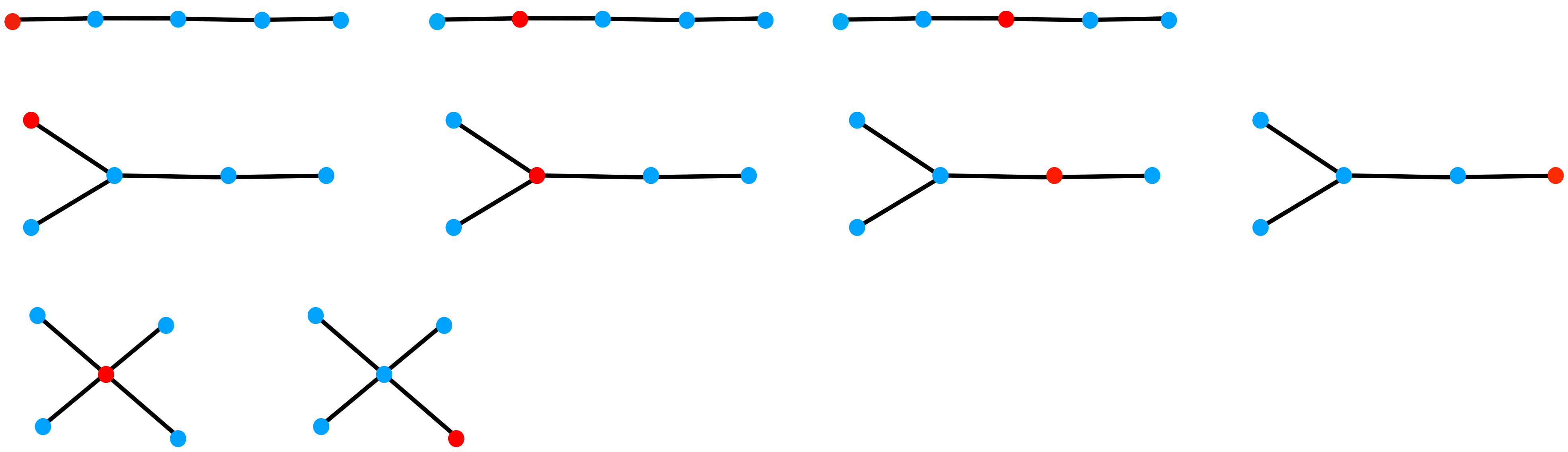
Trees with five vertices

- exactly 3 non-isomorphic connected acyclic graphs (trees):



