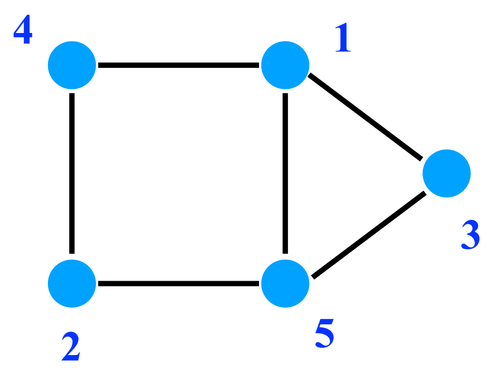
Lecture 10

Graph Theory

basics refresher & continuation

Graphs

-undirected



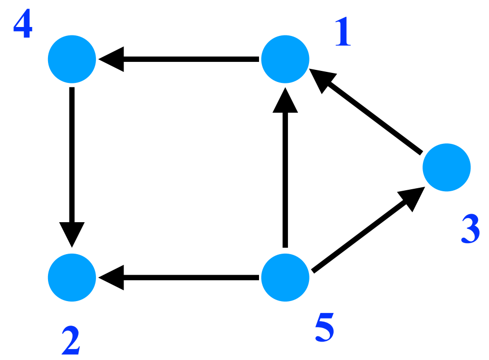
$V = \{1,2,3,4,5\}$

$E = \{\{1,3\},\{1,4\},\{1,5\},$
$\{2,4\},\{2,5\},\{3,5\}\}$

Definition. An undirected graph *G* is an ordered pair *(V,E)* where
- ***V = V(G)* is the set of vertices (or nodes)**
- ***E=E(G)* is the set of edges**
- ***an edge is a set of two vertices***

Graphs

-directed



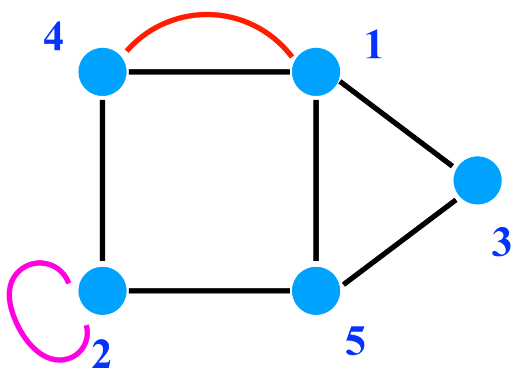
V = {1,2,3,4,5}

E= {(3,1),(1,4),(5,1),
(4,1),(5,2),(5,3)}

Definition. An undirected graph *G* is an ordered pair *(V,E) where*
- ***V = V(G) is the set of vertices (or nodes)***
- ***E=E(G) is the set of edges***
- ***an edge is an ordered pair of two vertices***

Graphs

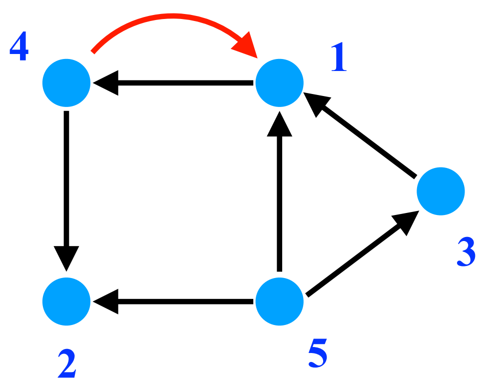
-simple graphs: no loops, no multiple edges



V = {1,2,3,4,5}

E= {{1,3},{1,4},{1,4},{1,5},
{2,4},{2,5},{3,5},{2,2}}

In directed graphs:
two edges of different
directions
are different!
Still simple graph



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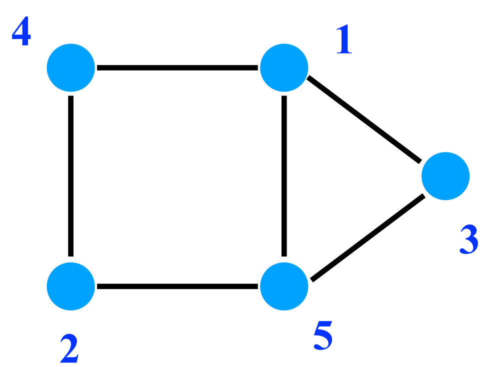
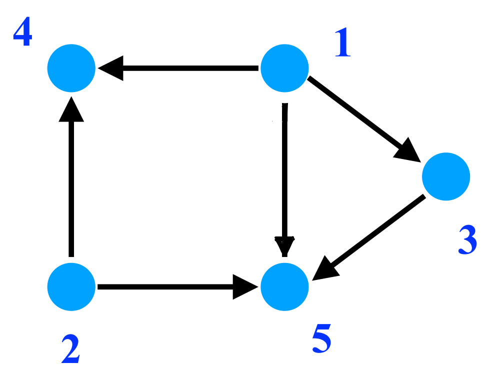
Graphs

-representations; directed graphs are binary relations
-can be represented as adjacency matrices

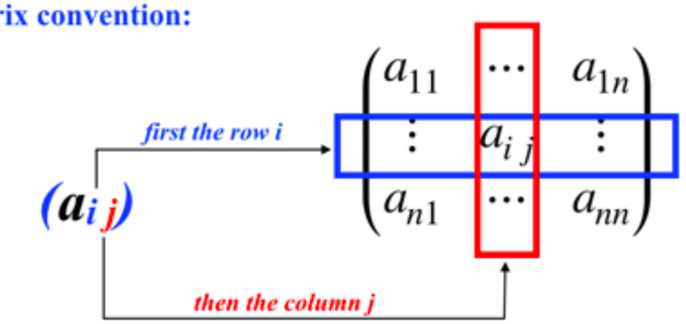


$$\text{from}\quad \begin{array}{cc} & \textit{to} \\ & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$



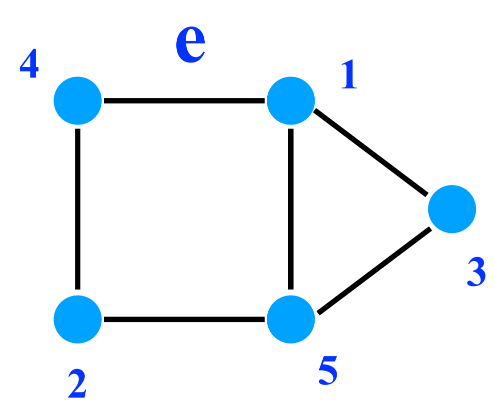
$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

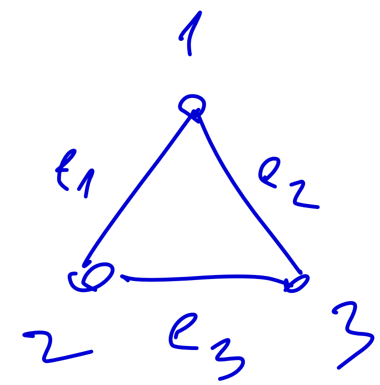
Matrix convention:



Graphs

-basic concepts: incidence (vertex-edge), neighbour(hood), adjacency





-degree of vertex deg(v)

$\|$

$\{(1,e_1),(1,e_2),(2,e_1),(2,e_3),(3,e_2),(3,e_3)\}$

Graphs

Degree-sum formula:

$$\sum_{v \in V} deg(v) = 2|E|$$

Proof: consider all incidence pairs (e,v)…

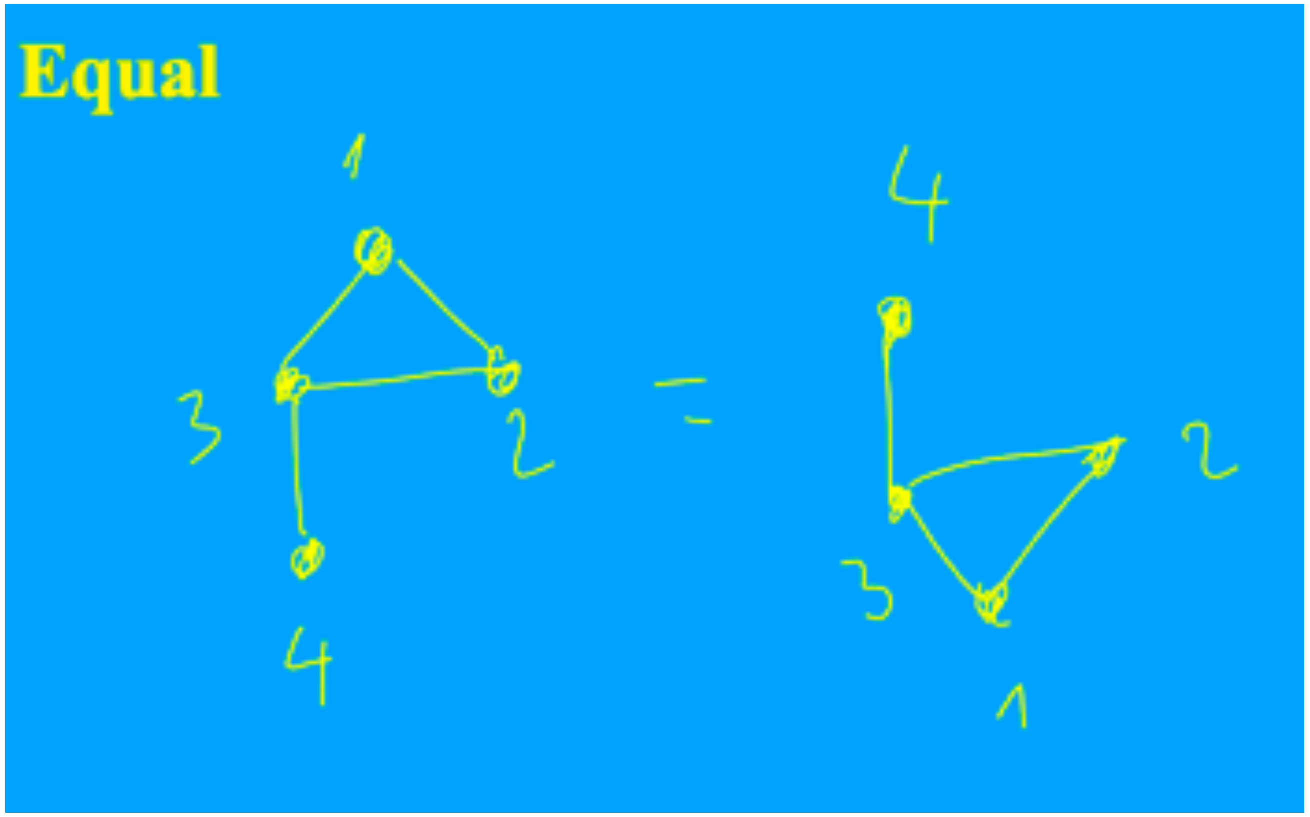
Handshaking lemma:
number of vertices with odd degree is even.

