Lecture 10

Graph Theory

refresher & continuation

Graph Theory: concepts so far

-definition; basic types (directed, undirected, simple)

-adjacency matrix, incidence matrix, degree

-sum-degree formula and handshaking lemma

-equality and isomorphism

-(induced) subgraph, vertex and edge removal

-path: simple, trail, closed, circuit, cycle

- connected components, bridge, cut vertex

- distance and diameter

Traversible and Eulerian Graphs

Def. Euler trail: a trail which uses each edge exactly once.

Traversible and Eulerian Graphs

Def. Euler trail: a trail which uses each edge exactly once.

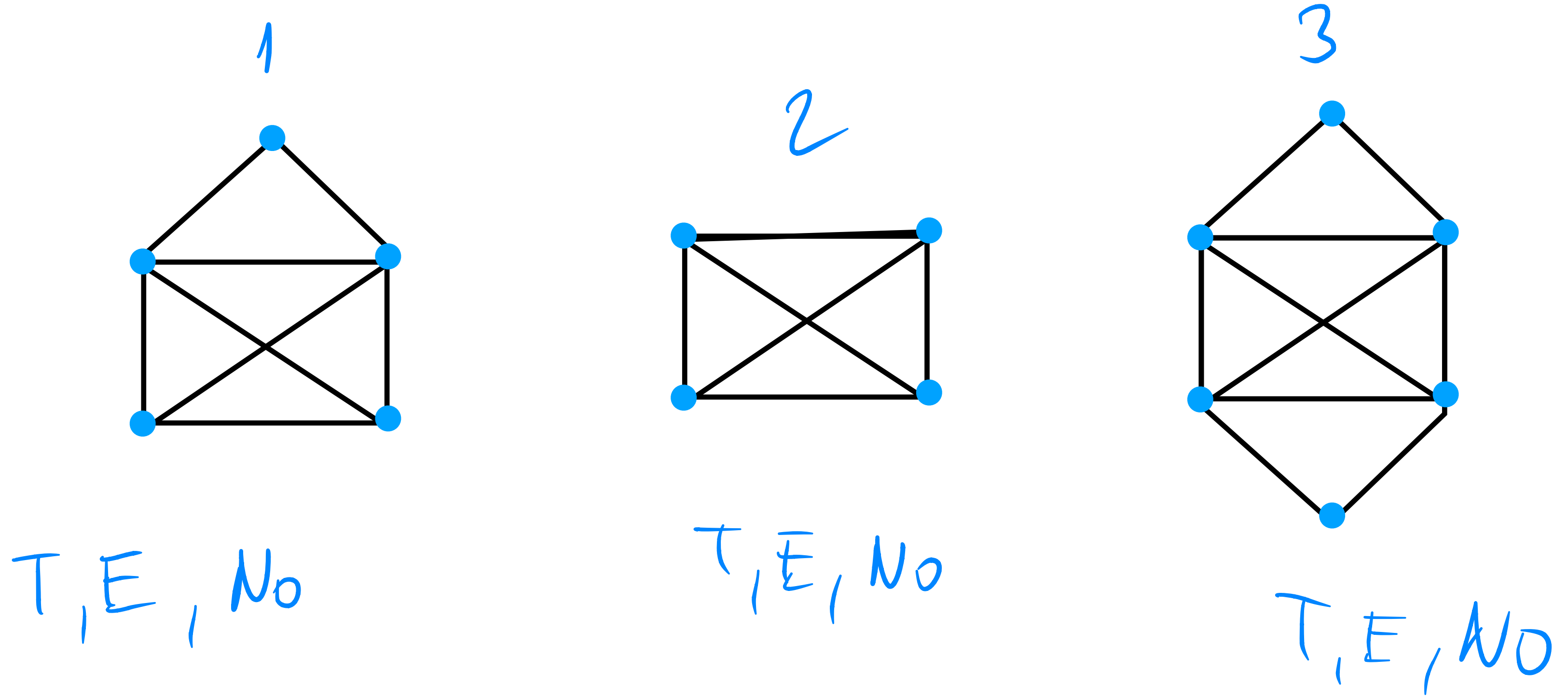
Graphs with an Euler (Eulerian) trail are called *traversible*

Traversible and Eulerian Graphs

Def. Euler trail: a trail which uses each edge exactly once.

Def. Euler circuit: a closed trail which uses each edge exactly once. (finish where you start)

Graphs with an Euler (Eulerian) circuit are called *Eulerian graphs*

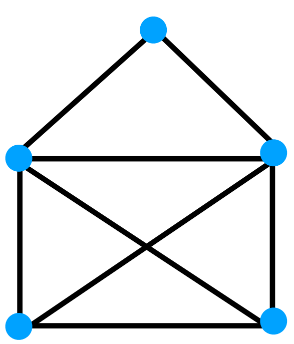


Traversible and Eulerian Graphs

Theorem 8.3 (Euler): a finite connected graph has an Euler circuit if and only if evey vertex has an even degree.

Corollary: a finite connected graph has an Euler trail if and only if evey vertex has an even degree or exactly two vertices have an odd degree

1



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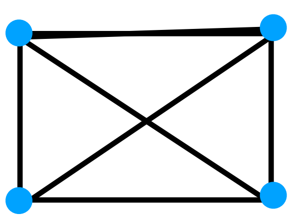

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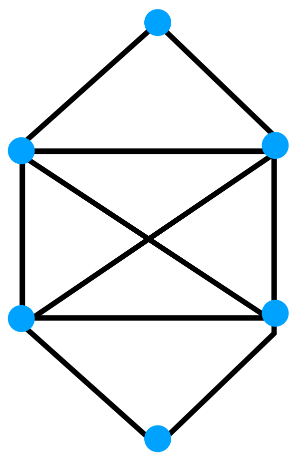
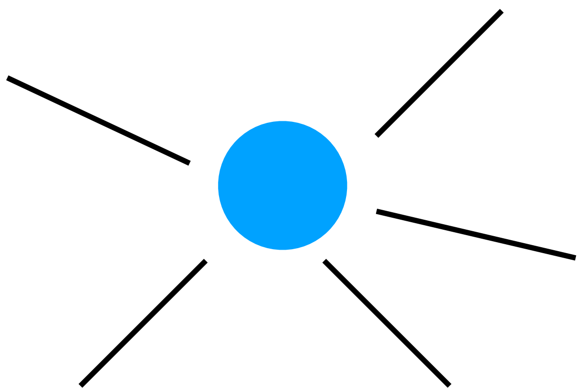


Traversible and Eulerian Graphs

Corollary. An undirected connected graph has an Eulerian trail if and only if exactly zero or two vertices have odd degree.

Almost a proof:

Imagine an "inner" vertex of the trail



if degree odd, you are *trapped*

intuition for sufficiency…

Assuming:

> Theorem 8.3 (Euler): a finite connected graph has an Euler circuit if and only if evey vertex has an even degree.

Can you prove:

> If a graph has two vertices v,w of odd degree (other even), then it has an Euler trail starting at v, ending at w?

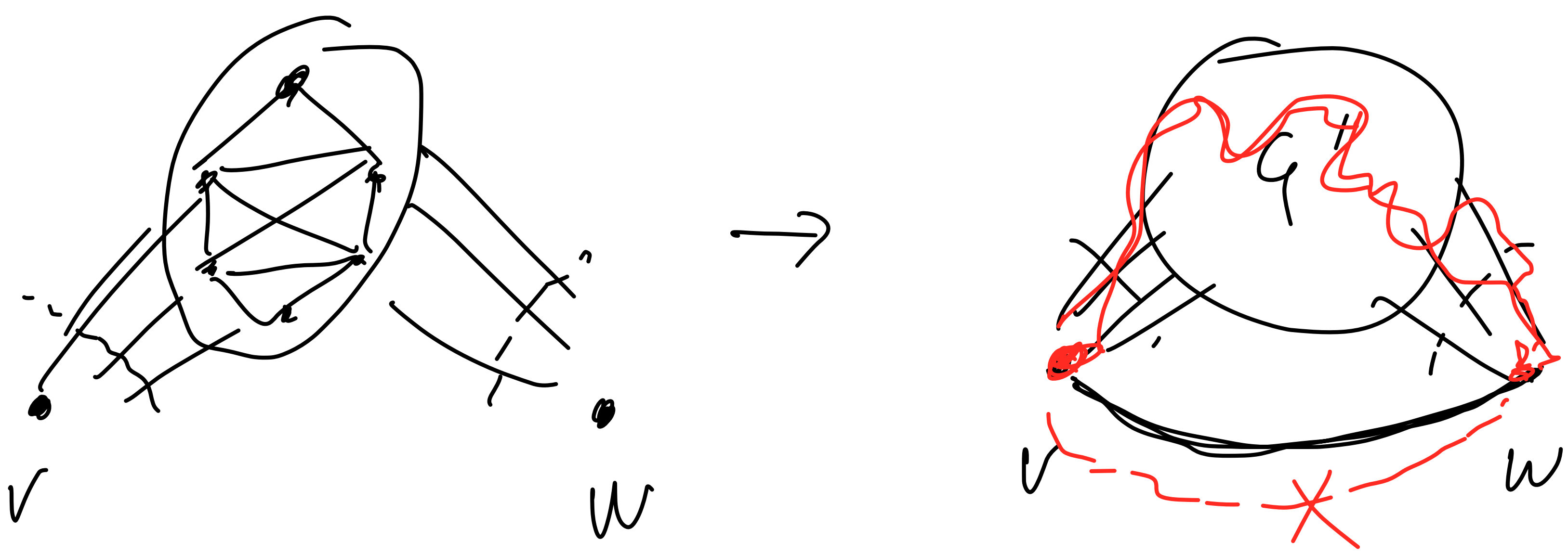
Assuming:

Theorem 8.3 (Euler): a finite connected graph has an Euler circuit if and only if evey vertex has an even degree. (1)

⇓

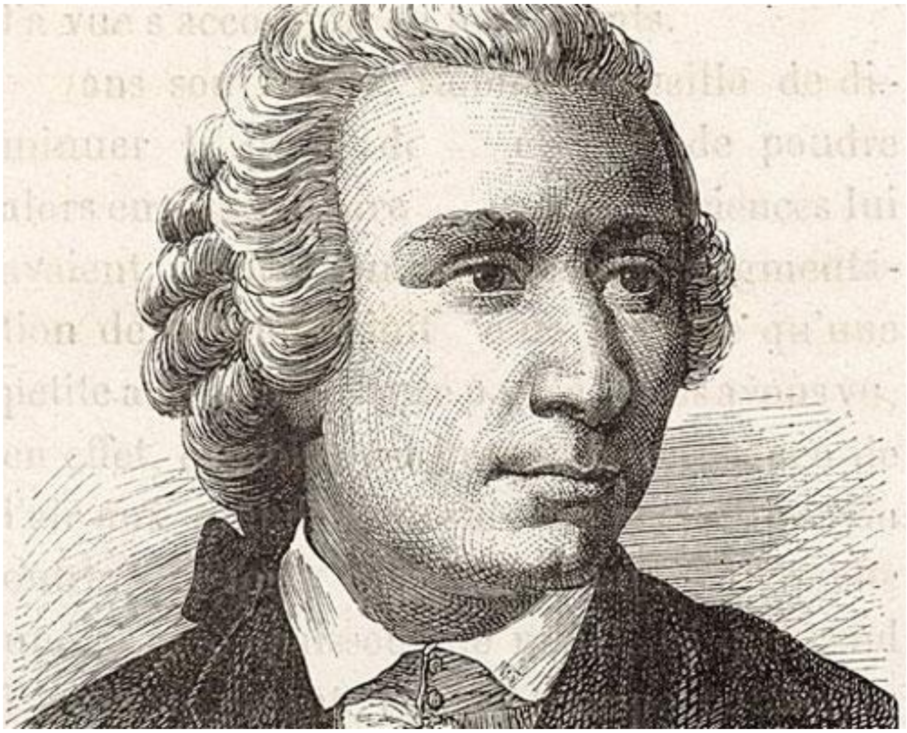
Can you prove:

