Lecture 10 & 11

Graph Theory

finishing

Graph Theory: concepts so far 1

-definition; basic types (directed, undirected, simple)

-adjacency matrix, incidence matrix, degree

-sum-degree formula and handshaking lemma

-equality and isomorphism

-(induced) subgraph, vertex and edge removal

-paths: simple, trail, closed, circuit, cycle

- connected components, bridge, cut vertex, graph distance and diameter

-Traversible and Eulerian circuits (+ characterization), Hamilton cycles

Graph Theory: concepts so far 2

-Traversible and Eulerian circuits (+ characterization), Hamilton cycles

-Special graphs: trees, bipartitie, complete, complete bipartite, clique, planar

-Counting edges ... more today!

-Labeled & Weighted graphs, (minimal) spanning trees (Prim's algorithm), shortest paths (Dijkstra's algorithm)

-Some specificities for directed graphs

Directed graphs, a.k.a "digraphs"

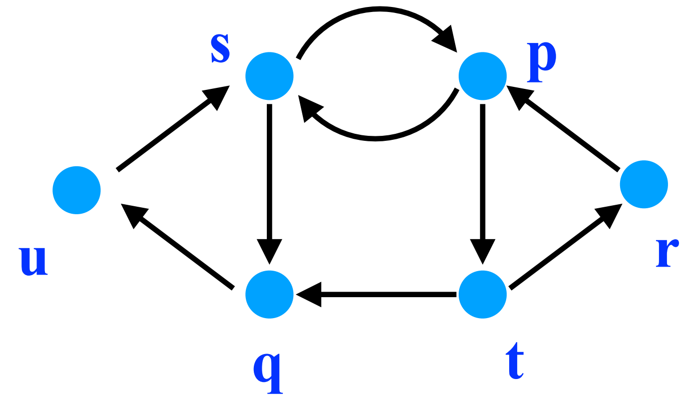
directed path: a sequence $v_1, e_1, v_2, e_2 \ldots, v_n$**, with** $e_k = (v_k, v_{k+1})$

all concepts analogous (simple directed path, directed circuit, directed cycle)

semipath (path relative to the underlying undirected graph)

Definition. A digraph is ***strongly connected*** *if* ***every pair of vertices*** *is* ***connected by a directed path.***

Definition. A digraph is ***weakly connected*** *if* ***every pair of vertices*** *is* ***connected by a semipath.***



(directed) spanning path: passes all vertices (recall Hamilton)

semipath: **undirected path; path in the underlying undirected graph**
($e_k = (v_k, v_{k+1})$ **OR** $e_k = (v_{k+1}, v_k)$ **)**

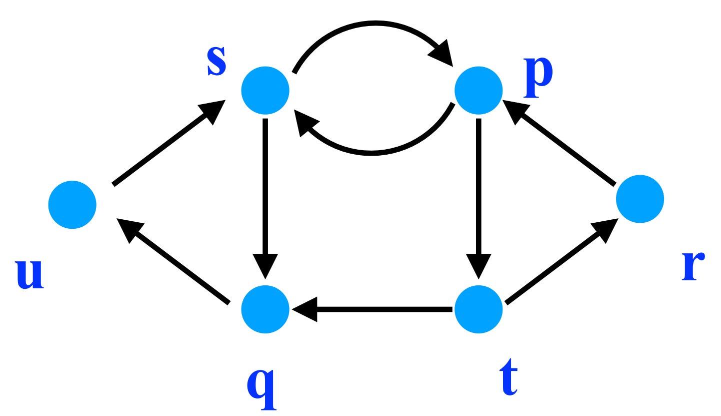
Degrees and directed sum-degree

out-degree *outdeg(v)*: number of outbound edges

in-degree *indeg(v)*: number of inbound edges

Source: vertex v with indeg(v)=0.

Sink: vertex v with outdeg(v)=0.



Theorem. In a directed graph *G* the following holds*:*

$$\sum_{v \in V} outdeg(v) = |E| = \sum_{v \in V} indeg(v)$$

“number of starts” = “number of ends"

Digraphs and connectedness

*Definitions A digraph is **strongly connected** if **every pair of vertices** is **connected by a directed path.***
*A digraph is **weakly connected** if **every pair of vertices** is **connected by a semipath.***

Theorem 9.2. a) strongly connected if and only a closed spanning path exists
b) weakly connected if and only if a spanning semipath exists

a) => has path u to v <u>and</u> v to u; <= "circle has two ways"