Proof: sum separately odd degree and even degree vertex degrees. Sum of odd degree must be even. But if a sum of odd numbers is even, there must be an even number of odd numbers…

Graphs

Representations, equality and isomorphism

Equal graphs: same set of vertices, and edges (can be *drawn* differently!)

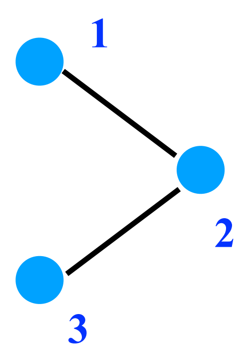


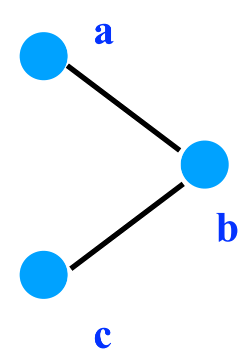
Graphs

Representations, equality and isomorphism

Equal graphs: same set of vertices, and edges (can be *drawn* differently!)

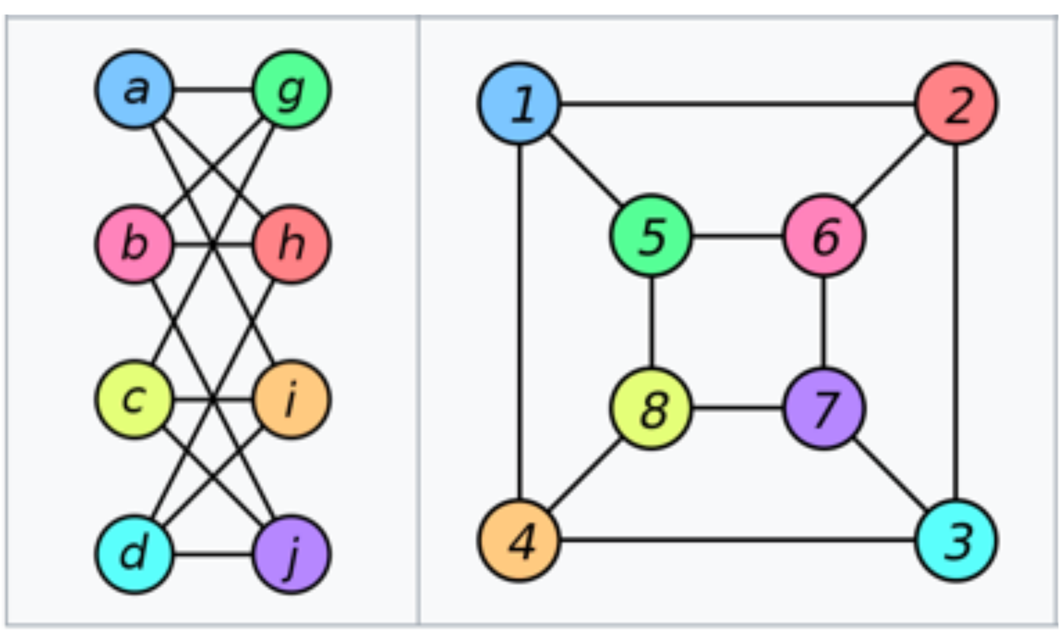
Isomorphic graphs: sets of vertices *of same size*, *connected in the same way*





G&H *Isomorphic: there exists a bijection f from V(G) to V(H), which* <u>preserves edges:</u>

{x,y} is an edge in G if and only if {f(x),f(y)} is amn edge in H



https://en.wikipedia.org/wiki/Graph_isomorphism

Representations, equality and isomorphism

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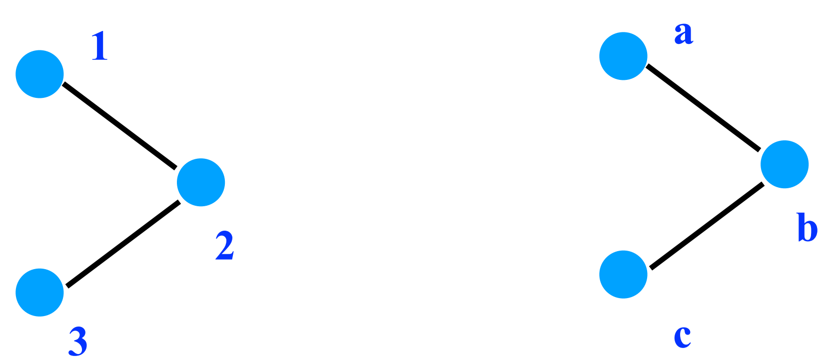
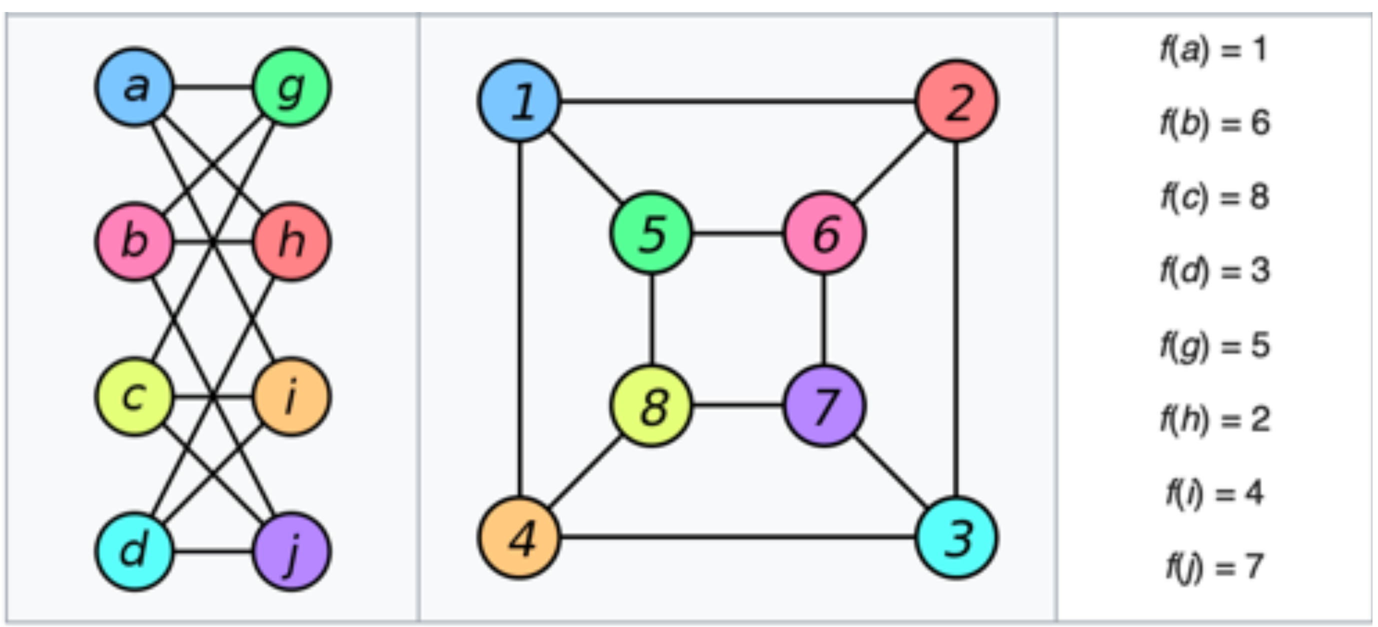


isomorphism



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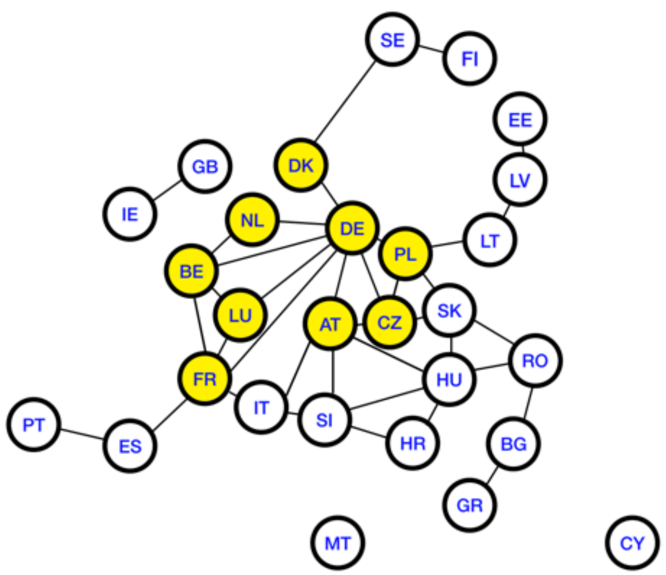
Graphs

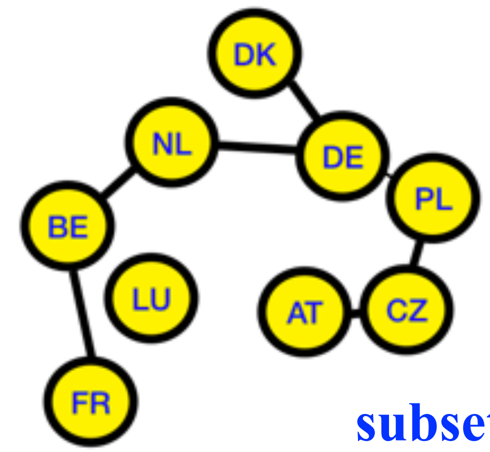
Subgraphs, and vertex induced subgraphs

(V,E)



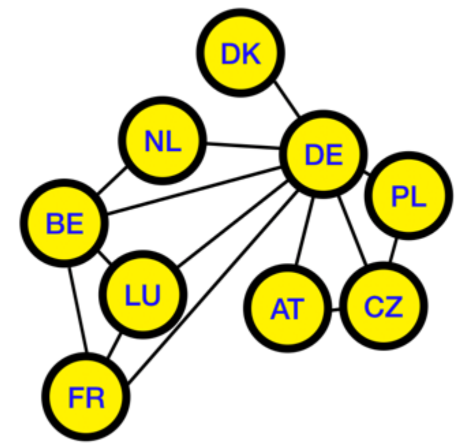
subgraphs

(V’,E’)



subset of vertices,
If {x,y} in E’ then {x,y} in E

vertex induced
subgraphs



subset of vertices,
all legal edges in E
(if x,y, in V” and
{x,y} in E then {x,y} in E”
and {x,y} in E” then {x,y}
in E)

(V”,E”)

Graphs

Edge and vertex removal: G-e = graph without edge e.
G-v = graph without vertex v and all edges incident with G

Walking the graph…

Paths

Path: a sequence $v_1, e_1, v_2, e_2 \ldots, v_n$**, with** $e_k = \{v_k, v_{k+1}\}$

Lenght of path = number of edges in path (n)