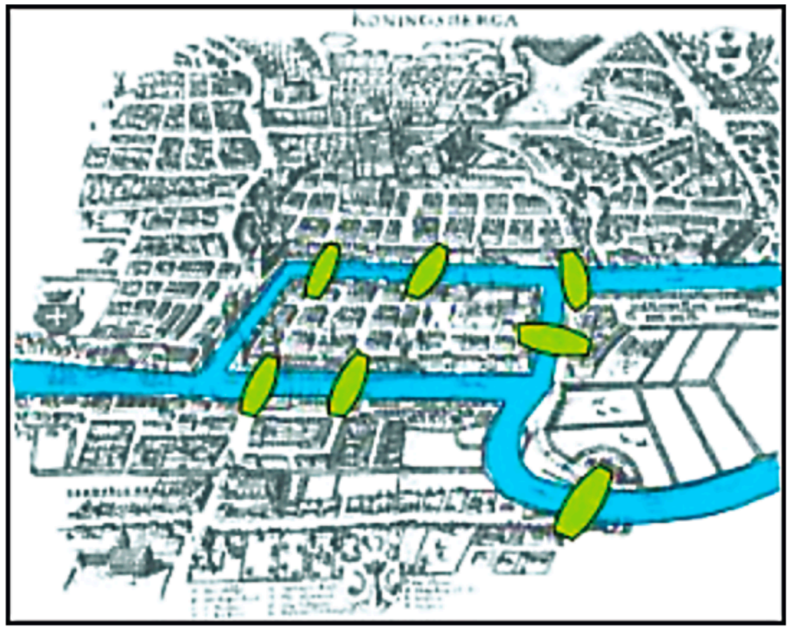
If a graph has two vertices v,w of odd degree (other even), then it has an Euler trail starting at v, ending at w? (2)

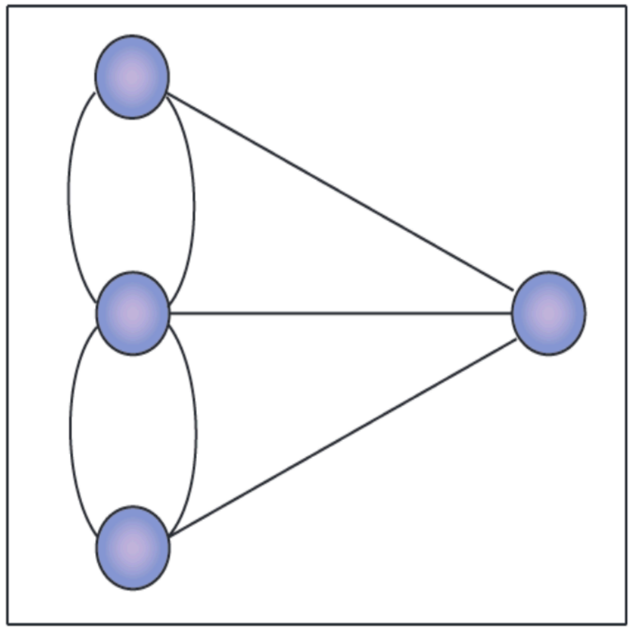


Traversible and Eulerian Graphs

Corollary. An undirected connected graph has an Eulerian trail if and only if exactly zero or two vertices have odd degree.



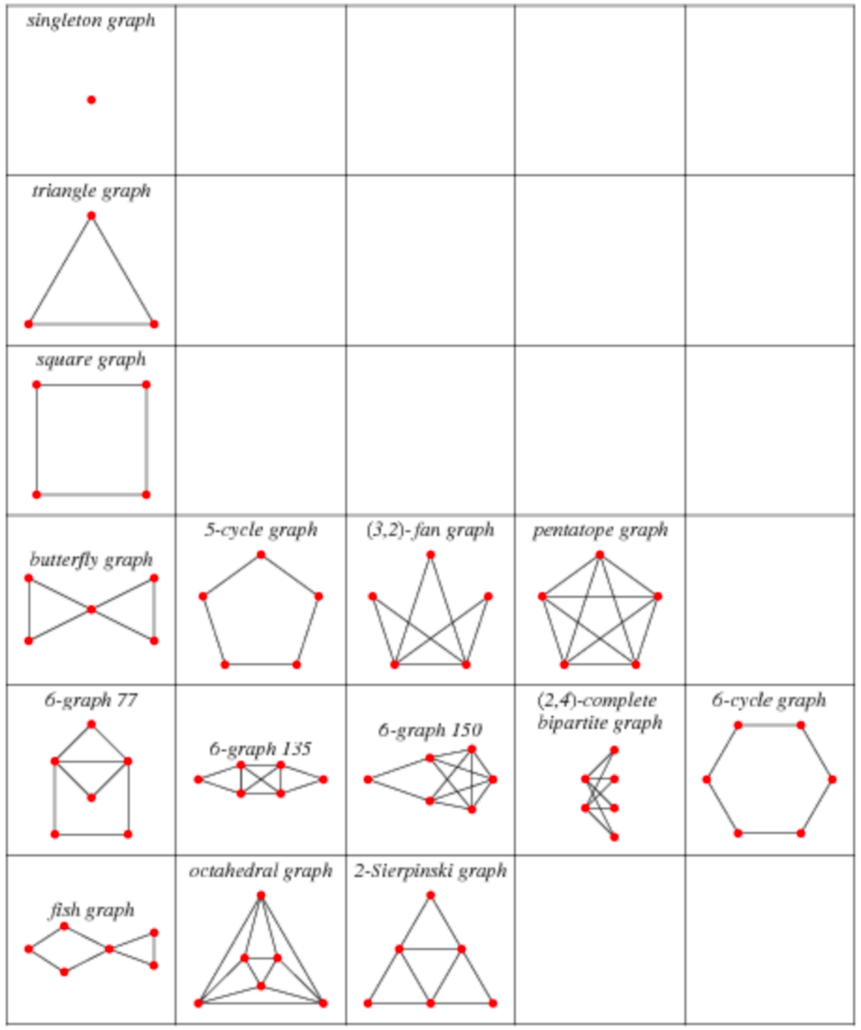




Traversible graph: an Eulerian trail exists

Eulerian graphs: an Eulerian circuit exists

Eulerian Graphs



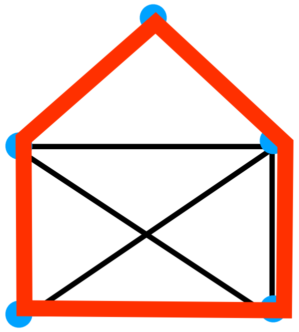
The numbers of (connected) Eulerian graphs with n nodes are 1, 0, 1, 1, 4, 8, 37, 184, 1782, ... OEIS A003049

http://mathworld.wolfram.com/EulerianGraph.html

Hamilton cycles

Def. Hamilton cycle: a closed path which uses each vertex exactly once.
(closed = starts where it ends)

->Travelling salesperson problem



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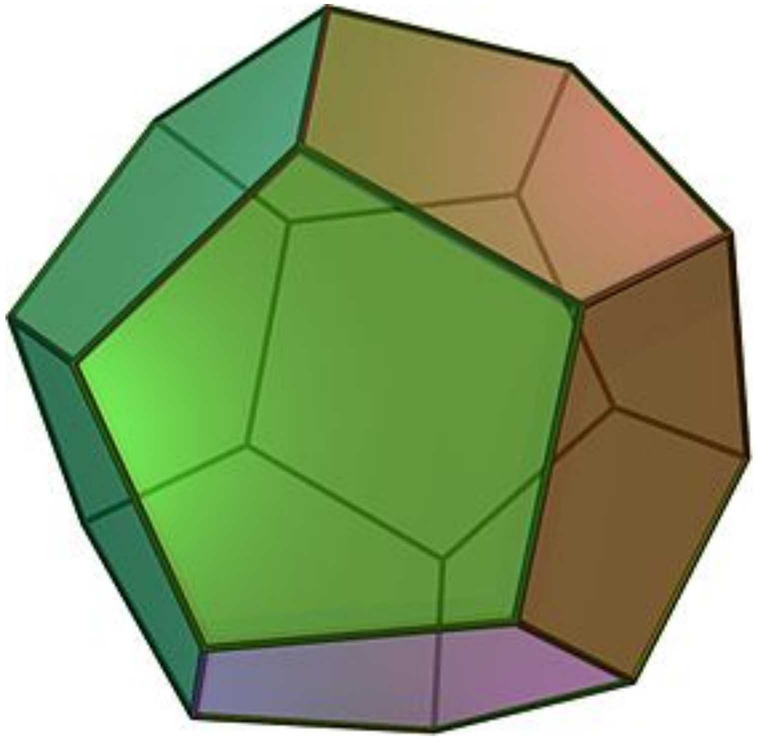

https://en.wikipedia.org/wiki/Regular_dodecahedron

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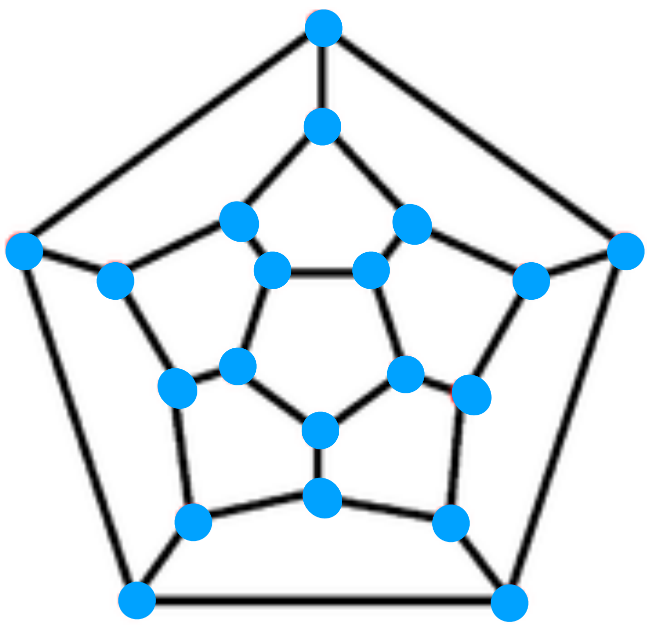

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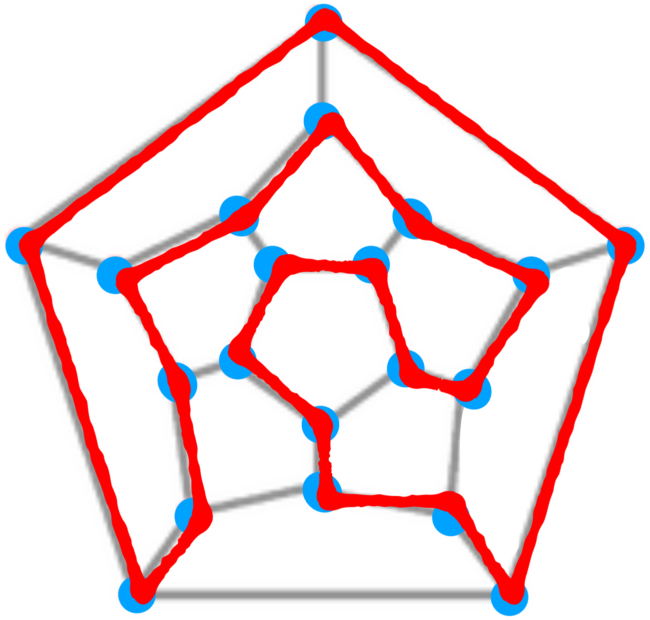