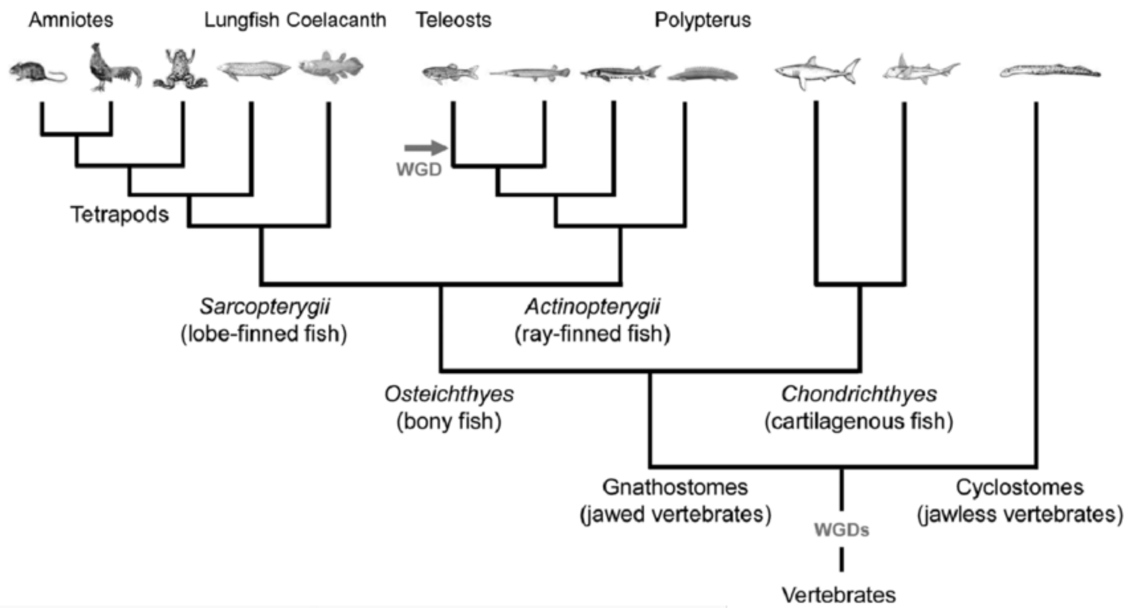
Trees with five vertices

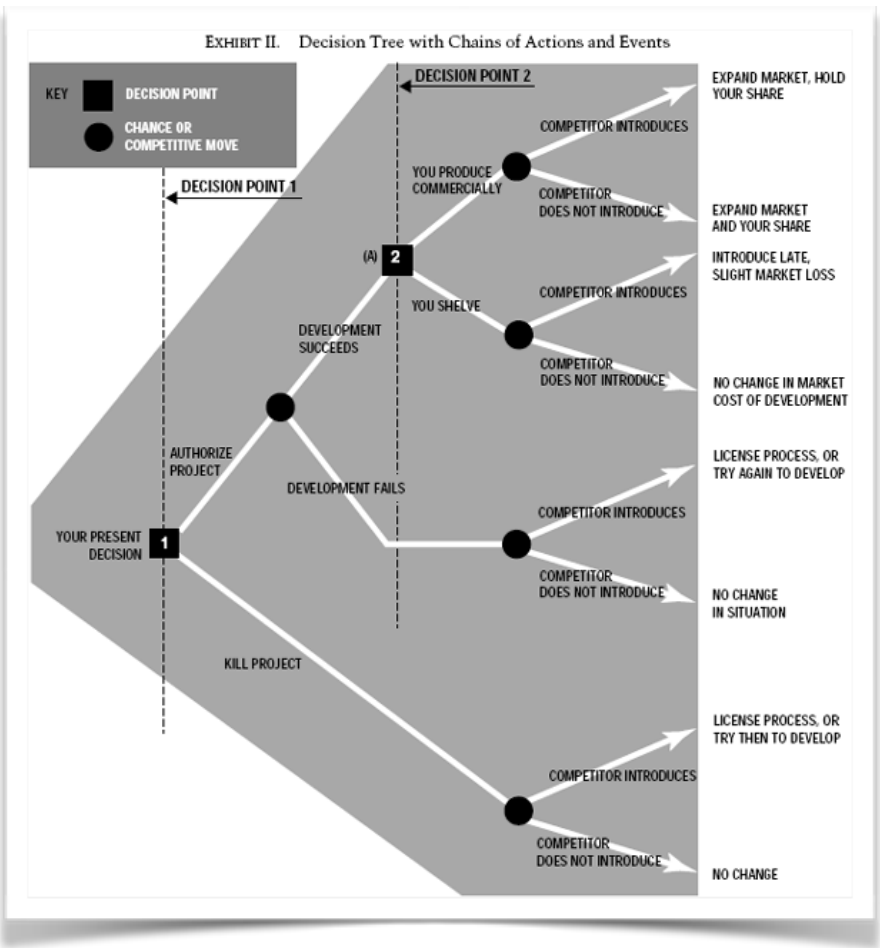
- more options for rooted trees

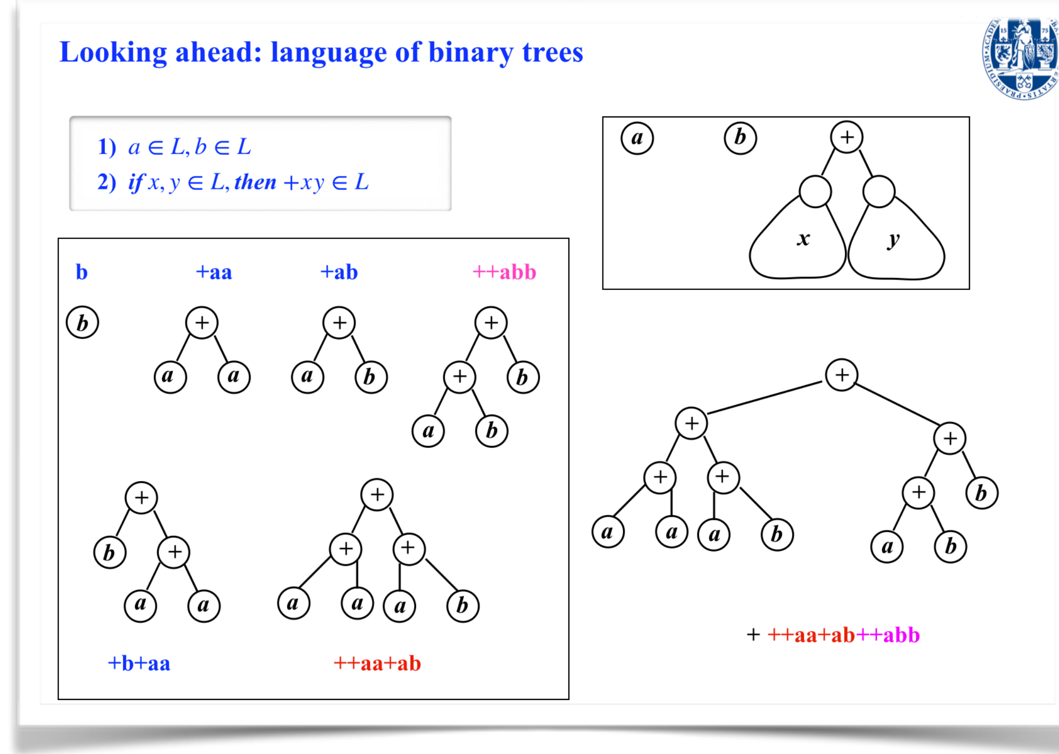


- each induces a different directed tree