We say: path from v_1 **to** v_2

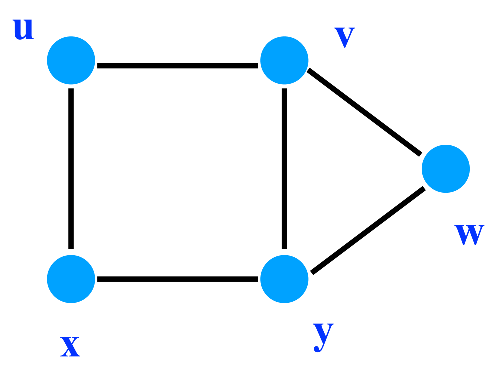
Closed path: $v_1 = v_n$

In graphs (not multigraphs), vertices suffice:

$v_1, e_1, v_2, e_2 \ldots, v_n \to (v_1, v_2, \ldots, v_n)$

$v_1 \to v_2 \to \ldots \to v_n$

$v_1 v_2 \ldots v_n$



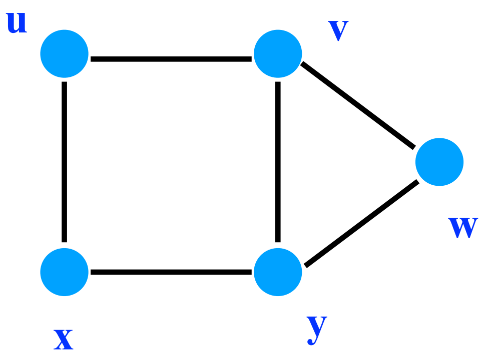
$u \to v \to w \to v \to u$

Paths: more concepts

simple path (*walk*): distinct vertices

trail: path with distinct edges

Closed path: $v_1 = v_n$



cycle: closed path of length > 2, all distinct vertices except first/last (essentially, closed simple path)

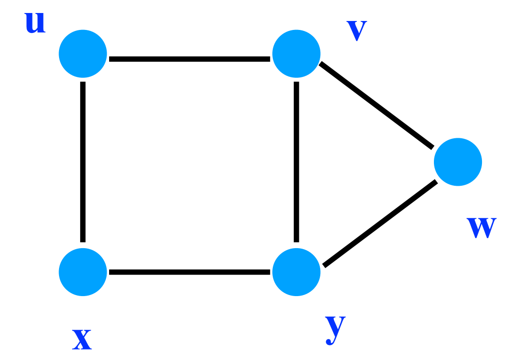
circuit: closed path, vertices may repeat, but edges cannot.

Same thing:

simple path:
Vertices may not repeat.
Edges may not repeat.

trail:
Vertices may repeat.
Edges cannot repeat. (a s.p. is a special trail.)

simple path:



$u \to v \to w \to y \to x$

cycle: closed path of length > 2
Vertices cannot repeat. Edges cannot repeat (Closed)

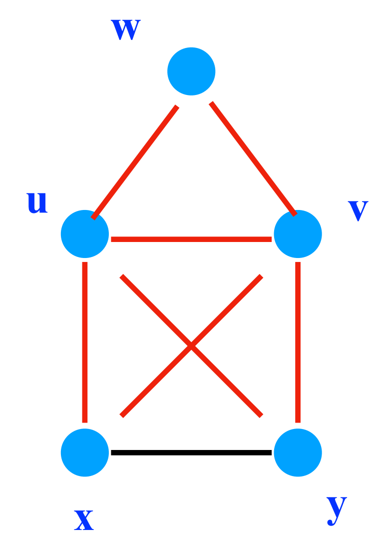
circuit:
(>2) Vertices may repeat. Edges cannot repeat (Closed)

Same thing:

simple path:
Vertices may not repeat.
Edges may not repeat.

trail:
Vertices may repeat.
Edges cannot repeat.

trail (& not simp. path)



$y \rightarrow v \rightarrow w \rightarrow u \rightarrow x \rightarrow v \rightarrow u \rightarrow y$

cycle: closed path of length > 2
Vertices cannot repeat. Edges cannot repeat (Closed)

circuit:
(>2) Vertices may repeat. Edges cannot repeat (Closed)

Same thing:

simple path:
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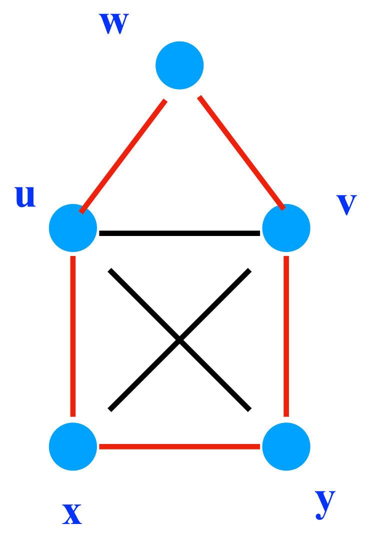
cycle:



$y \to v \to w \to u \to x \to y$

cycle: **closed path of length > 2**
Vertices cannot repeat. Edges cannot repeat (Closed)

circuit:
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Same thing:

simple path:
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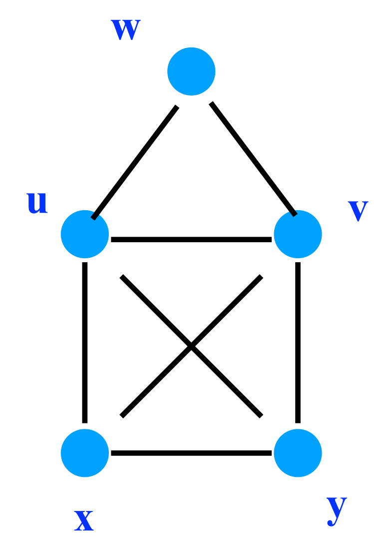
trail:
Vertices may repeat.
Edges cannot repeat.

cycle: closed path of length > 2
Vertices cannot repeat. Edges cannot repeat (Closed)

circuit:
(>2) Vertices may repeat. Edges cannot repeat (Closed)

circuit (& not a cycle)



$$y \to v \to w \to u \to x \to v \to u \to y \to x$$

Paths

Cycle and circuit not mutually exclusive.

Terminology *not* fully consistent between bodies of work. *Must be consistent within one work*
Check (and *give*) definitions

Useful math…

Every day you go from home to school, and your trail ("the way you take") is such that you pass by the same bakery twice. Is there a faster way to school?

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A theorem…

Graph theory: yes.

Theorem 8.2. (Schaum) If there is a path between vertices u and v, then there is a simple path between u and v.

since a simple path never sees the same vertex twice it is shorter by at least one.

Theorem 8.2. If there is a path between vertices *u* and *v,* then there is a simple path between *u* and *v.*

In graphs, you can avoid going through the same vertex twice…

Constructive proof!
(what are constructive proofs?)

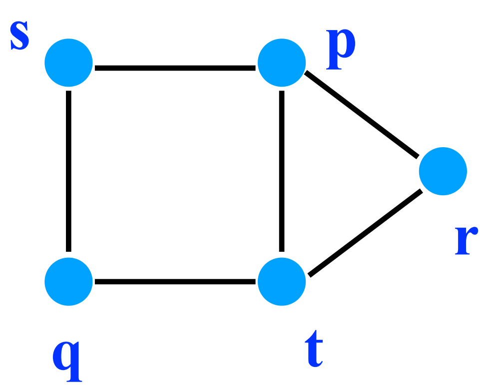
Take any path ... not simple? $\exists k$ s.t.

$$P_1 = u \to v_1 \to v_2 \to v_3 \to \dots \to v_k \underbrace{\to w_1 \to w_2 \cdots w_m \to}\, v_k \to v_{k+1} \dots \to v$$

$P_2 = u \to v_1 \cdots v_k \to v_{k+1} \cdots \to v$ is a path

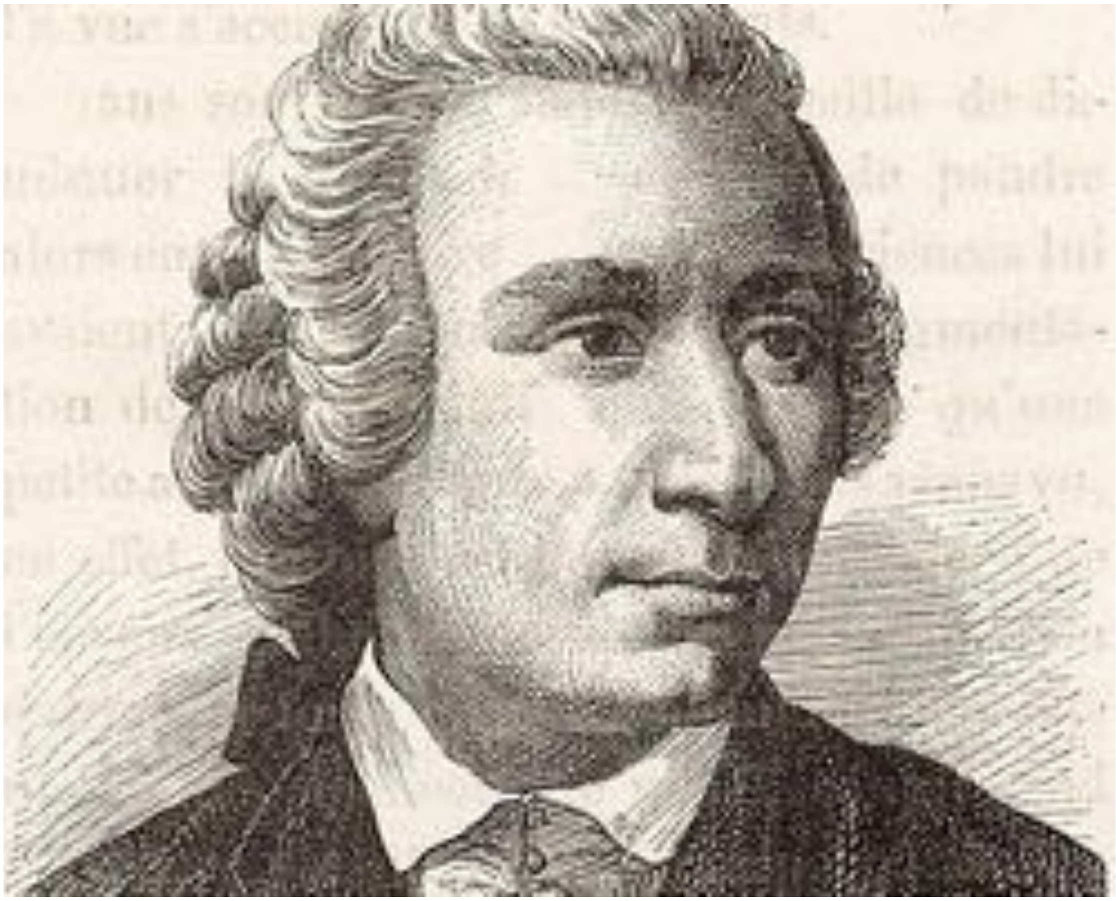
Repeat until simple... (each step decreases length ~ must terminate)

Paths: more examples



p→t→r→t→q	not simple,not trail (edge repeats)
p→r→t→p→s→q	not simple, trail (only vertex repeats)
p→r→t→q→s	simple trail
p	not cycle
p→r→p	not cycle (must be l>2)
r→t→p→s→q→t→r	not cycle, not circuit
r→t→q→s→p→r	cycle & circuit