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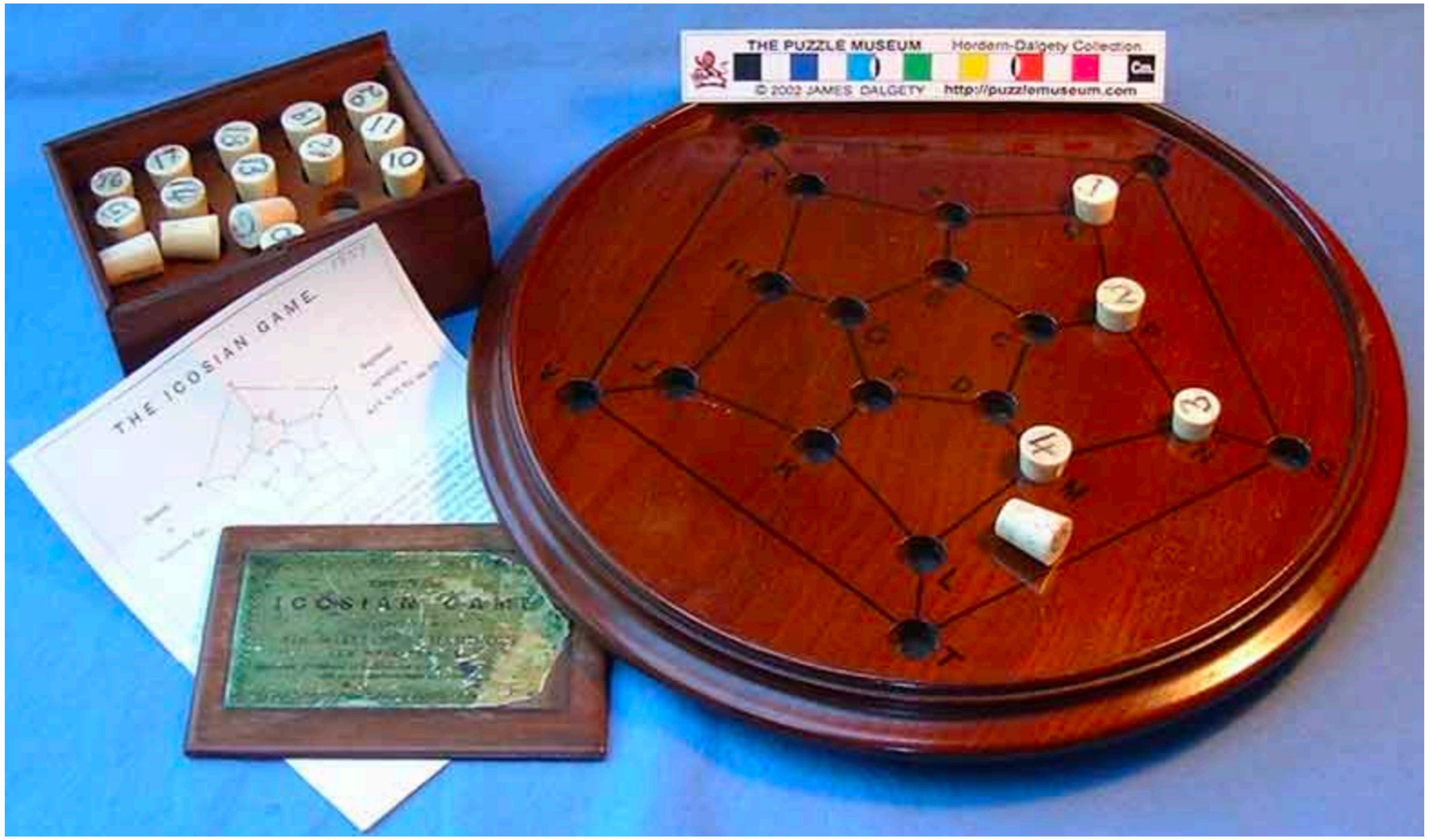
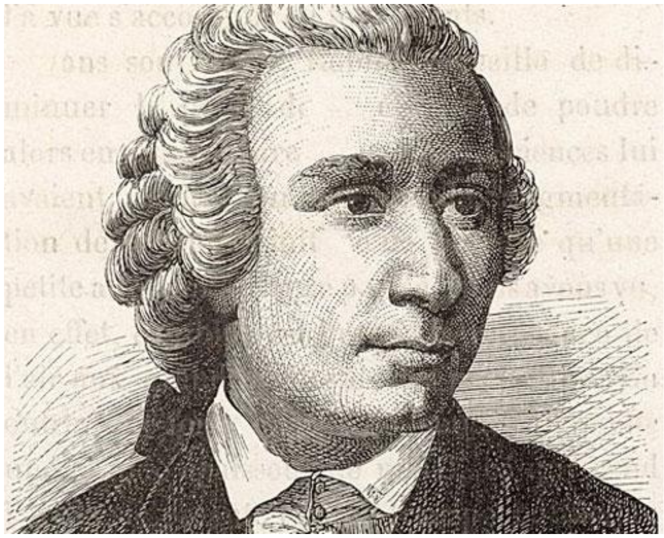


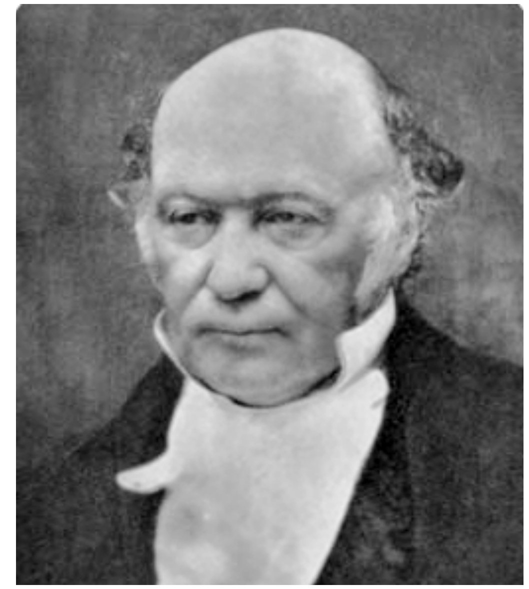
ICOSIAN GAME

https://www.puzzlemuseum.com/month/picm02/200207icosian.htm

Euler v.s. Hamilton



v



Leonhard Euler

Bridges of Königsberg
Closed, each line once

Simple characterization
Easy to detect

William Rowan Hamilton

Icosian game
Closed, each vertex once

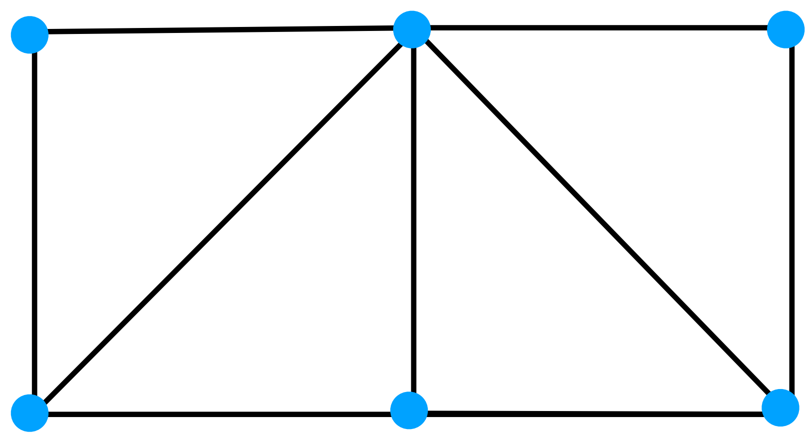
“Travelling salesperson problem”

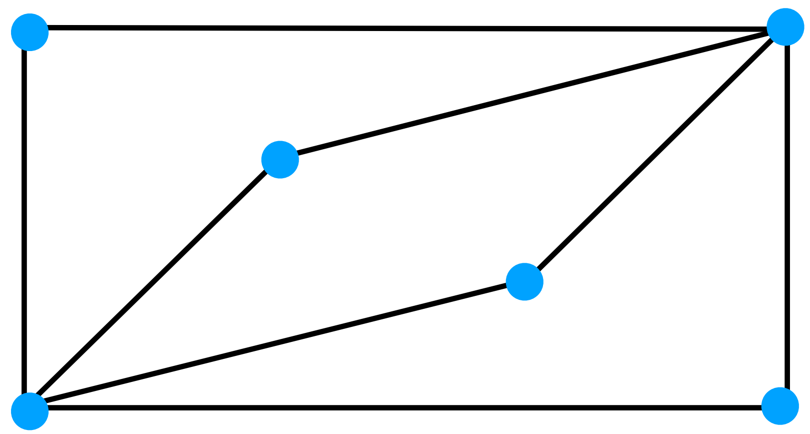
Ore (1960). A graph with n-vertices (n >3) is Hamiltonian if, for each pair of non-adjacent vertices, the sum of their degrees is n or greater.

If but not if and only if….NP-complete…

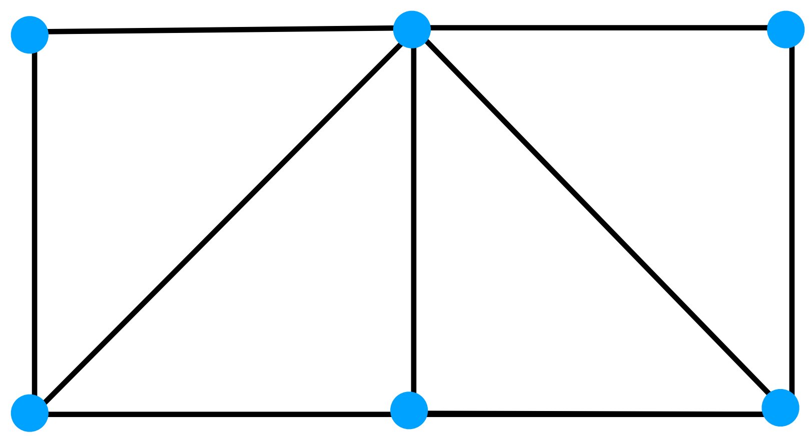
https://en.wikipedia.org/wiki/William_Rowan_Hamilton

Euler v.s. Hamilton (Schaum)

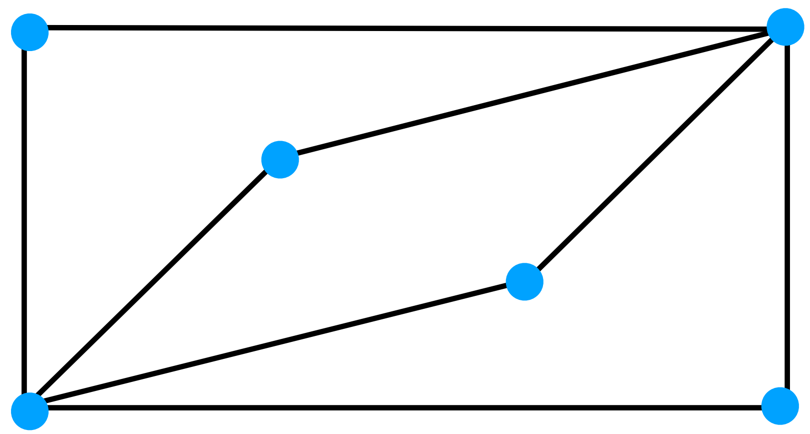




Euler v.s. Hamilton



Hamiltonian and non-Eulerian



Eulerian and non-Hamiltonian

“Note that an Eulerian circuit traverses every edge exactly once, but may repeat vertices, while a Hamiltonian circuit visits each vertex exactly once but may repeat edges.” Schaum p.161

Huh?