Some examples



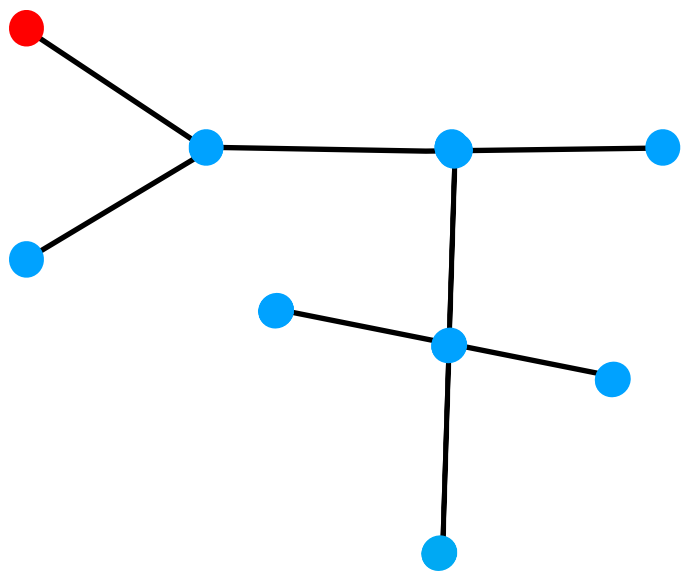




https://www.researchgate.net/figure/Phylogenetic-tree-of-vertebrates-A-simplified-phylogenetic-tree-focusing-on-the_fig1_316690258