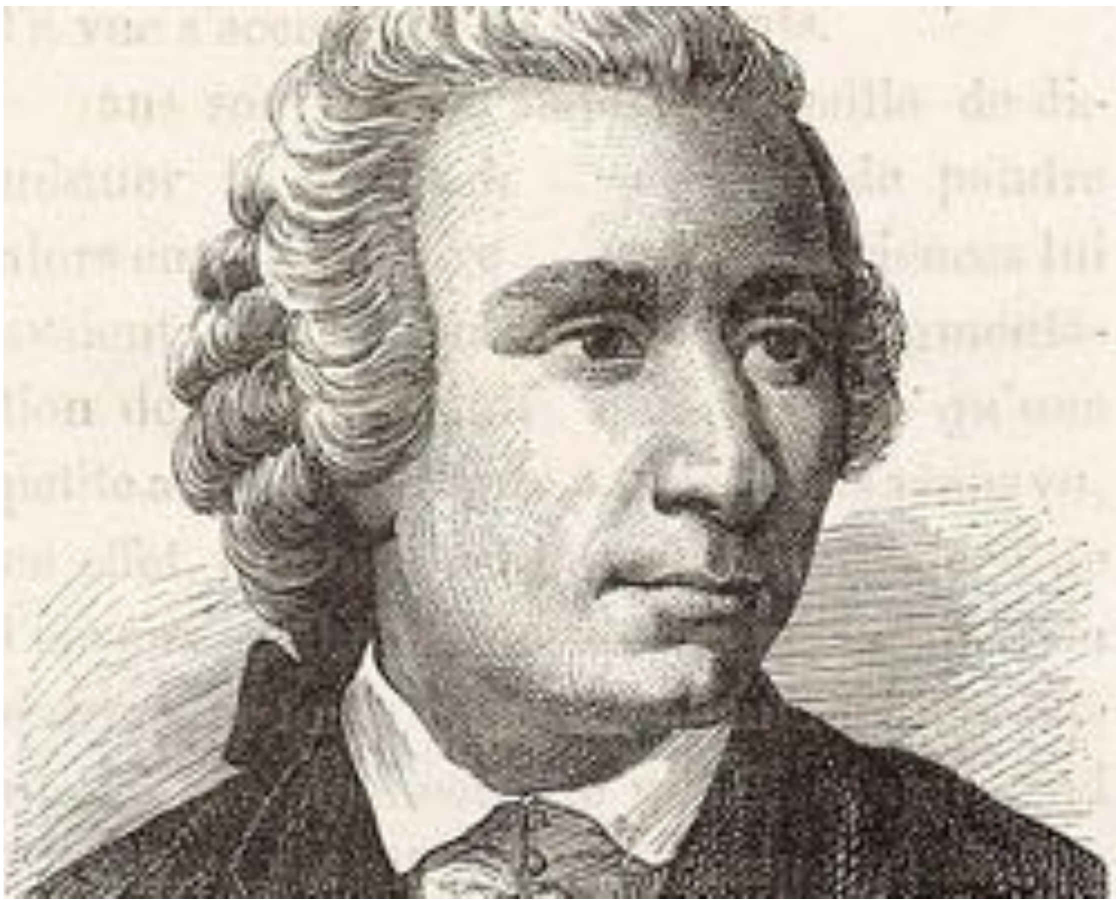
Math culture: Euler, Seven Bridges of Königsberg and beginnings of graph theory.

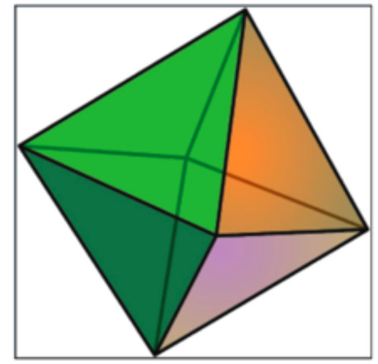


15 April 1707 – 18 September 1783

"Read Euler, read Euler, he is the master of us all." - Laplace

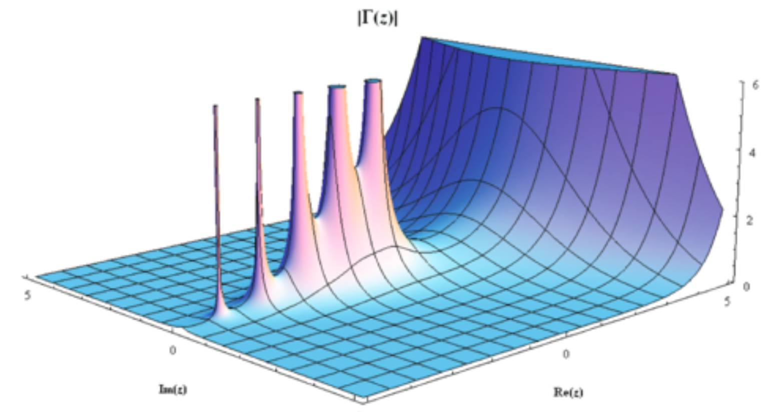
Polyhedral formula : $V - E + F = 2$





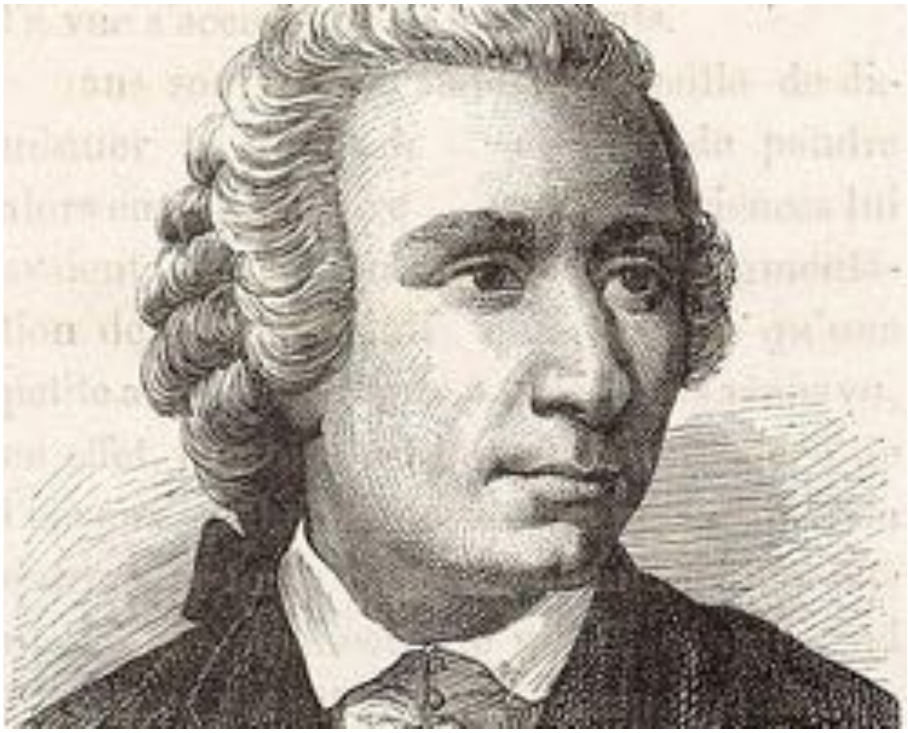
$$\prod_{p \in \mathcal{P}} \frac{1}{1 - 1/p^s} = \zeta(s), \quad s > 1,$$

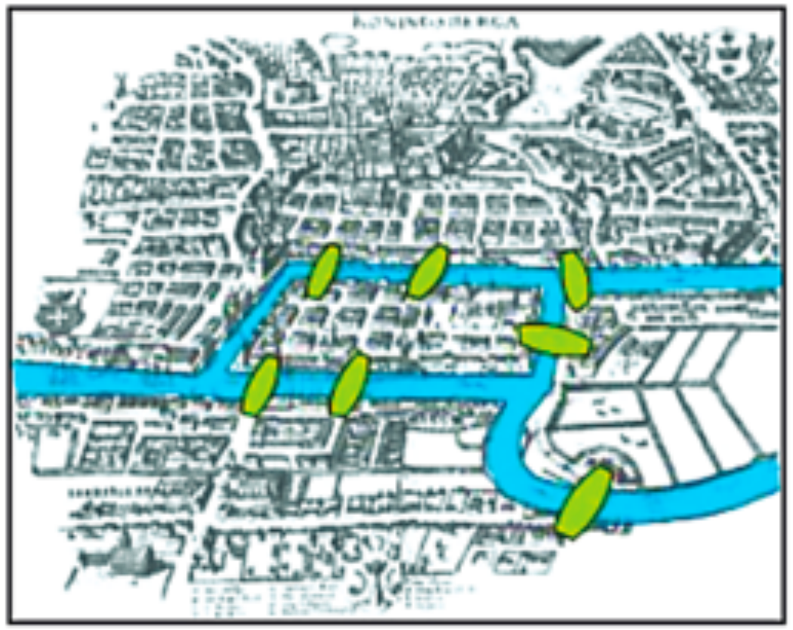
$$x! = \int_0^\infty \exp(-t) t^x \mathrm{d}t = \Gamma(x+1)$$

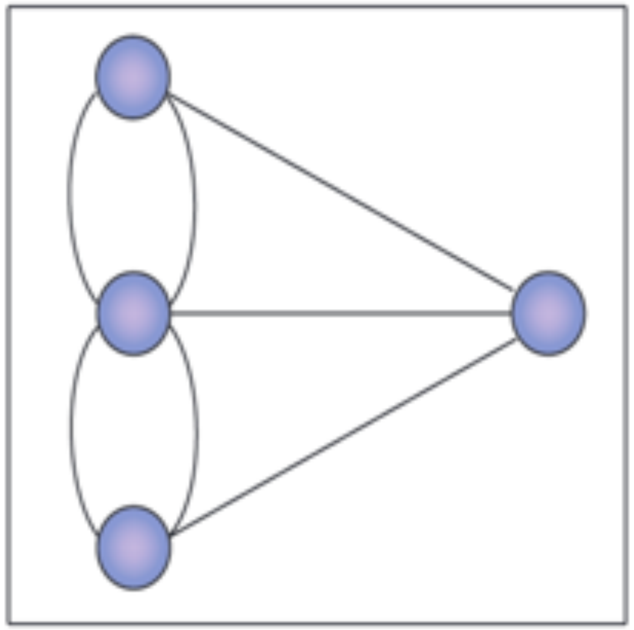


$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} \cdots + \frac{1}{n} - \ln n\right) = 0.57721\ldots.$$