Euler v.s. Hamilton

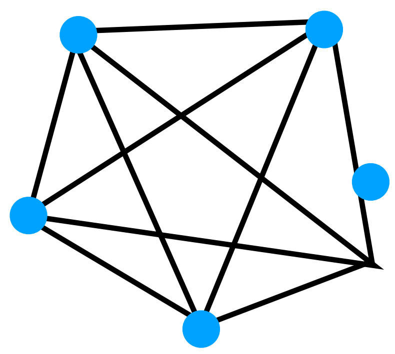
TYPOS and mistakes HAPPEN!

Schaum p.162 Theorem 8.5 (Dirac, 1952):
Let G be a connected graph with n vertices.
Then G is Hamiltonian if $n > 3$ and $n/2 \leq \deg(v)$ for each vertex v in G.

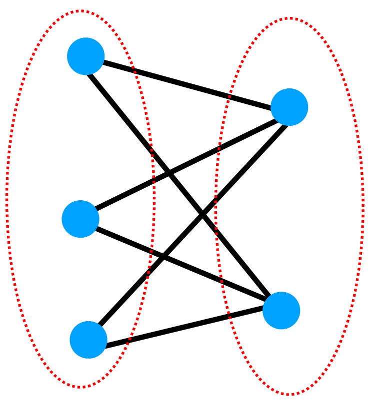
No need to know this theorem (for this course).
It is an illustration of the type of propositions
that have been obtained to encompass the concept of Hamiltonian.

Special graphs

complete graph K_n
bipartite graph
complete bipartite graph $K_{m,n}$ (or $K_{m \times n}$)

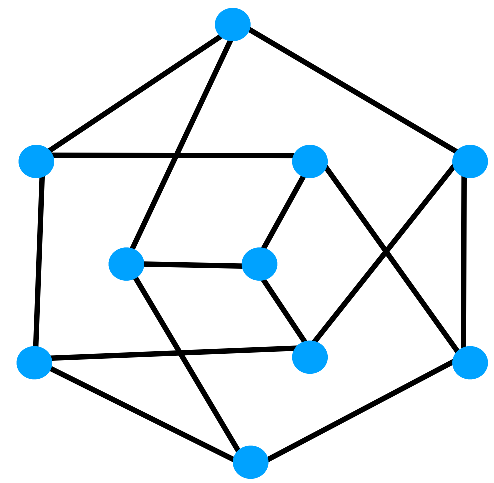


K_5

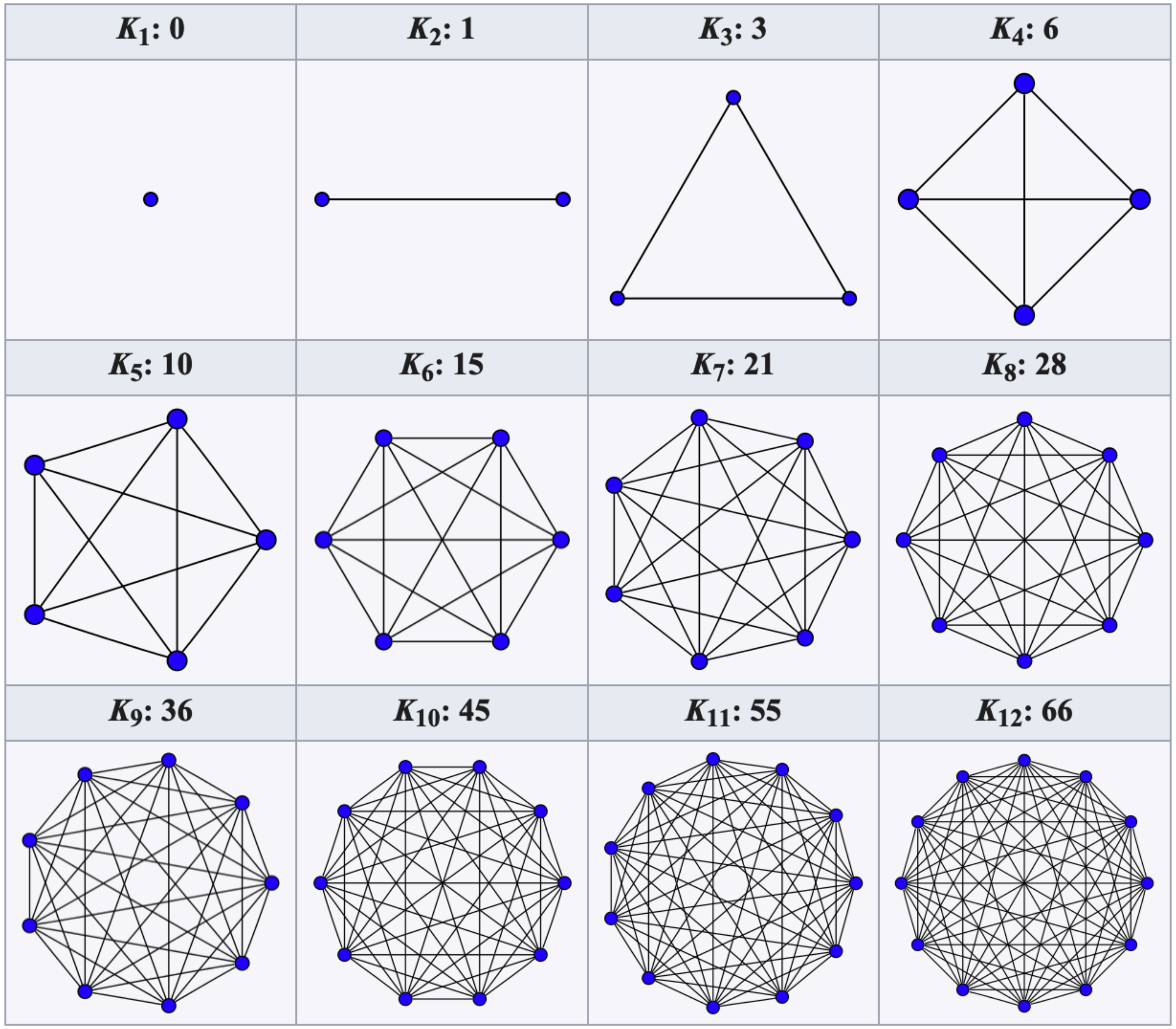


$K_{3,2}$

***k*-regular graph: all vertices degree k**



Complete graphs



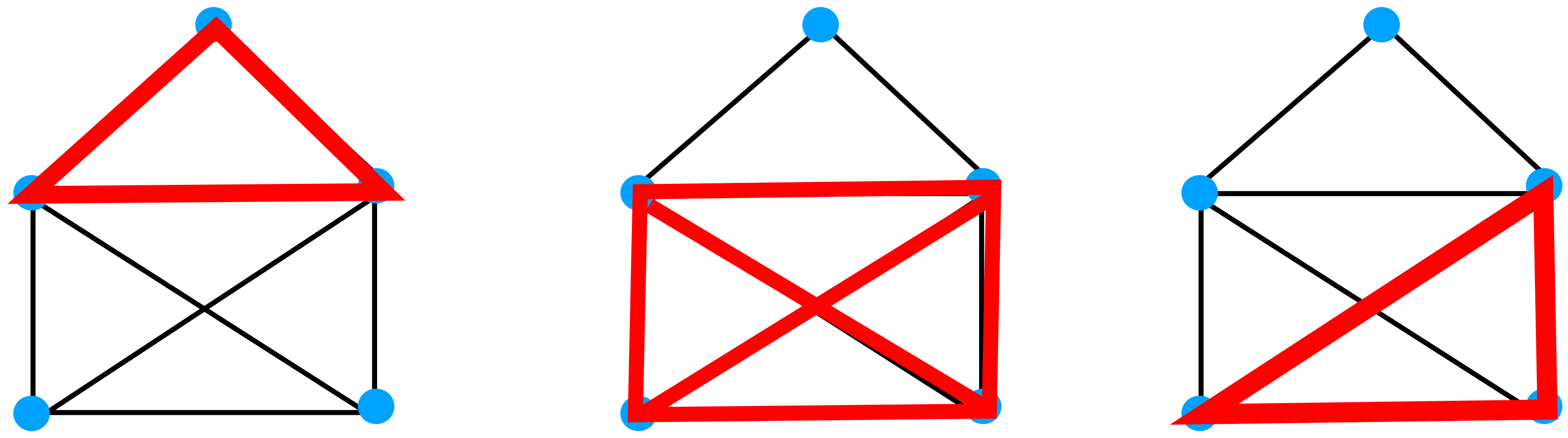
How many edges?

$$\frac{n(n-1)}{2} = \binom{n}{2}$$

https://en.wikipedia.org/wiki/Complete_graph

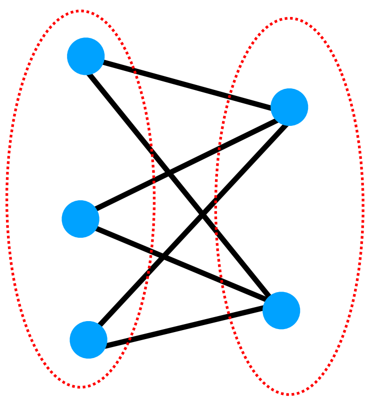
Complete graphs

Extra info: Complete induced subgraphs are called *cliques*



Bipartite graphs

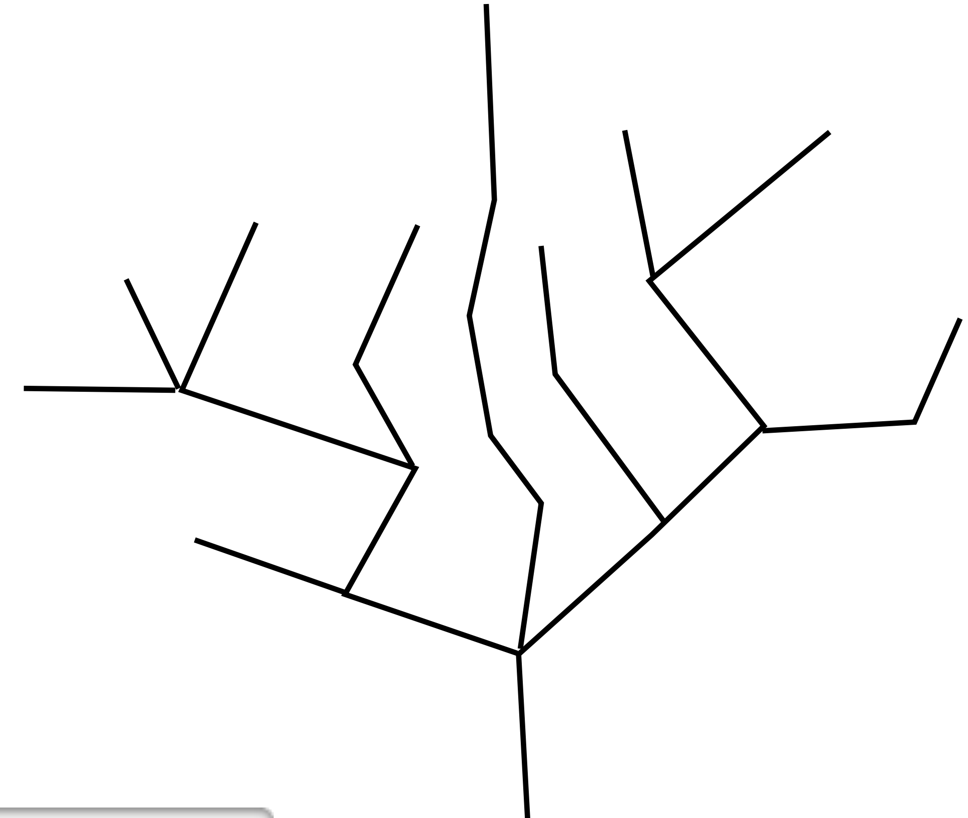
Def. A graph is bipartite if there exists a bipartition of the vertices s.t. (such that) there are no edges within the partitions.