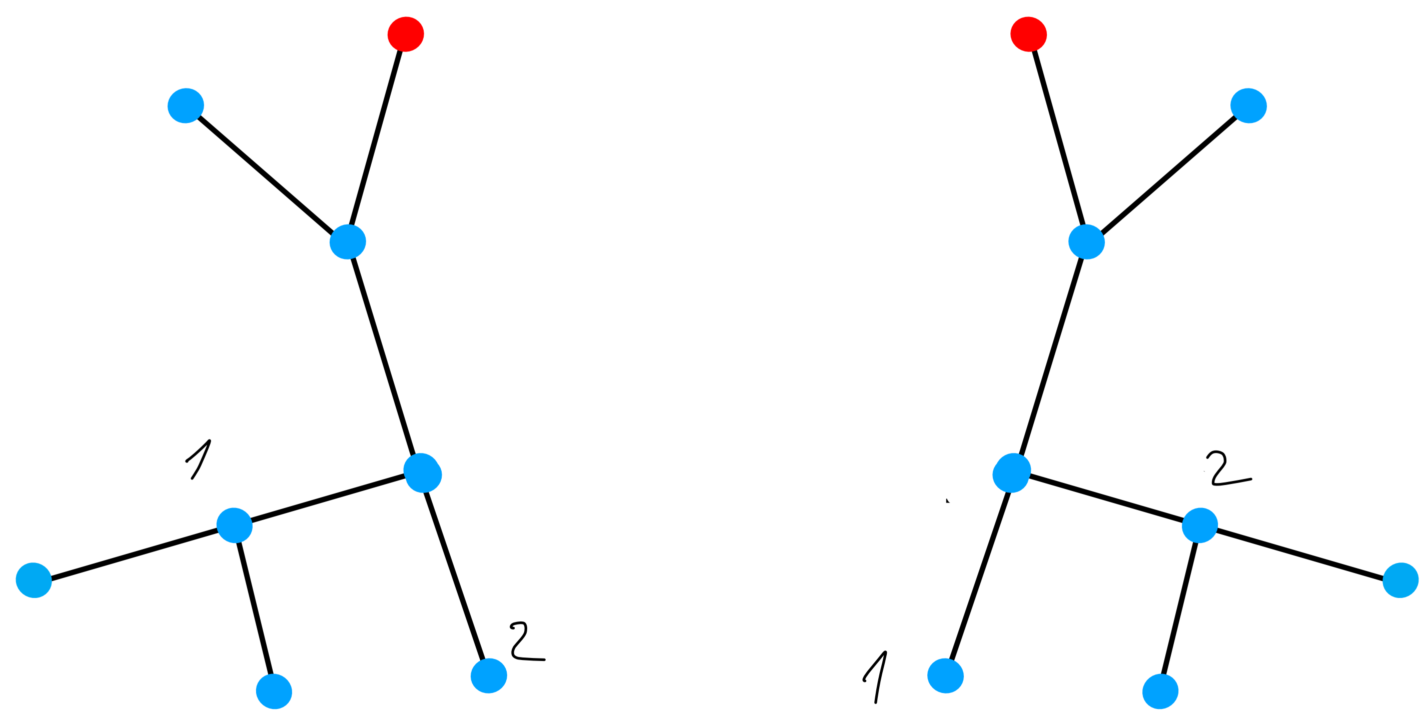
Trees: terminology and basic concepts contiued (Ch 10! — do read it)

- vertices (node):
  - leaf (degree 1)
  - internal vertex
- edge (branch)
- root (in rooted trees)
- child (in directed trees)
- sibling (in directed trees)
- parent (in directed trees)
- ancestor, descendant…



rooted tree

Terminology and basic concepts



ordered trees: children are ordered. “oldest” to “youngest”, first, second, … last… (think family tree)