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$





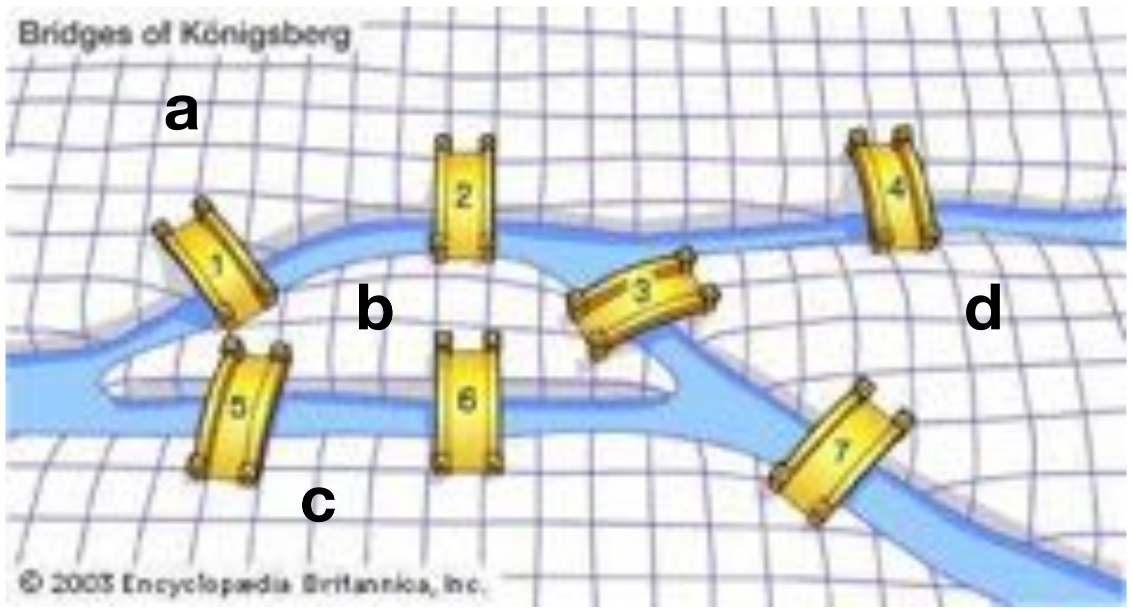


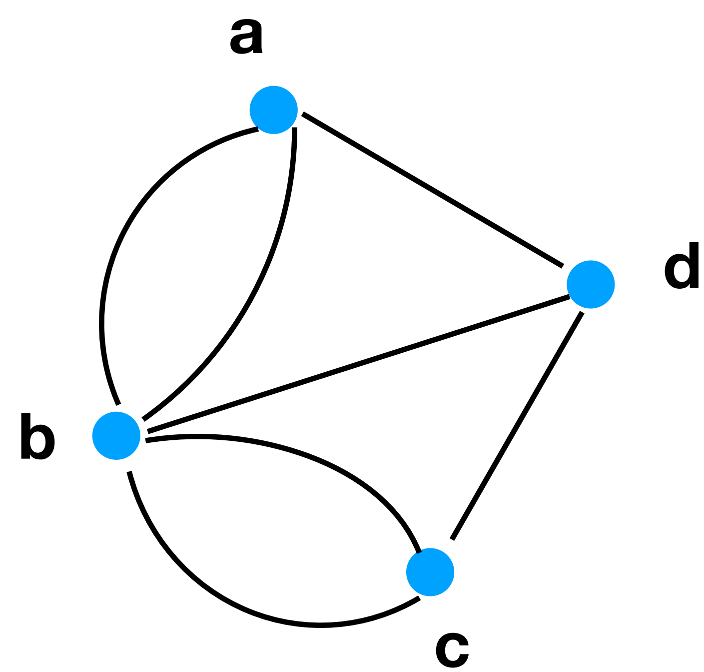
Can we cross that, and every other brige… but only once?

"This question is so banal, but seemed to me worthy of attention in that [neither] geometry, nor algebra, nor even the art of counting was sufficient to solve it."

Seven Bridges of Königsberg

A multigraph problem…





$$
\begin{array}{c c}
 & \begin{array}{cccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \end{array} \\
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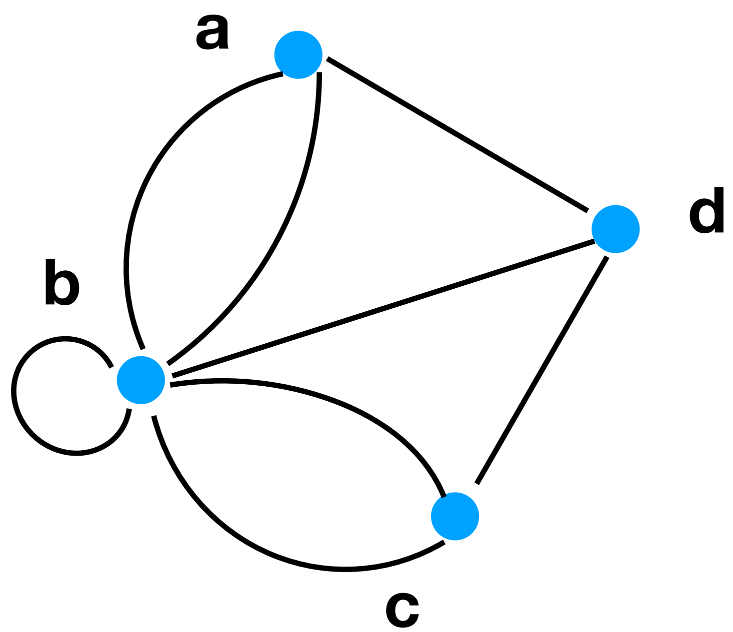
\end{array}
$$

Seven Bridges of Königsberg

A multigraph problems



Definition. A graph *G* is an ordered pair *(V,E)* where
- ***V = V(G) is the set of vertices***
- ***E=E(G) is the set of edges***

$$
\begin{array}{c|c}
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\end{array}
$$

Graphs do not suffice…

Seven Bridges of Königsberg

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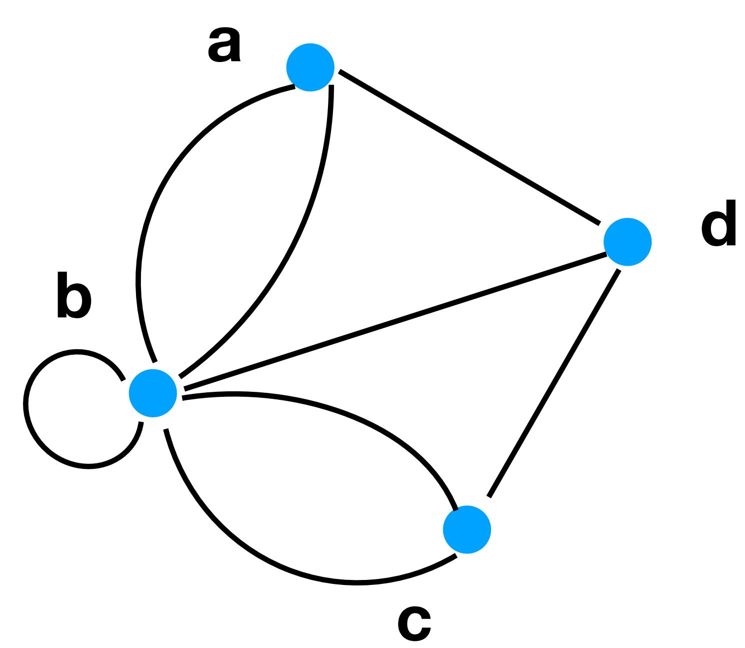


$$\begin{array}{c c} & \begin{array}{cccc} \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \end{array} \\ \begin{array}{c} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{array} & \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & \boxed{1} & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix} \end{array}$$

Euler proved impossibility, and tackled generalizations; the result is graph theory as we know it…solution soon!

Main contribution: observation that only "connectedness" (topology), rather than actual positions matter…

A moment's thought

concepts, "mathematical objects", formalization, modelling, abstraction…

Bridges in reality, bridges on a map

v.s.

Graph — abstract concept

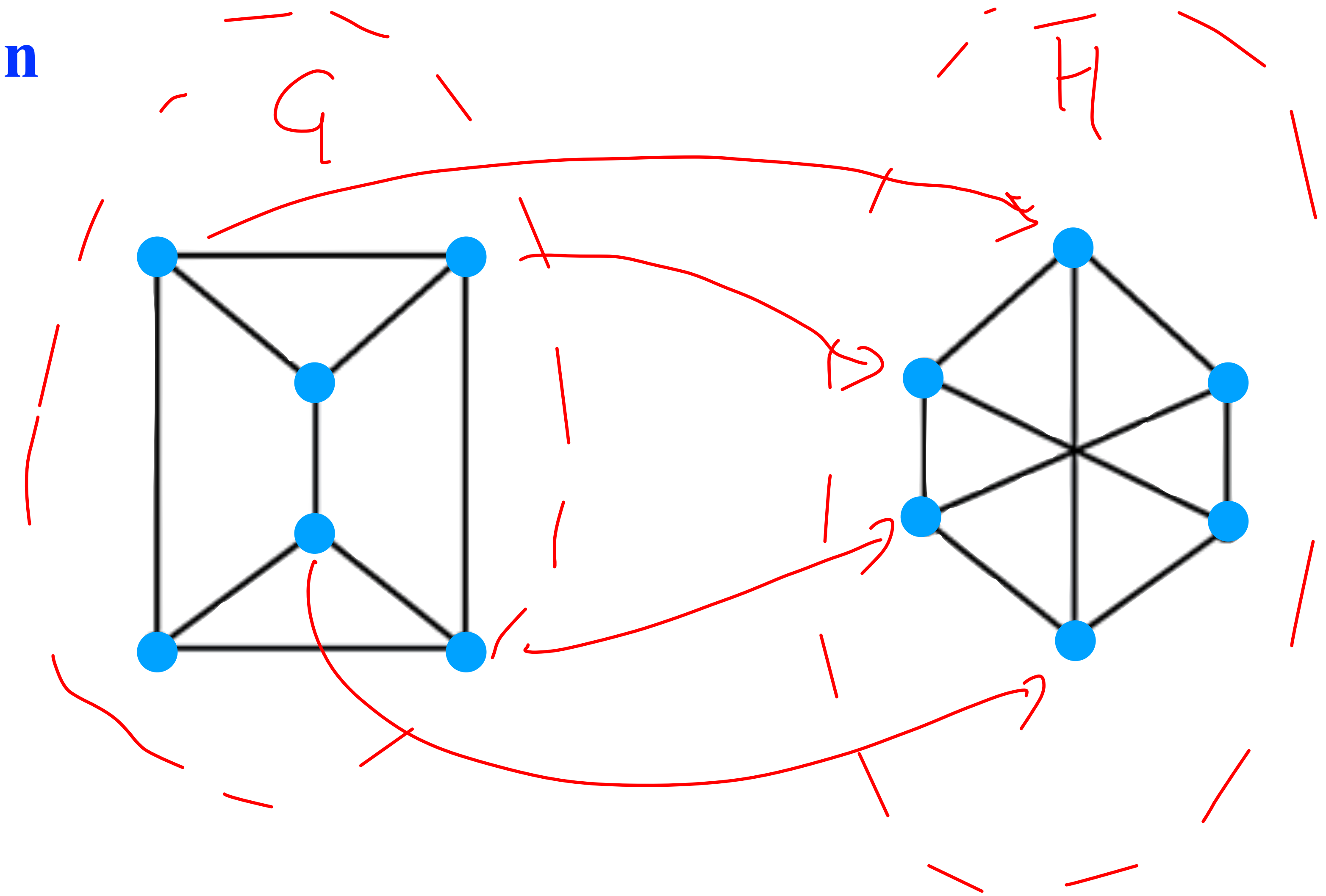
v.s.