$K_{3,2}$

Theorem. A graph is bipartite if it has no cycles of odd lenght.

Trees (graphs)



Def. Tree is a connected graph with no cycles.

The following are equivalent:

1) *G* is a tree; (over n vertices)

2) *G* has no cycles and *n-1* edges;

3) *G* is connected and has *n-1* edges;

WILL DO THIS EXTENSIVELY ..

https://commons.wikimedia.org/wiki/File:Tree_without_leaves_2.jpg

Bipartite graphs

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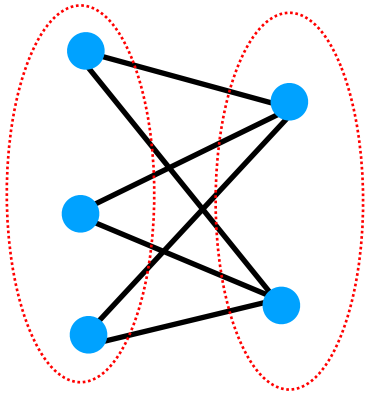

$K_{3,2}$

Theorem. A graph is bipartite if it has no cycles of odd lenght.

One way is easy…

Bipartite graphs

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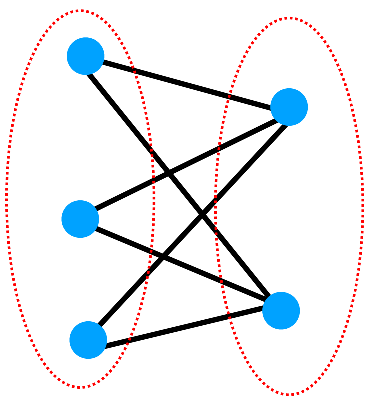


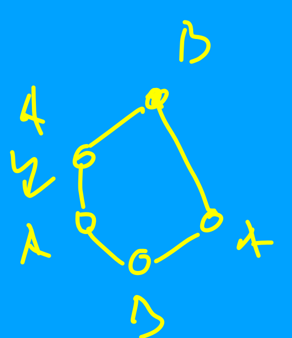
$K_{3,2}$

Theorem. A graph is bipartite if and only if it has no cycles of odd lenght.

One way is easy…

1.) ONLY NEED TO CONSIDER CONNECTED GRAPHS

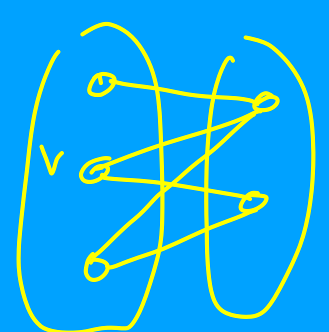
2) ODD CYCLE ⇒ NOT BIPARTITE.
LENGTH n.



3) No odd cycles.
Choose vertex v.
- put all Even distance in B
- all odd distance in A

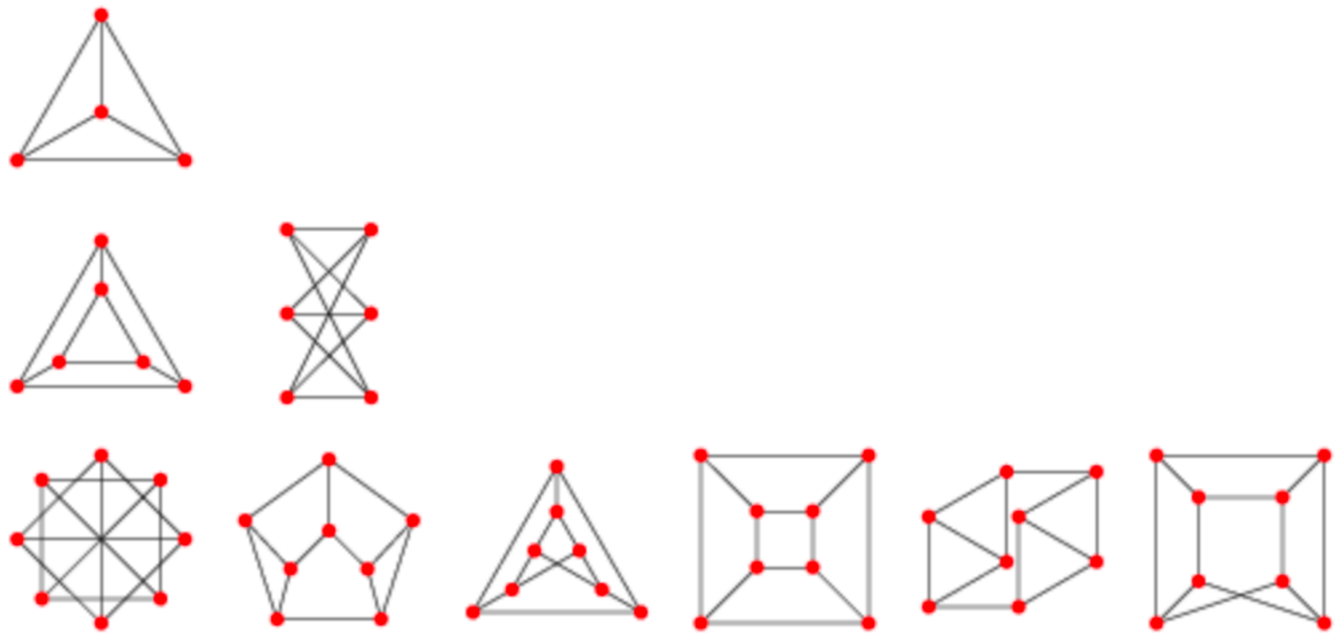
$v \in B$, $A \cap B = \emptyset$

$a_1, a_2 \in A$, adjacent $\Rightarrow$ $\left.v_1 \cdots a_1, a_2 \cdots v_1\right\}$ odd walk $\Rightarrow$ (can be proven) $\exists$ odd cycle



Digression

3-regular graphs are also called cubic graphs…

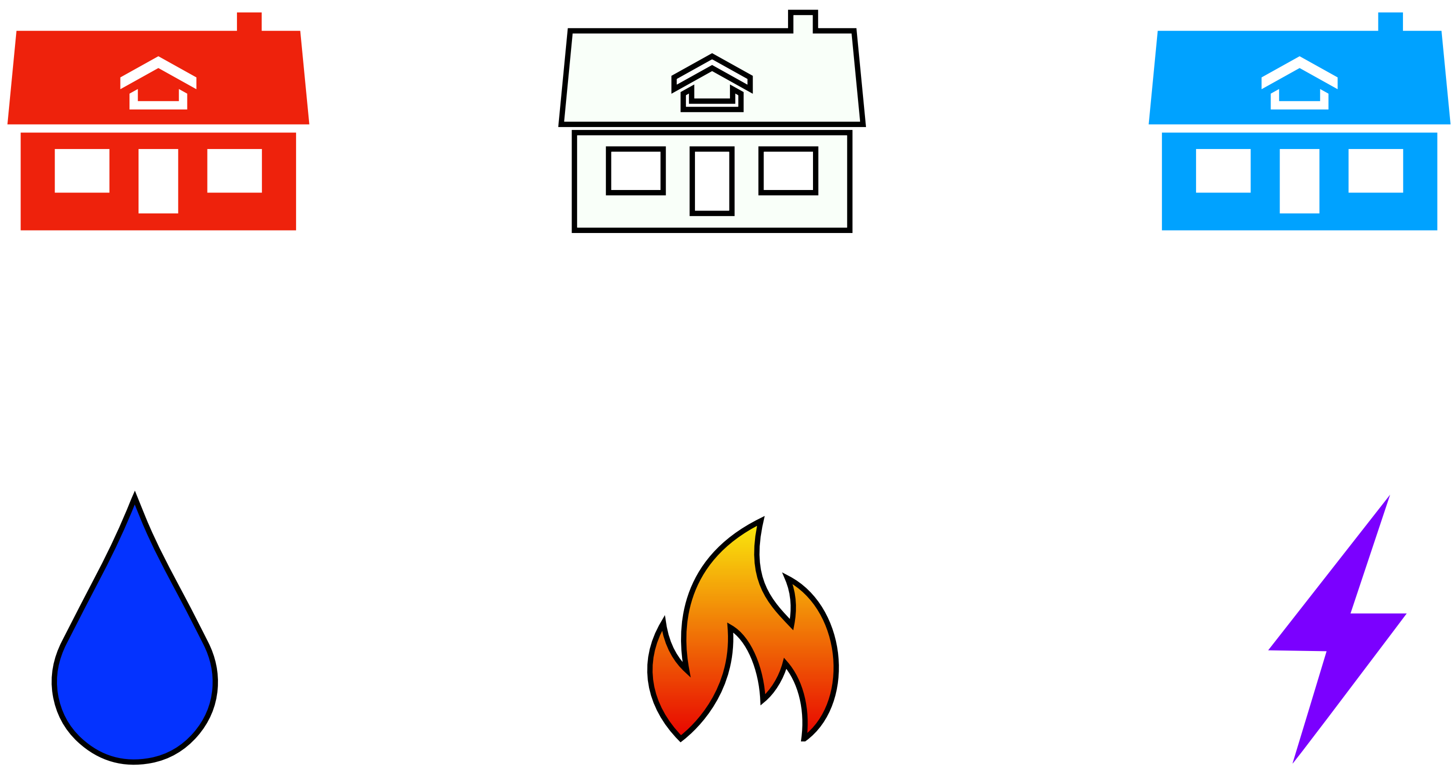


0, 1, 2, 5, 19, 85, 509, 4060, 41301, ... (OEIS A002851).