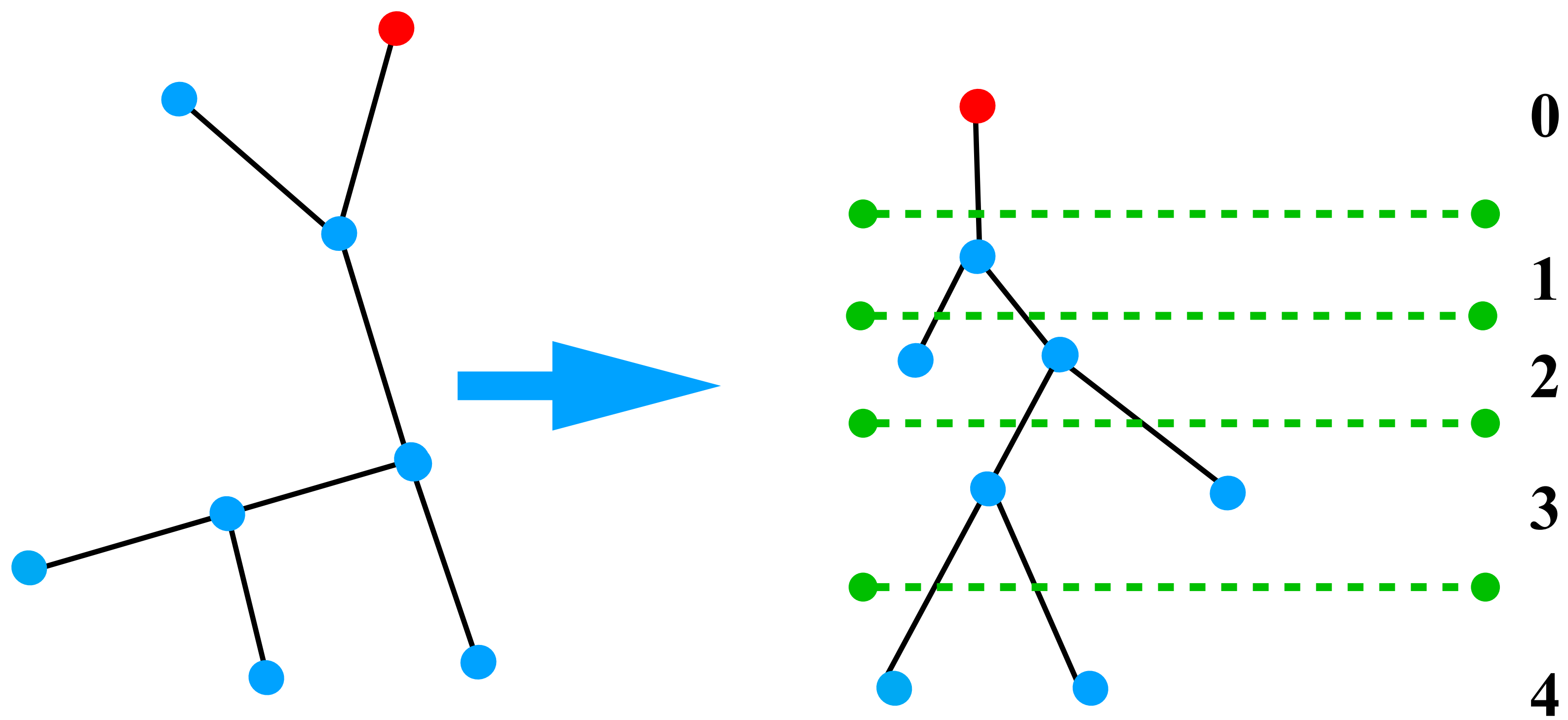
example: isomorphic as unordered trees, not isomorphic as ordered.

Terminology and basic concepts (rooted tree): level and depth

depth of vertex:
distance from the root = lenght of path from root to vertex

height of vertex:
lenght of path from vertex to furthest child

height of tree:
height of root, equiv. depth of deepest vertex;

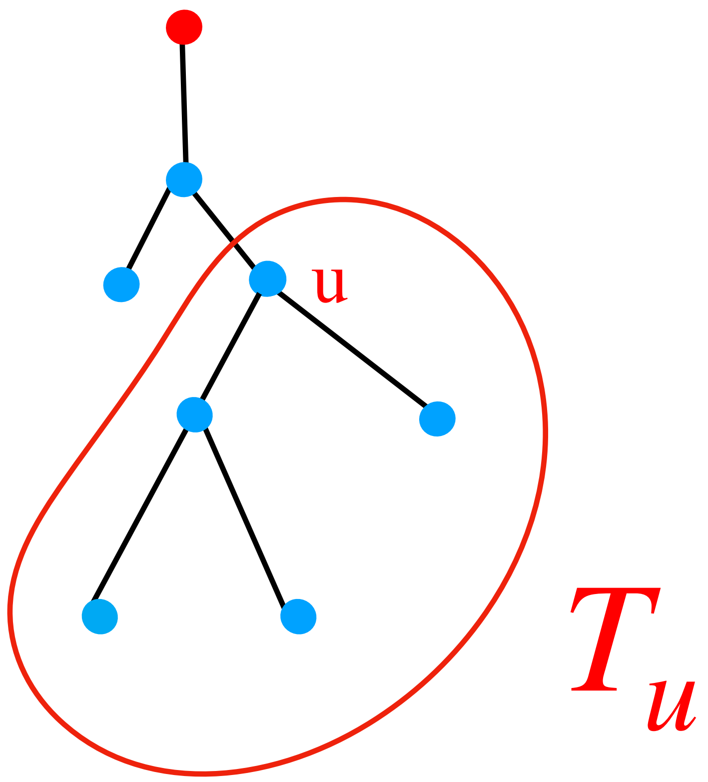


mostly depth is used.