graph (picture on paper),
set-theoretical notation for a graph,
adjacency matrix,
incidence matrix,

v.s.

graph as a binary relation
function (e.g. characteristic)

Digression



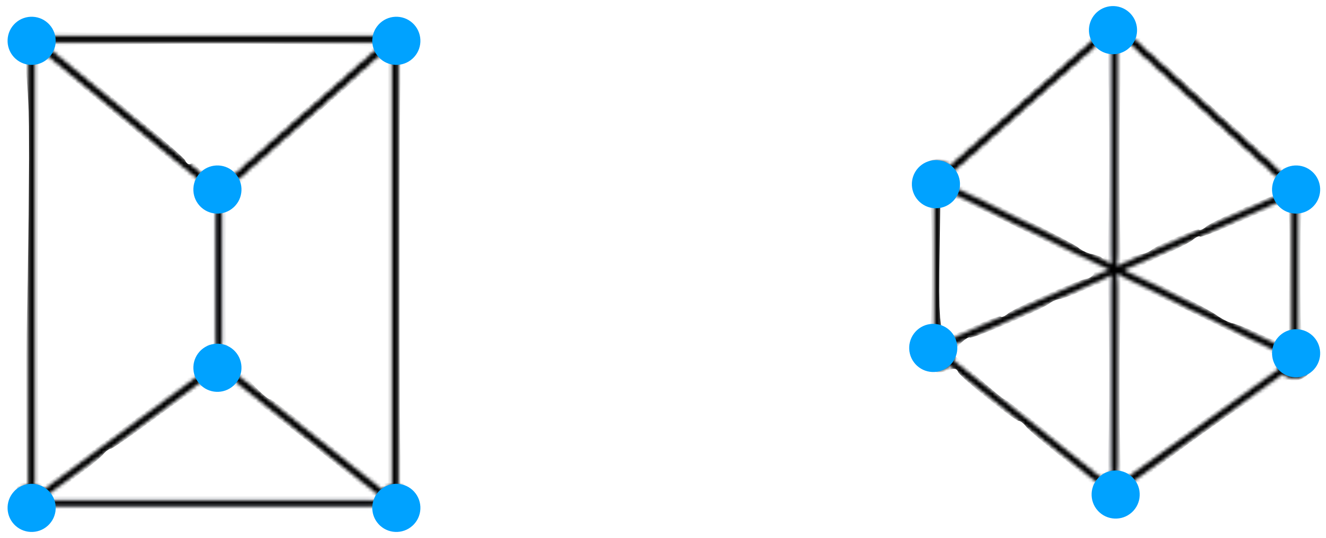
Isomorphic?

NO. G HAS TRIANGLES
H DOES NOT

Triangle = three pair-wise connected vertices,
an isomorphism would preserve this

Check main properties…

Digression



Showing *non-isomorphism can be hard*

Note: isomorphism is a relation on Graphs.
What kind?

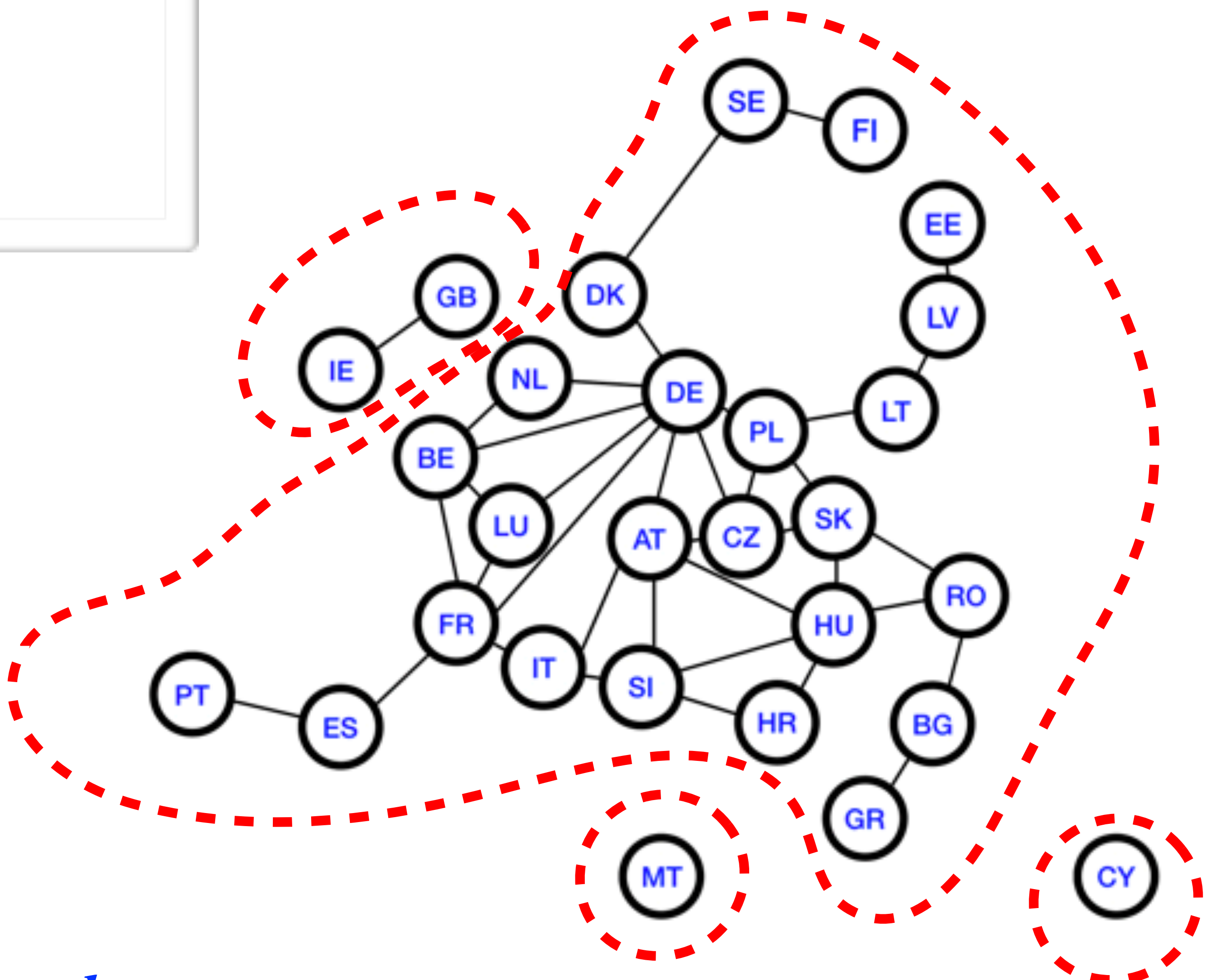
Graphs and connectedness

Definition. A graph is ***connected*** *if* ***every pair of vertices*** *is* ***connected by a path.***

$(\forall\ u, v \in V, \exists\ path(u, v))$

"there exists a path between" is a relation on $V \times V$

reflexive
symmetric
transitive



Connected components:
intuitive- "maximal" connected subgraphs

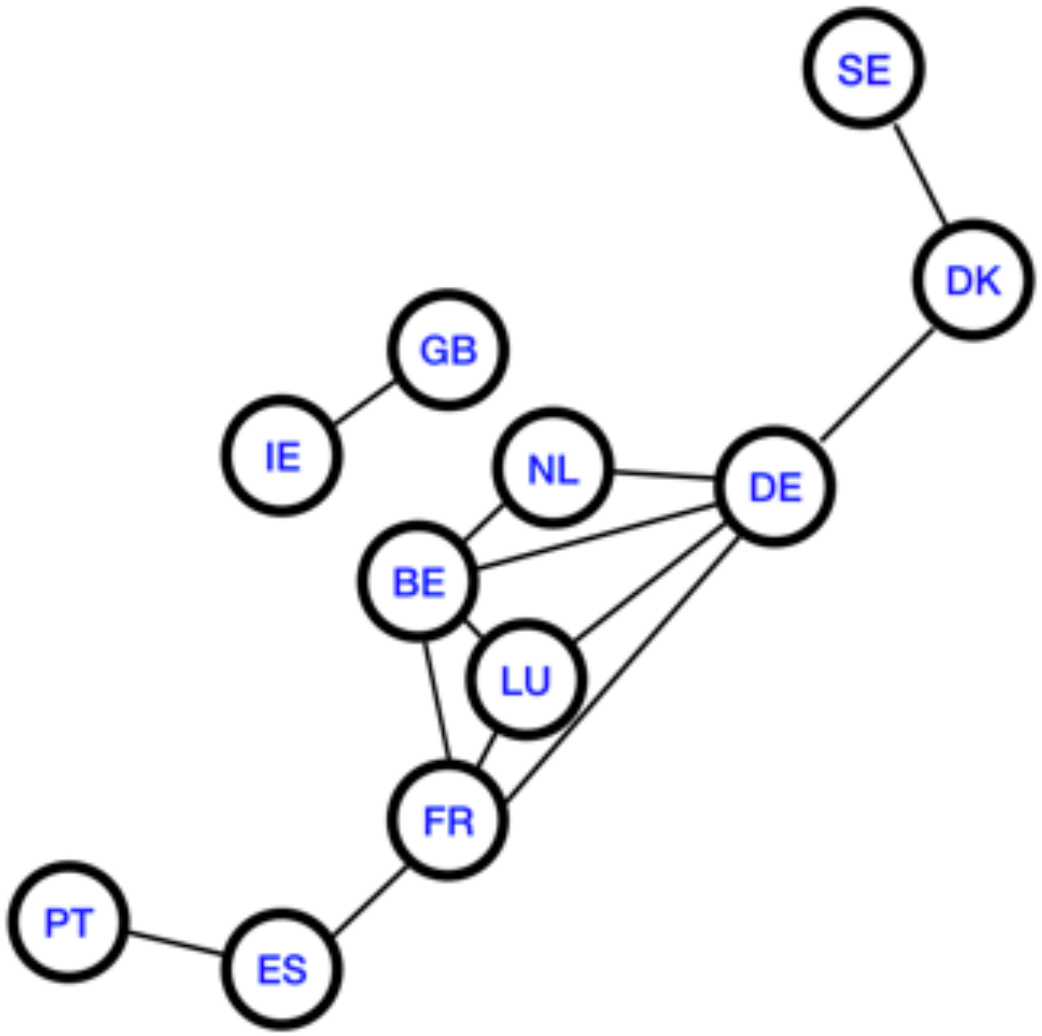
Graphs: bridge and cutpoint

Def. ***<u>Bridge</u> is an edge e whose removal increases the number of connected components.***