http://mathworld.wolfram.com/CubicGraph.html

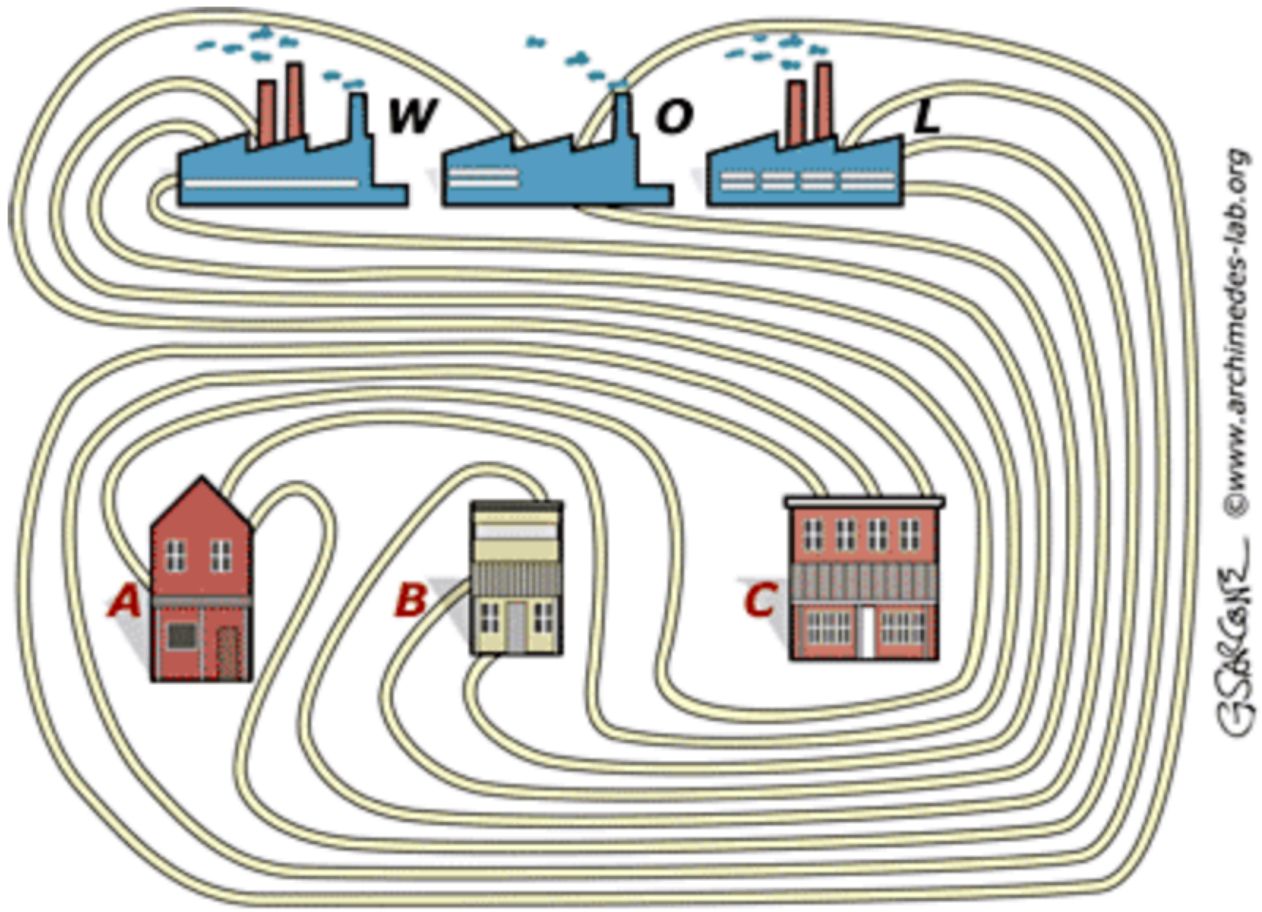
Planar graphs

Water, Gas and Electricity



Connect each house to source… no lines crossing!

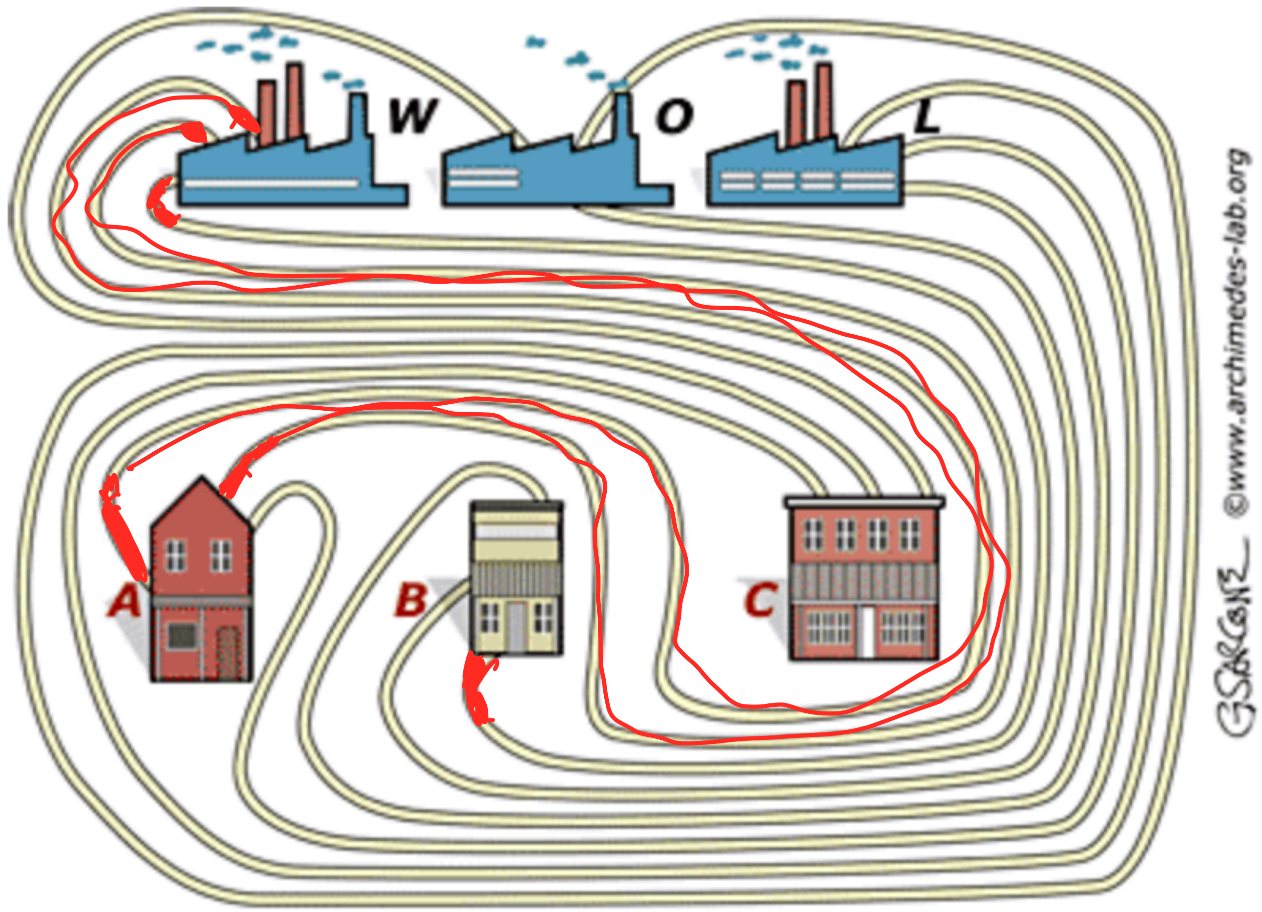
Planar graphs



?

http://www.archimedes-lab.org/How_to_Solve/Water_gas.html

Planar graphs



?

http://www.archimedes-lab.org/How_to_Solve/Water_gas.html

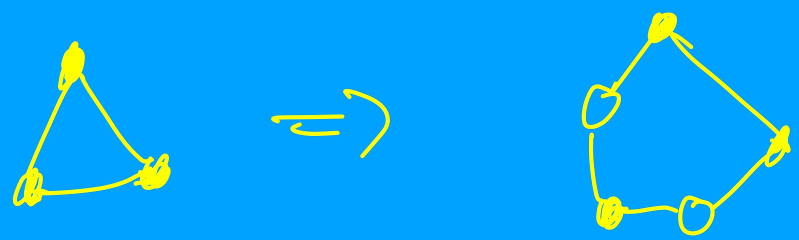
Planar graphs

Planar graphs can be drawn (on a plane) without intersecting edges.
Euler proved the following relationship for planar graphs:
$|V| - |E| + r = 2;$
where r stands for the *faces:* "regions" the plane is cut into, including the outermost.

Kuratowski: A finite graph is planar if and only if it
does not contain a subgraph that is a subdivision of the complete graph K_5
or the complete bipartite graph $K_{3,3}$

Explain a bit…

Subdivision = graph expansion by subdividing

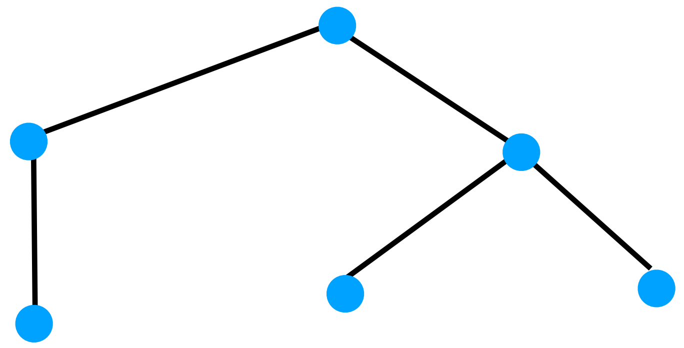


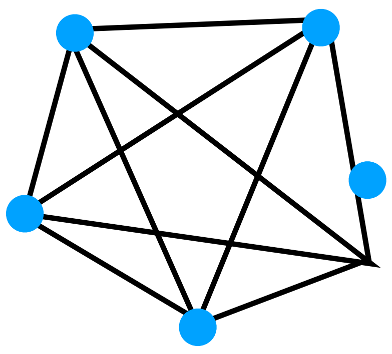
See Wikipedia
for pretty pictures

Counting edges

A connected graph with n vertices has:

- **at least *n-1* edges**
- **at most** $\binom{n}{2} = \frac{n(n-1)}{2}$ **edges**





K_5

Labeled graphs & weights

Labeled graph: information on the edges
Weighted graph: values (numbers) on the edges

$w : E \to$ **Labels; or** $w : E \to \mathbb{R}; \quad w(e)$

Can mean: capacity (conductance, diameter), cost (time, distance)

- **weight of a path: sum of weights across a path**
- **minimal spanning tree: *Prim's algorithm, Kruskal's algorithm***
- **shortest ("cheapest") paths; *Dijksta's algorithm***