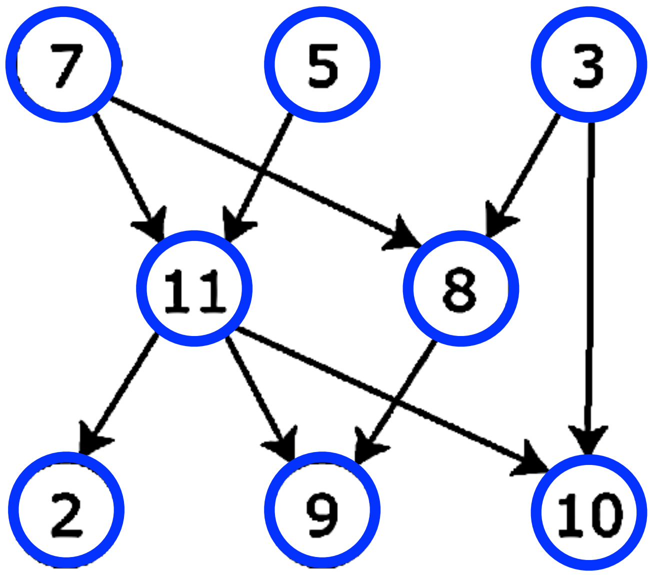
Theorem 9.3. A directed graph G without cycles has a source and a sink.

Essentially, see longest path. First and last vertex must be source and sink. See Schaum

Digraphs: topological ordering

A topological ordering ***(topological sorting***) of a directed graph G = (V, E) is a sequence (an enumeration) $v_1, v_2, \ldots, v_n$ of all the vertices of of G such that $(v_i, v_j) \in E, \Rightarrow \; i < j$

Or: you can draw the vertices of the graph in such a configuration that the arrows always point left to right.



a topological ordering:
7,5,11,2,3,10,8,9

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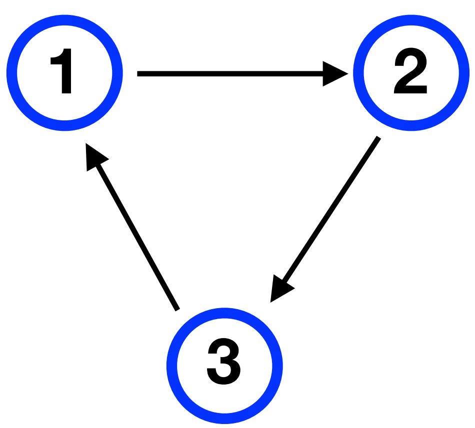

Rock, Paper, Scissors, Lizard, Spock?

Digraphs: topological ordering

Theorem 9.8. If G is a directed graph without cycles, then there exists a topological ordering of G (and converse)

How many edges in an n-vertex complete graph?

How many graphs over n-vertices are there (including isomorphic)?

How many isomorphic complete graphs are there (over n vertices)?

Elementary combinatorics

(a reminder of counting)

Counting ordered lists

How many different sequences (of lenght n) of numbers from 1 to 5 are there?

Counting ordered lists

How many different sequences (of lenght n) of numbers from 1 to 5 are there?

Counting ordered lists

How many different sequences (of lenght n) of numbers from 1 to 5 are there?

11
12
13
14
15
21
⋮
25
31
⋮
35
41
⋮
45
51
⋮
55

Each digit increases nuber of options 5 fold.

$$\underset{1}{5} \times \underset{2}{5} \cdots \underset{n}{5} = 5^n$$

$$f(n) = 5 \cdot f(n-1) \qquad f(1) = 5$$

answer: 5^n

Compare to counting subsets:

{ A, B, C }

0	0	0	→	{ }
0	0	1	→	{ C }
0	1	0	→	{ B }
0	1	1	→	{ B, C }
1	0	0	→	{ A }
1	0	1	→	{ A, C }
1	1	0	→	{ A, B }
1	1	1	→	{ A, B, C }

binary counting $\mathscr{P}(\{A, B, C\})$

2^n because each new element x introduces two new options $x \in S$, $x \notin S$

= "how many graphs incl. isomorphisms"

="do I include an edge or not…" ... we will see...

Counting different orderings: permutations

How many different sequences of length 5 can you make with the numbers 1 to 5 if each number can only occur once?
Or: how many permutations of sequences of numbers 1 to 5 are there?