bridge, isthmus, cut-edge

Def. ***<u>Cut vertex</u> is a vetex v whose removal increases the number of connected components.***

cut point, cut vertex, articulation points



Graphs: bridge and cutpoint

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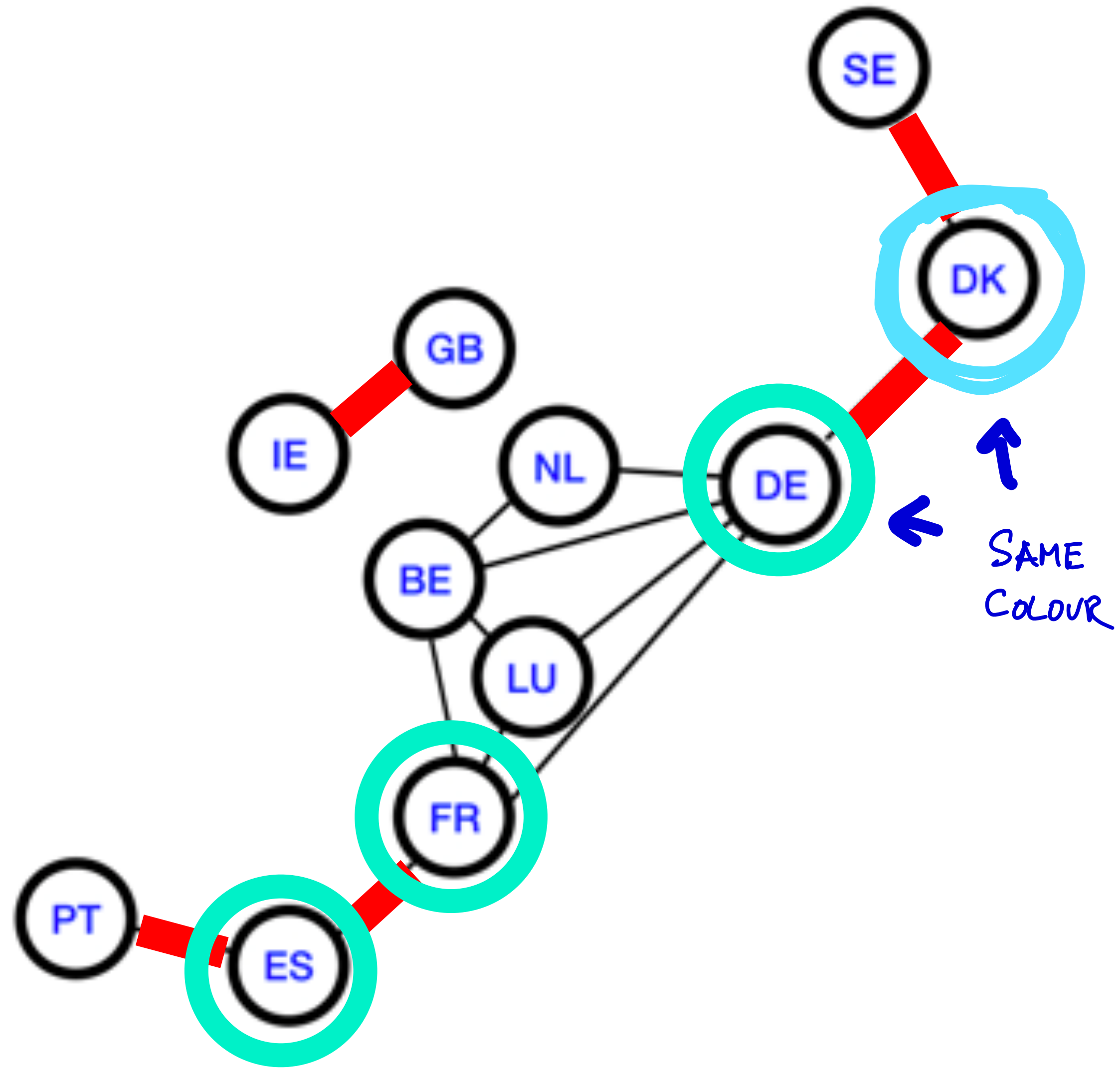
Def. Bridge is an edge e whose removal increases the number of connected components.

Def. Cut vertex is a vetex v whose removal increases the number of connected components.

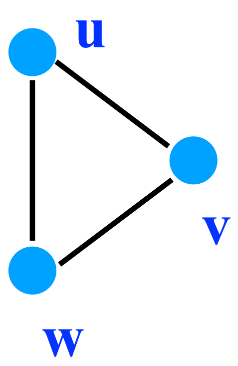


Graphs: distance

Def. The <u>distance</u> $d(u, v)$ between vertices u,v in graph G is the lenght of the <u>shortest path</u> between u and v in G.

Comment. If there is no path (u and v are in distinct connected components) we say the distance is infinite, $d(u, v) = \infty$

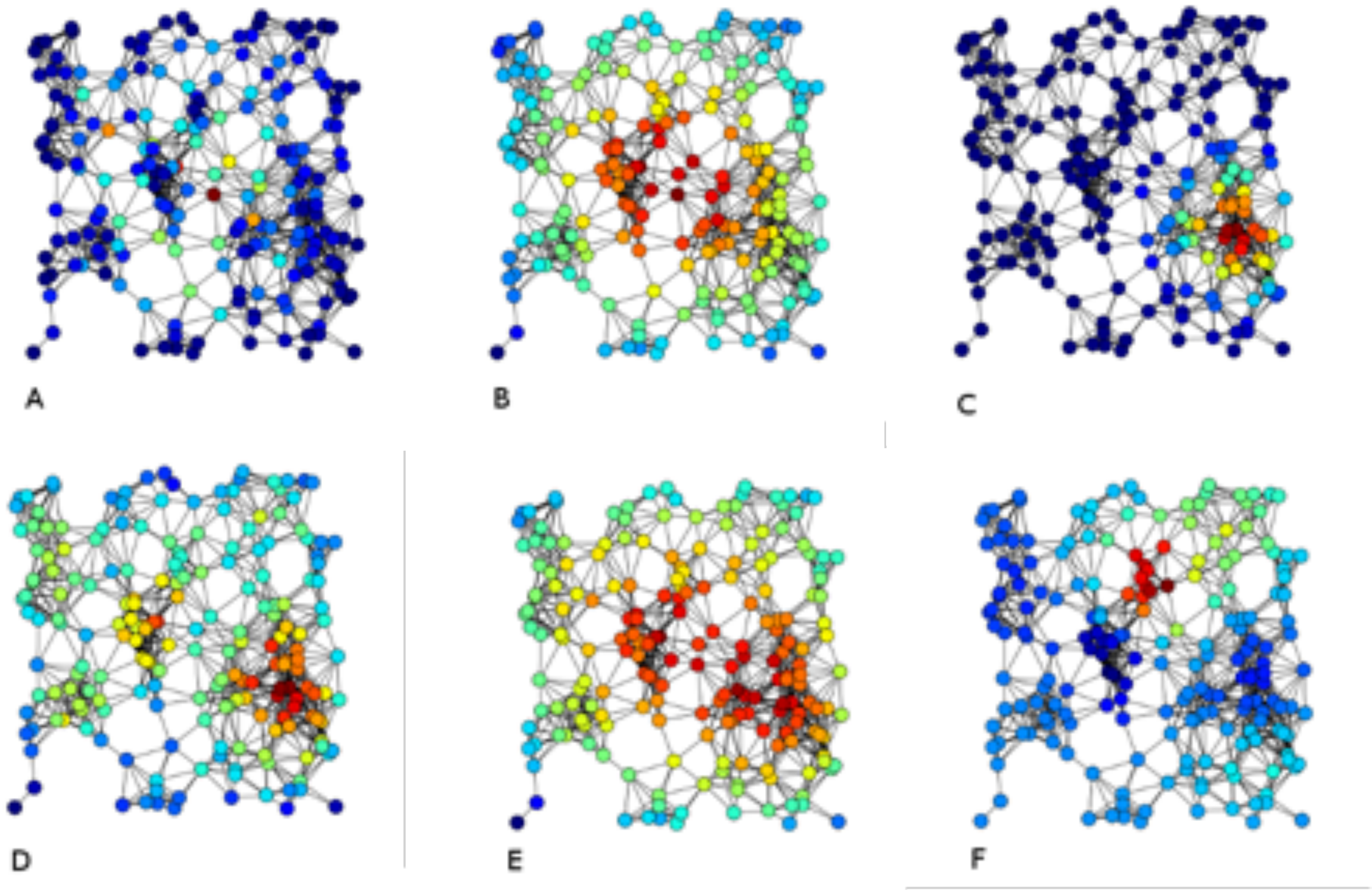
Theorem. Vertex distance obeys the triangle inequality:
For all $u, v, w \in V,\ d(u, v) + d(v, w) \geq d(v, w)$



Proof. Cannot be more. Can be less.

Def. The <u>diameter of the graph G</u> , denoted <u>diam(G)</u>, is the distance between two maximally distant nodes which are connected by a path