Labeled graphs & weights

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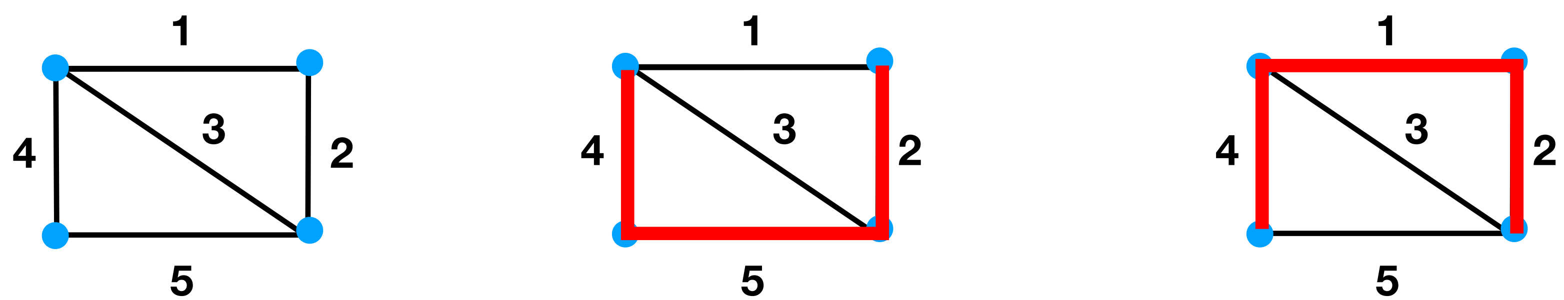
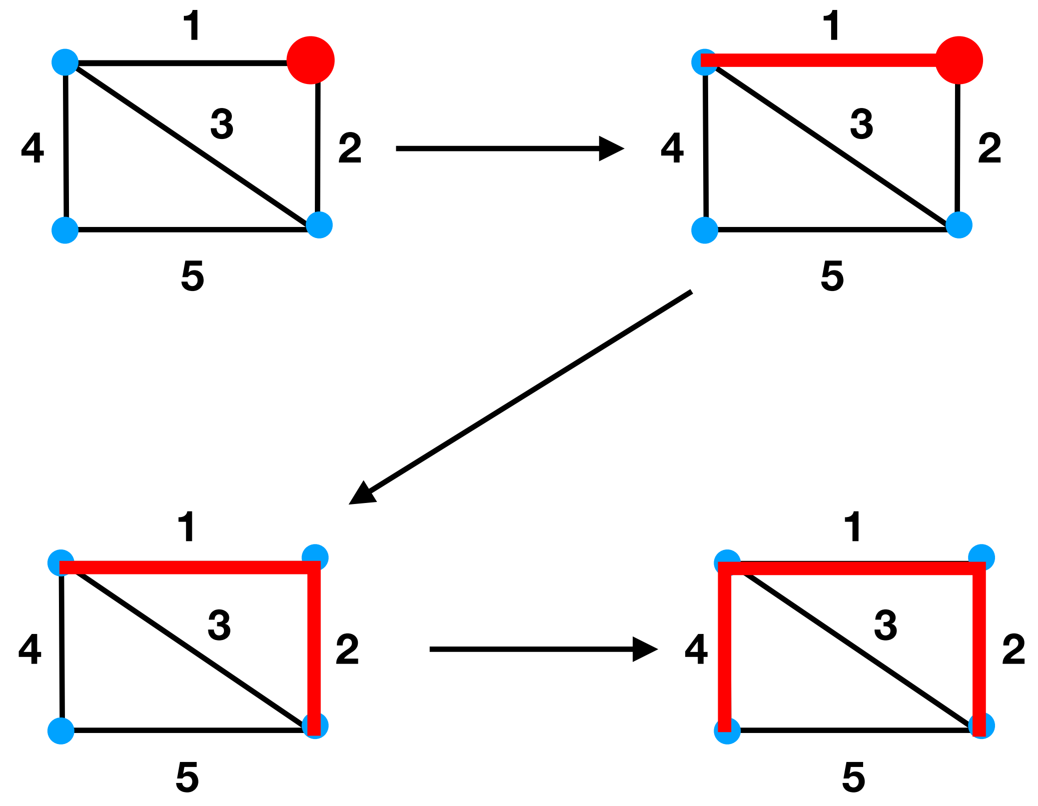
- **minimal spanning tree and *Prim's algorithm***



- *Prim's greedy algorithm…*



- *add lightest tree edge to the tree*

Labeled graphs & weights

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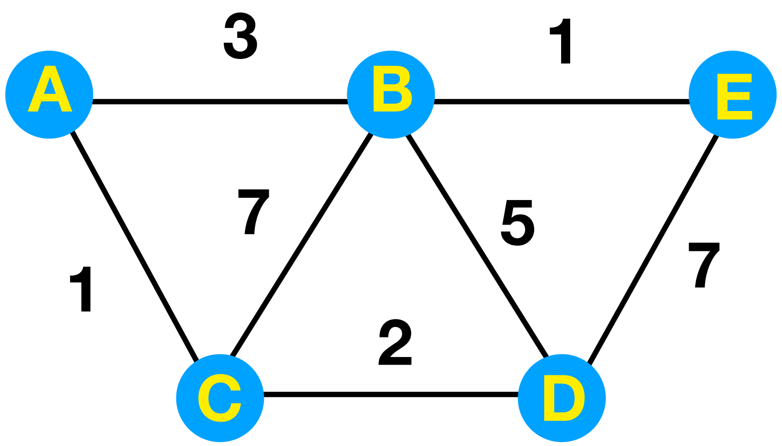
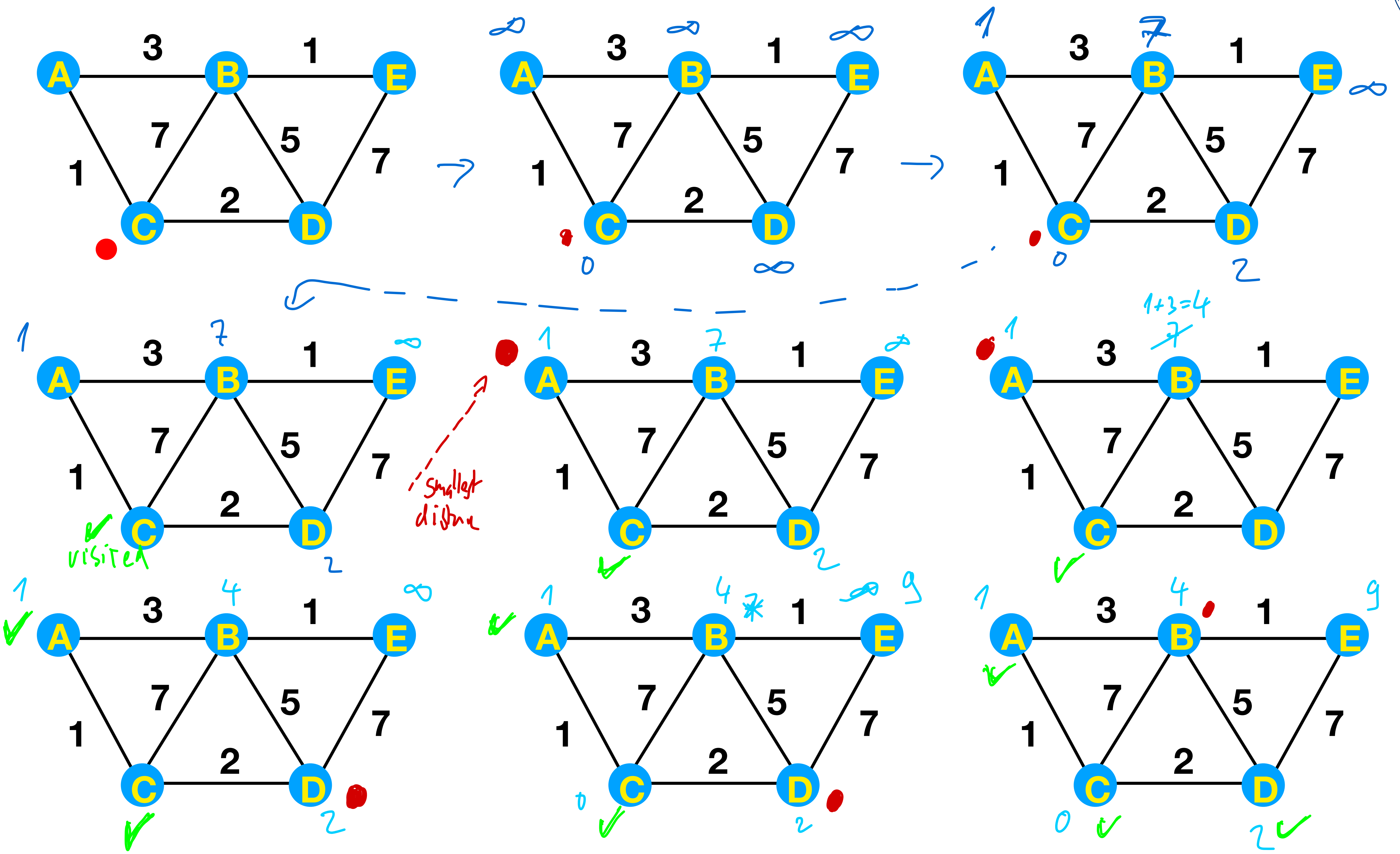
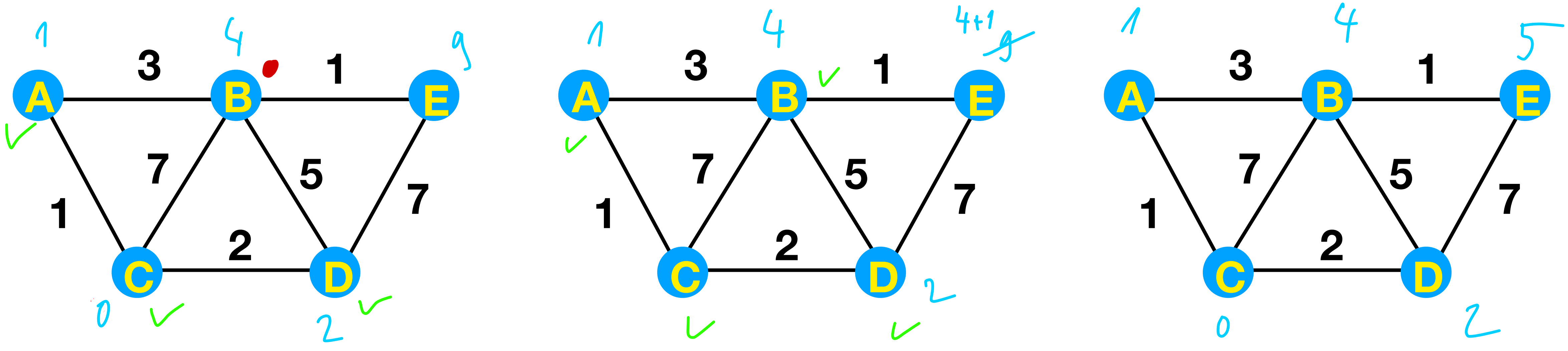
- **shortest ("cheapest") paths; *Dijksta's algorithm***



- ***Dijksta's algorithm***



- ***Dijksta's algorithm***



① MARK INITIAL NODE, ZERO, REST INFINITY.

② SET NON-VISITED WITH CURRENT SMALLEST DISTANCE AS CURRENT NODE C

③ ∀ neighbour n of c: add current distance of c with the weight n→c
If smaller than current, set as new current.

④ MARK C as visited

⑤ If there are no current. non-visited nodes, goto 2.

Directed graphs

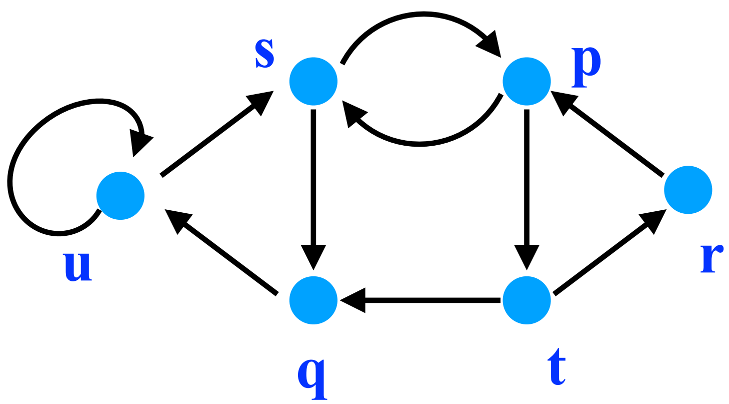
Definition. A directed graph *G* — *a digraph* — is an ordered pair *(V,E) where*

- ***V = V(G) is the set of vertices (or nodes)***
- ***E=E(G) is the set of directed edges (or arrows, or arcs)***

directed edges e=(u,v) are ordered pairs of vertices, *<u>from u to v</u>*

we also say the edge begins in <u>u</u> and ends in <u>v</u>, *u precedes v, or v follows u*

Loops possible. Parallel edges not (anti-parallel yes!).



$$E = \{\boxed{(p,s)}, (p,t), (q,u), (r,p), \boxed{(s,p)}, (s,q), (t,q), (t,r), (u,s), \boxed{(u,u)}\}$$

Caveat: mistakes and inconsistencies happen

In example 9.1 in Schaum (p 202), graph (a) contains two parallel arrows: (B, A) appears twice in the set E (G). That is not in line with the definition of a set. So this graph is actually a directed multigraaf. Oh well. Note that for defining a directed or undirected multigraph for E, we could use the concept of a multiset. In the book and the lecture, multigraphs (directed or undirected) are used informally.