Terminology and basic concepts: subtree

Subtree: induced subgraph which is a tree. More often in **rooted settings:**

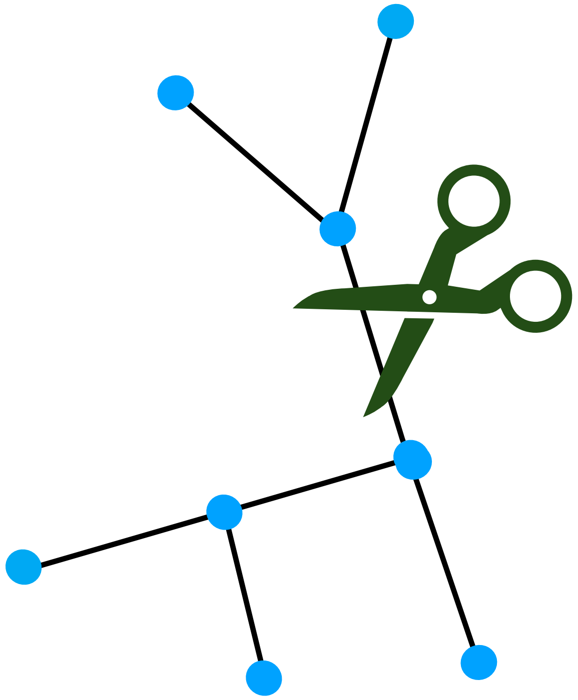
Let T = (V, E) a tree and u a vertex in T.
Then T_u is the sub-tree of T consisting of the vertex u,
all its successors and all (directed) branches between the vertices.



Basic properties of trees

recall: tree is an undirected acyclic graph

> *Lemma. Let G=(V,E) be a tree with n>1 vertices, and let e be any edge. Then G-{e} is not connected.*



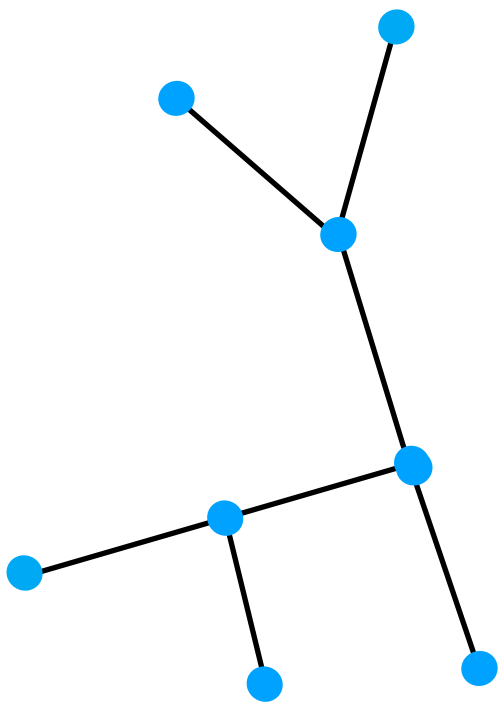
Proof: let e = {u,v}, and suppose G-{e} is connected; Then there is a path, and hence a simple simple path from u to v. Adding e makes a cycle. So G has a cycle. contradiction.