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$$\underbrace{* \quad * \quad * \quad *}_{4 \text{ present}} \quad \circ$$

5th has 5 options

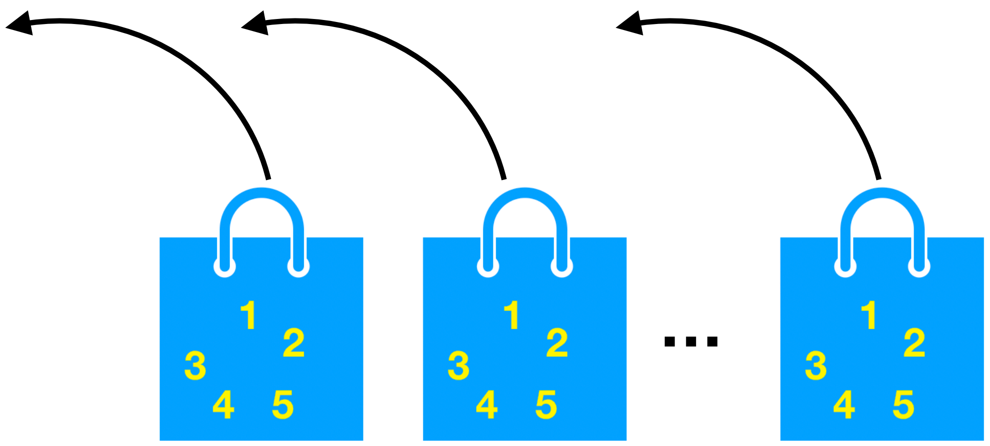
$1 \times 2 \times 3 \cdots \times 5 = 5!$

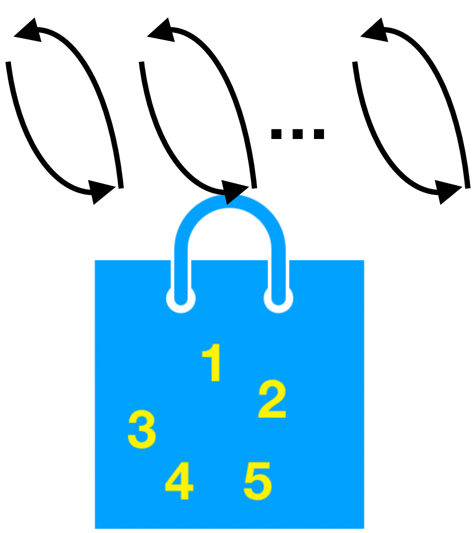
$f(n) = n \times f(n-1) \;;\quad f(1) = 1$

Answer: $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120 = 5!$

Ordered lists v.s. Permutations

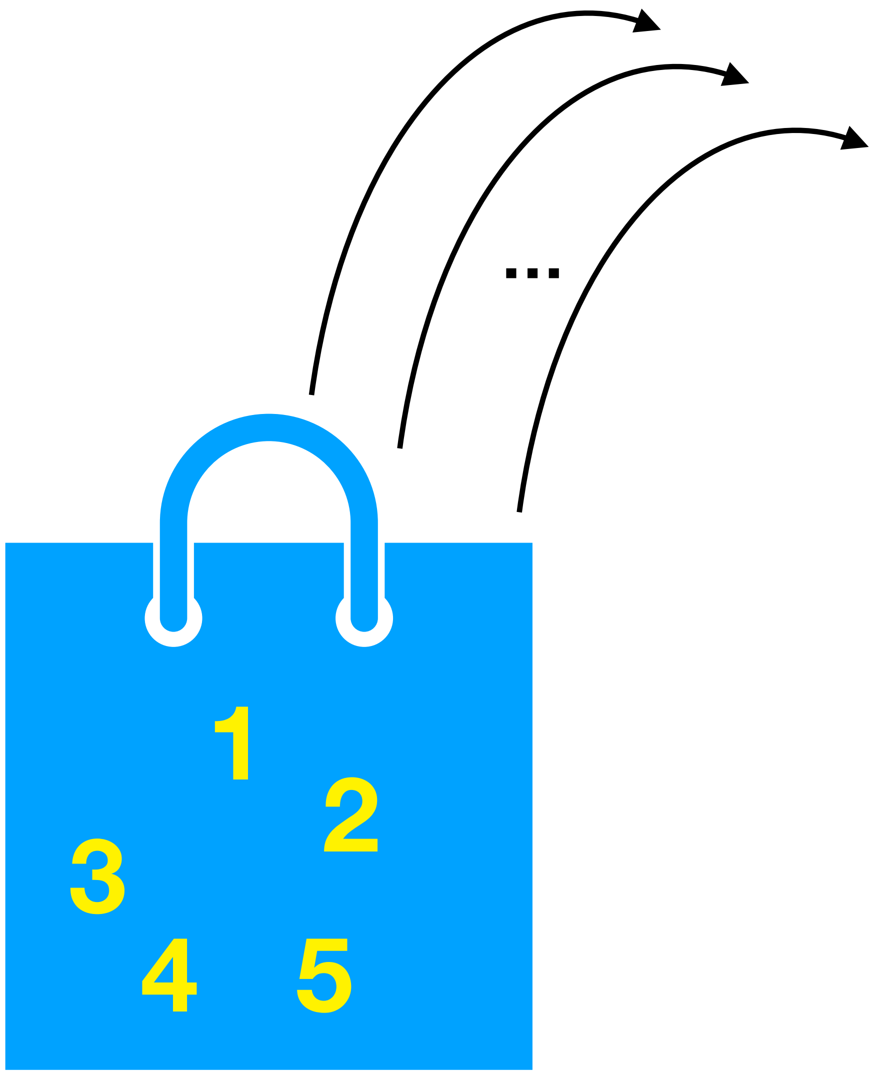
Ordered lists





"With replacement"