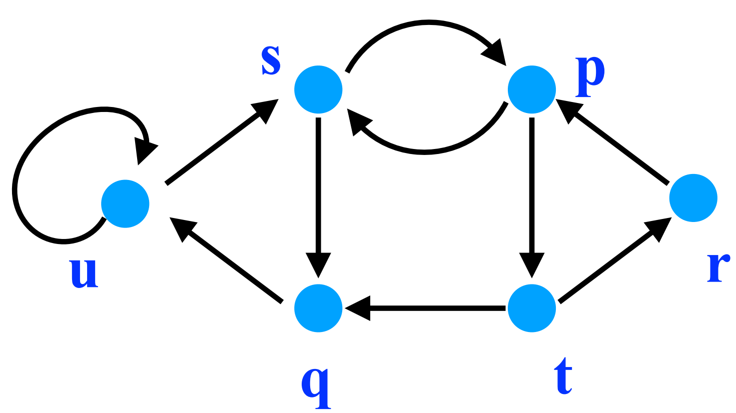
Directed graphs: main concepts

out-degree *outdeg(v)*: number of outbound edges

in-degree *indeg(v)*: number of inbound edges

Source: vertex v with indeg(v)=0.

Sink: vertex v with outdeg(v)=0.



to…

from…

$$
\begin{array}{c}
\\ p \\ q \\ r \\ s \\ t \\ u
\end{array}
\begin{array}{c}
\begin{array}{cccccc} p & q & r & s & t & u \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}
\end{array}
$$

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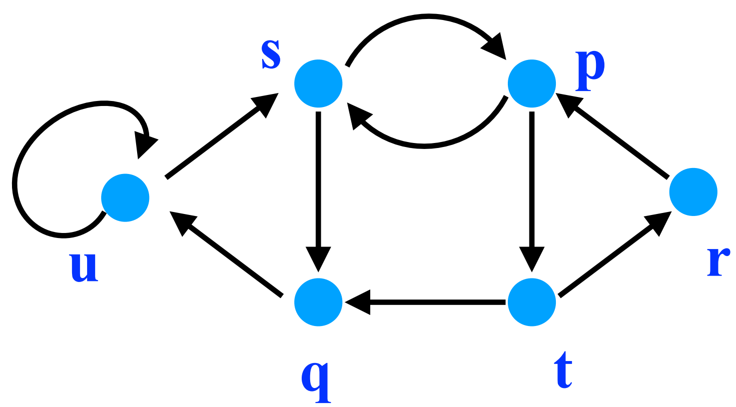
Source: vertex v with indeg(v)=0.

Sink: vertex v with outdeg(v)=0.



to…

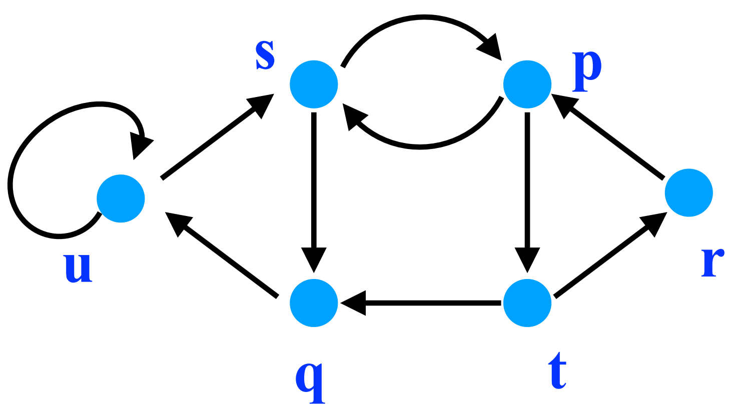
from…

$$\begin{array}{c|c} & \begin{array}{cccccc} p & q & r & s & t & u \end{array} \\ \begin{array}{c} p \\ q \\ r \\ s \\ t \\ u \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

Directed graphs: main concepts

out-degree *outdeg(v)*: number of outbound edges
in-degree *indeg(v)*: number of inbound edges

Source: vertex v with indeg(v)=0.
Sink: vertex v with outdeg(v)=0.



Theorem. In a directed grap *G* the following holds*:*

$$\sum_{v\in V} outdeg(v) = |E| = \sum_{v\in V} indeg(v)$$

"number of starts" = "number of ends"

Directed graphs: main concepts

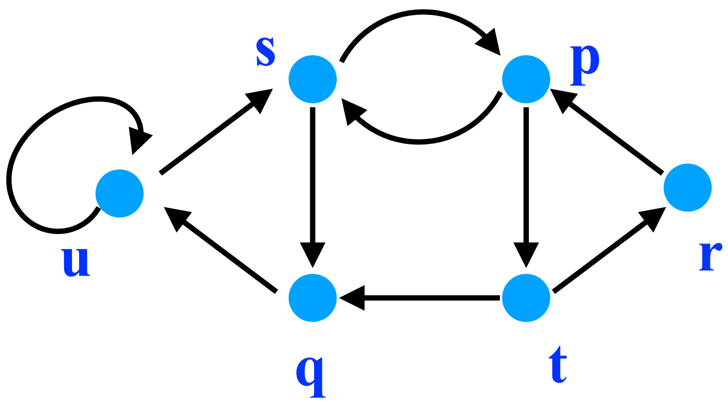
directed path: a sequence $v_1, e_1, v_2, e_2 \ldots, v_n$, with $e_k = (v_k, v_{k+1})$

Lenght of path = number of (directed) edges in path *(n)*

simple: differing vertices
cycle: closed path (first vertex = last vertex)
trail: differing edges
circuit: closed trail

spanning path: passes all vertices (recall Hamilton)

***semipath:* undirected path; path in the underlying undirected graph**
($e_k = (v_k, v_{k+1})$ OR $e_k = (v_{k+1}, v_k)$)



path: q→u→s→p→t→r
semipath : p→s→q←t→r

Caveat: mistakes and inconsistencies happen

In Schaum, the term cycle is not dealt with very consistently. According to the definition, loops and a closed path such as s, p, s in the example of the previous previous would be cycles. After all, Schaum does not limit the length of the closed path, as was the case with undirected graph. However, in the example on page 221 (problem 9.1 (d)), Z, W, Z is not counted as a cycle. Oh, well.

Digraphs and connectedness

*Definition. A digraph is **strongly connected** if **every pair of vertices** is **connected by a directed path.***

*Definition. A digraph is **weakly connected** if **every pair of vertices** is **connected by a semipath.***

Theorem 9.2. a) strongly connected if and only a closed spanning path exists
b) weakly connected if and only if a spanning semipath exists

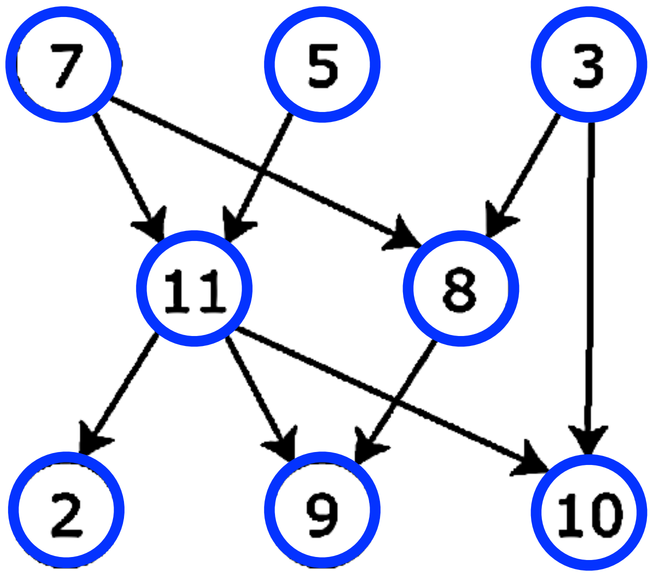
Theorem 9.3. A directed graph G without cycles has a source and a sink.

Theorem 9.3. If G is a directed graph without cycles, then there exists a topological ordering of G (and converse)

Digraphs: topological ordering

A topological ordering ***(topological sorting***) of a directed graph G = (V, E) is a sequence (an enumeration) $v_1, v_2, \ldots, v_n$ of all the vertices of of G such that $(v_i, v_j) \in E, \Rightarrow \; i < j$

Or: you can draw the vertices of the graph in such a configuration that the arrows always point left to right.

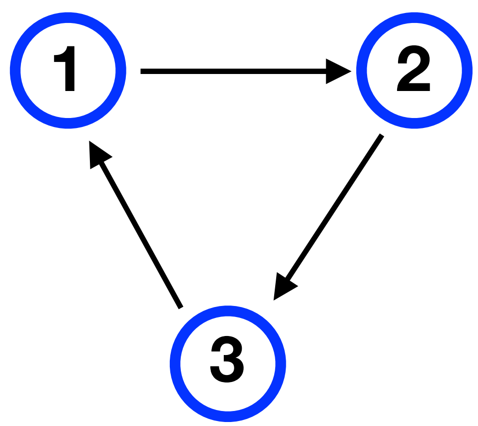


a topological ordering:
7,5,11,2,3,10,8,9

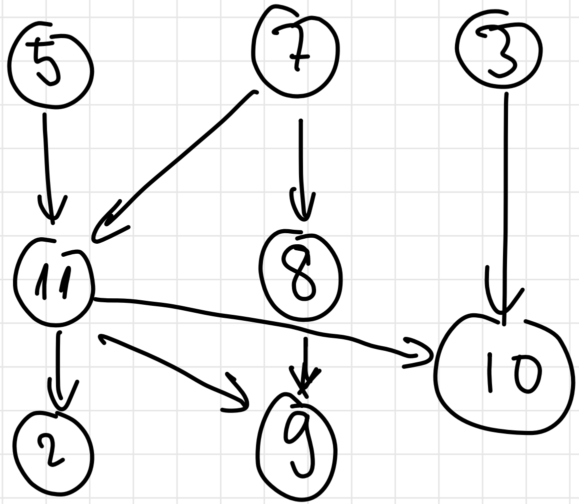
Digraphs: topological ordering

A topological ordering ***(topological sorting***) of a directed graph G = (V, E) is a sequence (an enumeration) $v_1, v_2, \ldots, v_n$ of all the vertices of of G such that $(v_i, v_j) \in E, \Rightarrow \;\; i < j$

Or: you can draw the vertices of the graph in such a configuration that the arrows always point left to right.



Rock, Paper, Scissors, Lizard, Spock?



5
7
3
11
8
10
2
9

5, 7, 3, 11, 8, 2, 9, 10

3, 5, 7, 8, 11, 2, 9, 10

5, 7, 3, 8, 11, 10, 9, 2