Basic properties of trees

WE STOPPED HERE

recall: tree is an undirected acyclic graph

> ***Lemma. Let T=(V,E) be an [undirected] tree. Then |E| = |V|−1***



Comment: when dealing with tree graphs we customarily denote then “T” instead of “G”

Properties of trees

recall: tree is an undirected acyclic graph

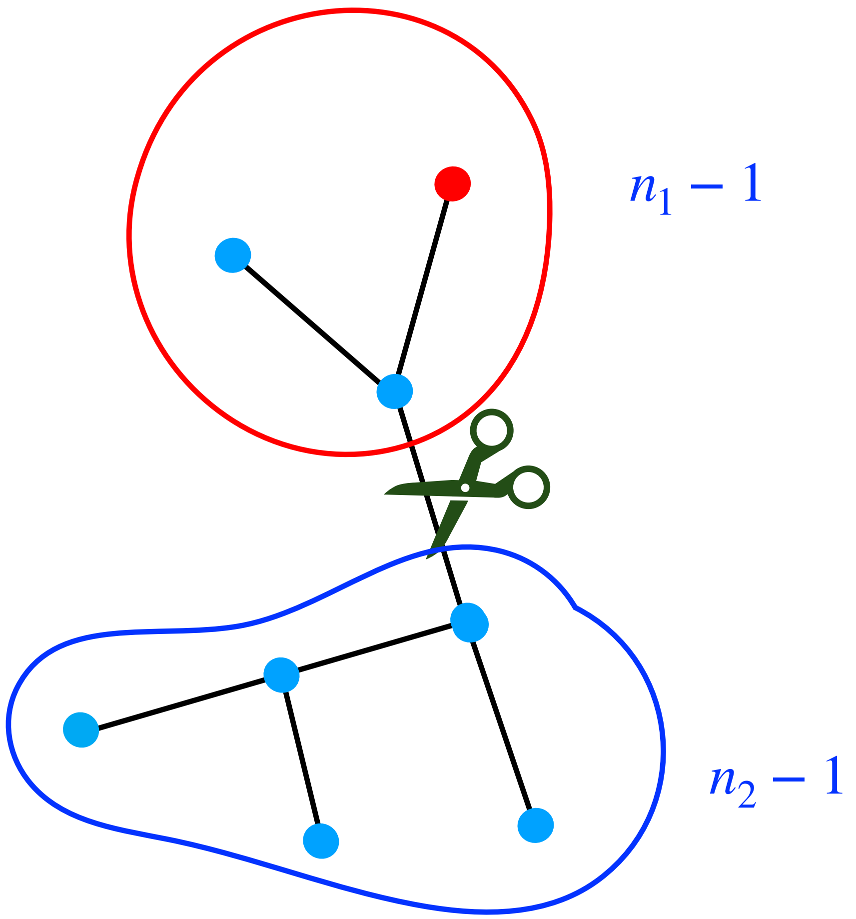
Lemma. *Let G=(V,E) be an [undirected] tree. Then |E| = |V|–1*

Proof 1: *induction over the number of vertices.*

(i) basis: n=1, works.

(ii) assume holds for all k<n.

take any tree of n vertices, and cut any edge.
Now we have two trees with $n_1, n_2, n_1 + n_2 = n$ *vertices,*
hence, by previous lemma, $n_1 - 1 + n_2 - 1 = n - 2$ *edges in total.*
Since you cut one edge, the initial graph must have had n-1 edges