Permutations



"Without replacement"

Permutations: a variant

How many different sequences of length 3 can you make with the numbers 1 to 5
if each number can only occur once?

Permutations: a variant

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$5 \to 4 \to 3$

$5 \times 4 \times 3 \ldots$

Alternatively

$* \; * \; * \; \underbrace{* \; *}_{2!}$

$$\frac{5!}{2!}$$

order of last 2 doesn't matter

Answer: $5 \cdot 4 \cdot 3 = 60$

Combinations

How many different triplets can you choose from the numbers 1 to 5, with each number occurring only once (so distinct sets of 3 elements)?

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How many different triplets can you choose from the numbers 1 to 5,
with each number occurring only once (so distinct sets of 3 elements)?

From previous:

$5 \times 4 \times 3$ sequences

but order does not matter

$= \dfrac{5 \times 4 \times 3}{3!}$

OR.

$\underbrace{* \quad * \quad *}_{3!} \quad \underbrace{* \quad *}_{2!} \qquad 5!$

$\Rightarrow \dfrac{5!}{3!\,2!}$

Answer: $(5 \cdot 4 \cdot 3)/(3 \cdot 2 \cdot 1) = 10$

Basic elements of combinatorics

Number of sequences of lenght *k* from set of size *n*: n^k

(order matters, repetition ok — with replacement)

Number of permutations of sequence *n distinct elements*: $n! = n \cdot (n-1) \cdots 1$

(order matters, repetition not ok, all distinct— no replacement)

Number of ways to select *k sized subsets out of n elements*: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

(order irrelevant, repetition not ok, all distinct — no replacement)

distinct objects in a bag…

Basic elements of combinatorics

Also: number of <u>functions</u>, $f : A \to B, |A| = k, |B| = n,$ **is** n^k

Number of <u>bijections</u> (bijective functions) $f : A \to A, |A| = n,$ **is** $n!$

Binomial coefficients: lotto problem $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

there are interesting connections between the three..

How many edges in an n-vertex complete graph?

each edge is a set of two non-identical vertices
so this is number of ways you can select two-element sets from a set of all vertices.

$$= \binom{n}{2} = \frac{n!}{(n-2)!2!} = \frac{n(n-1)}{2}$$

How many graphs over n-vertices are there (including isomorphic)?

=each graph is a subset of the set of all vertices $= 2^{\binom{n}{2}}$

How many isomorphic complete graphs are there?

=all complete graphs are isomorphic.
so this number all bijections from V **to** $V = n!$

Basic elements of combinatorics

binomial coefficients (Schaum 5.3, 5.5.)

$$\binom{n}{k} = \frac{n!}{n!(n-k)!}$$ **pronounced "n choose k".**

$$n! = n \cdot (n-1) \cdots 1$$ **pronounced "n factorial".**

Example: $(a+b)^4 = 1 \cdot a^4 + 4 \cdot a^3 b + 6 \cdot a^2 b^2 + 4 \cdot ab^3 + 1 \cdot b^4$

Pascal's triangle…

Basic elements of combinatorics

Pascal's triangle…

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
\boxed{1 \quad 4 \quad 6 \quad 4 \quad 1} \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}
$$

$$\binom{n}{k} = \frac{n!}{n!(n-k)!}$$

$$(a+b)^4 = 1 \cdot a^4 + 4 \cdot a^3b + 6 \cdot a^2b^2 + 4 \cdot ab^3 + 1 \cdot b^4$$

Basic elements of combinatorics

Recursive relations!

$$\binom{n}{k} = \binom{n}{k-1} + \binom{n-1}{k-1}$$

$$n! = n \cdot (n-1)!$$

Comment "n!" looks funky, but it is actually !(n), and "!" is a function…

For home: practice…

Leidsche Flesch needs a new board consisting of 5 people.

There are 8 candidates: 4 women and 4 men.

1. How many different boards can we elect?

2. Same question, but with more balance… we need boards with 2 men and 3 women

3. The same question as 1 , but now we also must assign the position on the board! (president, ab-actis, Quaestor, Education assessor, External assessor).

4. The same question as 2, but also with assigning the position…