basis

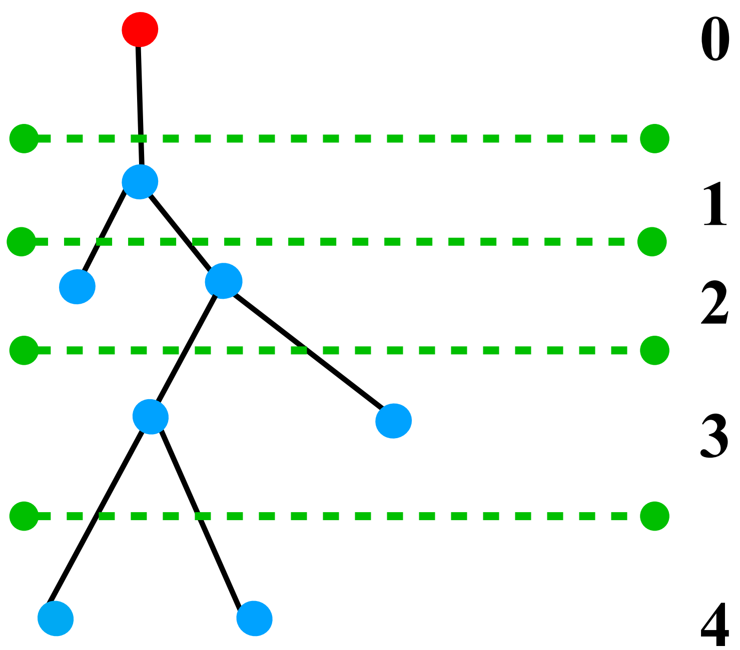
Properties of trees

recall: tree is an undirected acyclic graph

***Lemma.* Let G=(V,E) be an [undirected] tree. Then |E| = |V|−1**

Proof 2:

Choose a root and look at levels of rooted tree.



Each vertes has exactly one prececessor
in previous level…except the root.
So n-1 edges.

Properties of trees

Lemma. Let G=(V,E) be an [undirected] tree. Then |E| = |V|--1

Proof 3 (very alternative):

Recall a simple recursive definition of undirected "TrEe":
(1) a vertex is a tree
(2) a graph obtained by adding a vertex and connecting it to one vertex of a tree is a tree.
(3) nothing else is a tree

Induction over vertices: Basis: 1 vertex tree true.
Any n+1 vertex tree is obtained from an n vertex tree by adding an edge.
Done.

NOTE: we have not yet proven that "TrEe" is the same as an undirected tree…
is easy to see though.

Properties of trees

Lemma. A connected graph with no cycles and at least one edge as at least two vertices of degree 1 (See exercise 8.38).

Proof 1:
We have seen this. Consider longest path. What is the degree of first and last?

Properties of trees

> ***Lemma.** A connected graph with no cycles and at least one edge as at least two vertices of degree 1 (See exercise 8.38).*

Proof 2:

Assume that this is not true, so all but one vertex have degree 2 or higher.
But then the number of degrees t_{tot} ***is at least 2(n-1) +1 = 2n-1, so*** $t_{tot} \geq 2n-1$

By sum-degree formula we know $2|E| = t_{tot}$. ***So*** t_{tot} ***must be even so it is 2n, implying that*** $|E| = n$

But for trees we know that $|E| = n-1$. ***Contradiction.***

Properties of trees

Theorem (Characterization of trees 1).

For a graph G (over n vertices) the following are equivalent

(1) G is a tree
(2) G is maximally acyclic : adding an edge to G creates a cycle
(3) G is minimally connected: removing an edge makes it unconnected

Properties of trees

Proofs

(1) => (2): a) is acyclic ✓

b) adding edge makes a cycle:

add new $e = \{v, w\}$; But G was connected

=> $\exists$ simple path $v, v_1 \ldots w$. (and $\{v, w\}$ is not in)

So $v, v_1 \ldots w, v$ is a cycle.

(2) => (1) (2) => - connected. otherwise connecting two unconnected components doesn't make a cycle

- connected + acyclic => tree per definition

Properties of trees

Proofs

(1) => (3) tree => connected. & acyclic.

=> removing edge disconnects (or cycle!) [seen before]

=> minimally connected

(3) => (1) minimally connected => connected

need acyclic.

assume connected & cyclic => cut cycle

=> not disconnected. => acyclic. tree.

Properties of trees

Theorem (Characterization of trees 2).

For a graph G (over n vertices) the following are equivalent

(1) G is a tree
(2) G is acyclic and has n-1 edges
(3) G is connected and has n-1 edges

(1) => (2) acyclic by definition. n-1 edges proven before.

(2) => (3) acyclic + (n-1) edges => connected

Assume k components, all acyclic => trees

$\Rightarrow n_1 + n_2 .. n_k = n$ & $n_1 - 1 + n_2 - 1 + .. n_k - 1 = n - 1$

$\Rightarrow n - k = n - 1 \Rightarrow k = 1$. 1 tree => connected

Properties of trees

Proofs

(3) => (1) Assume connected, n-1 edges & cycle.

=> can remove edge.

=> connected graph over n vertices with n-2 edges (see 8.39)

Connected => has spanning tree => n-1 edges $\lightning$.