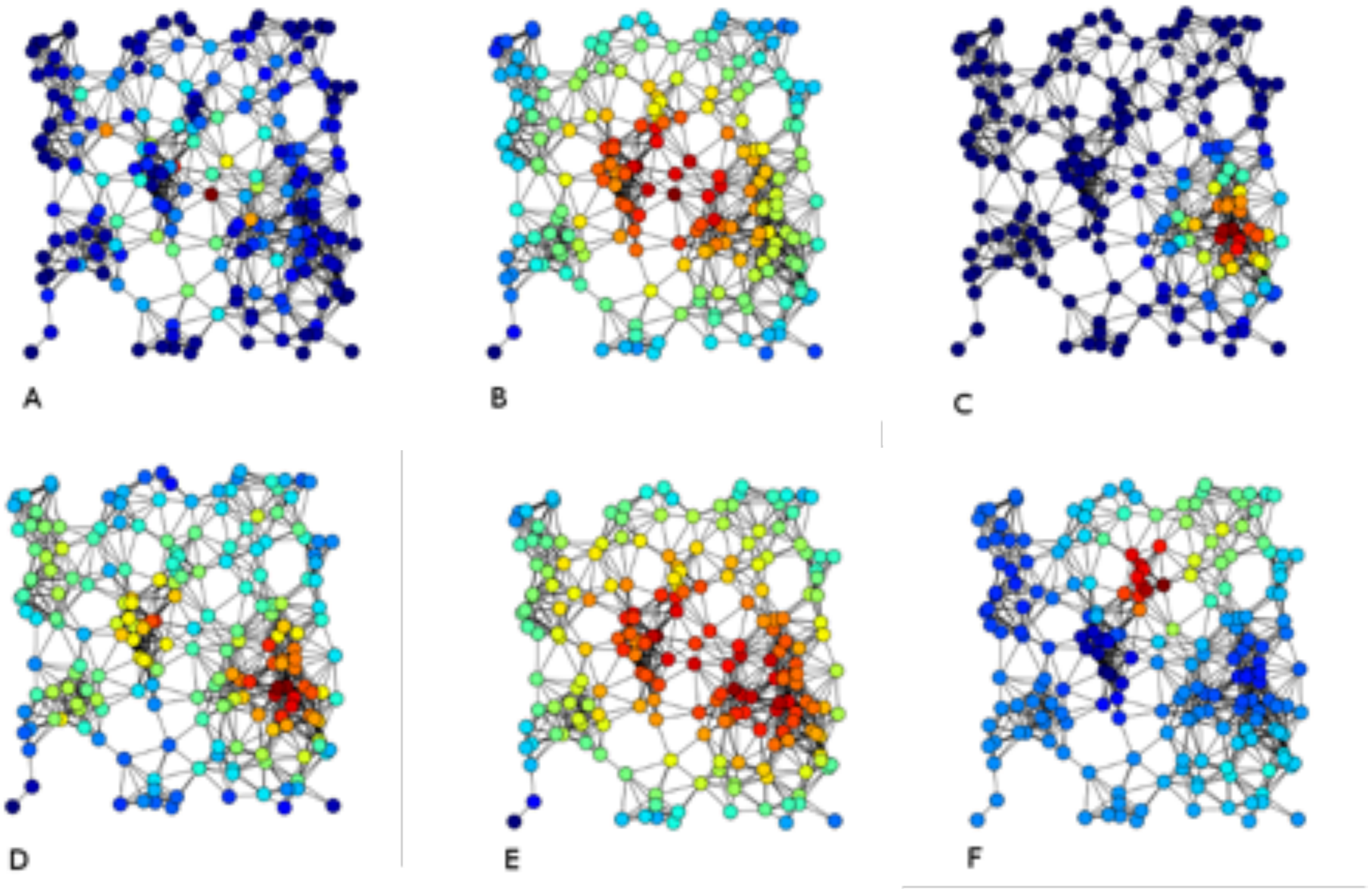
Digression: graph centrality



A) Betweenness centrality, B) Closeness centrality, C) Eigenvector centrality,
D) Degree centrality, E) Harmonic Centrality and F) Katz centrality

https://en.wikipedia.org/wiki/Centrality

Digression: graph centrality

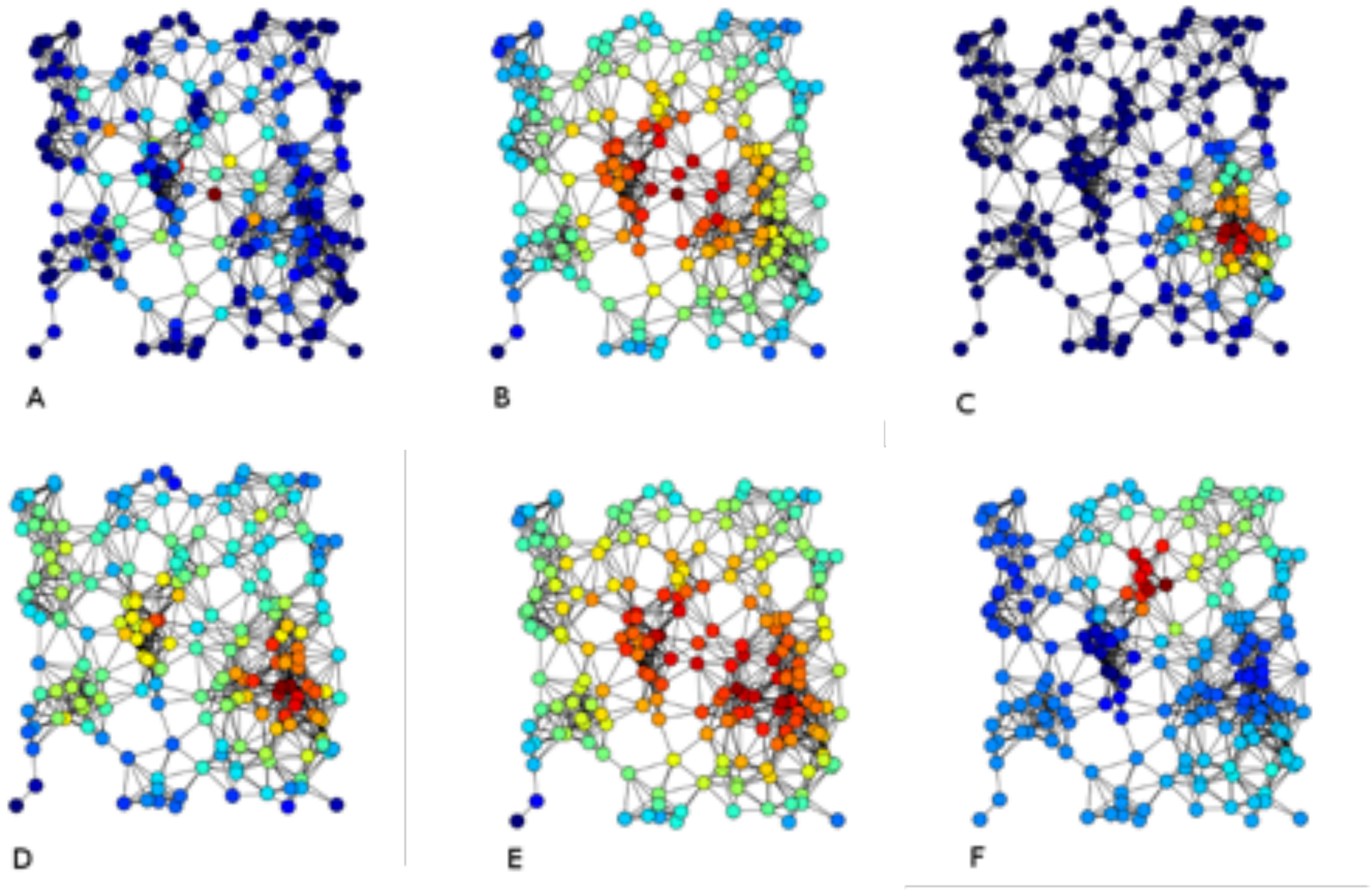


A) Betweenness centrality, B) Closeness centrality, C) Eigenvector centrality,
D) Degree centrality, E) Harmonic Centrality and F) Katz centrality

https://en.wikipedia.org/wiki/Centrality

Social network analysis: how "central" nodes are in a graph.
measure of importance.
That can be, for example, a measure that indicates
(a) how many paths run through that node,
(b) the total / average distance to the other nodes,
(c) the degree of the node

The red side of spectrum indicates “more central”



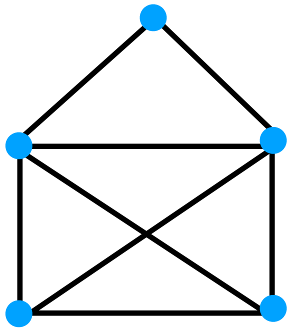
http://liacs.leidenuniv.nl/~takesfw/SNACS/

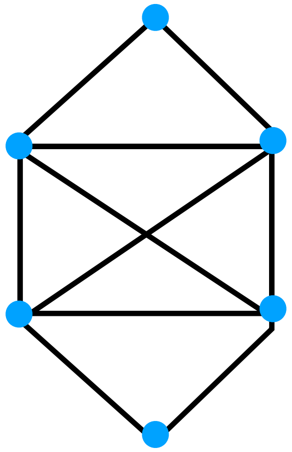
Traversible and Eulerian Graphs

Def. Euler circuit: closed trail which uses each edge exactly once.

"Can you draw the following without crossing any edge twice, ending where you started from?"

Aslo Seven bridges on multigraph!



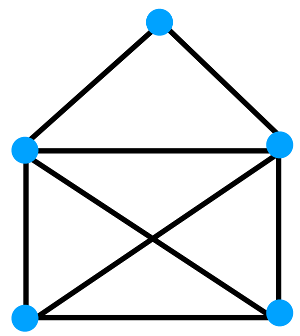


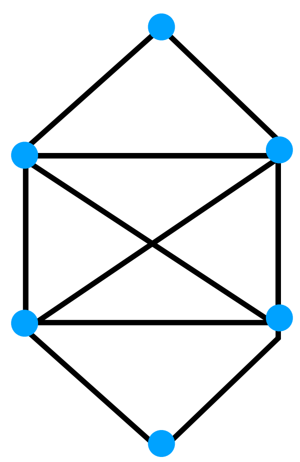
Traversible and Eulerian Graphs

Def. Euler trail: a trail which uses each edge exactly once.

Graphs with an Euler (Eulerian) trail are called Traversible

Theorem 8.3 (Euler): a finite connected graph has an Euler circuit if and only if evey vertex has an even degree.





Traversible and Eulerian Graphs

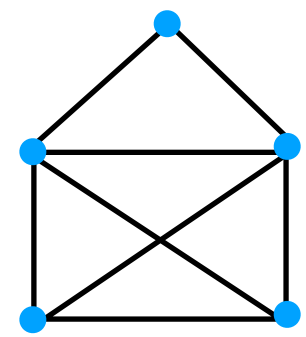
Def. Euler trail: a trail which uses each edge exactly once.

"Can you draw the following without crossing any edge twice?"

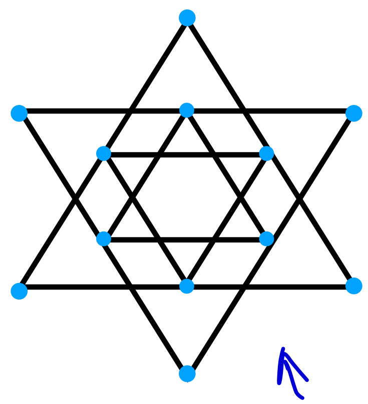
Corollary. An undirected connected graph has an Eulerian trail if and only if exactly zero or two vertices have odd degree.

Euler trail $\Leftrightarrow$ $\emptyset$ or 2 odd degree

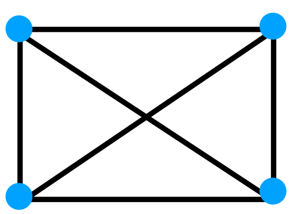
Euler circuit $\Leftrightarrow$ all even degree



E. trail
not
E. circuit



E. circuit!!



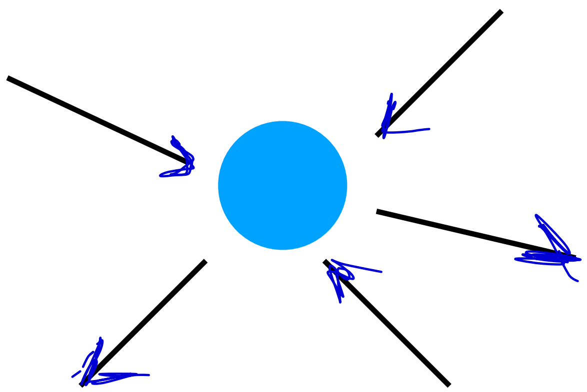
Nope!

Traversible and Eulerian Graphs

Corollary. An undirected connected graph has an Eulerian trail if and only if exactly zero or two vertices have odd degree.

Almost a proof:

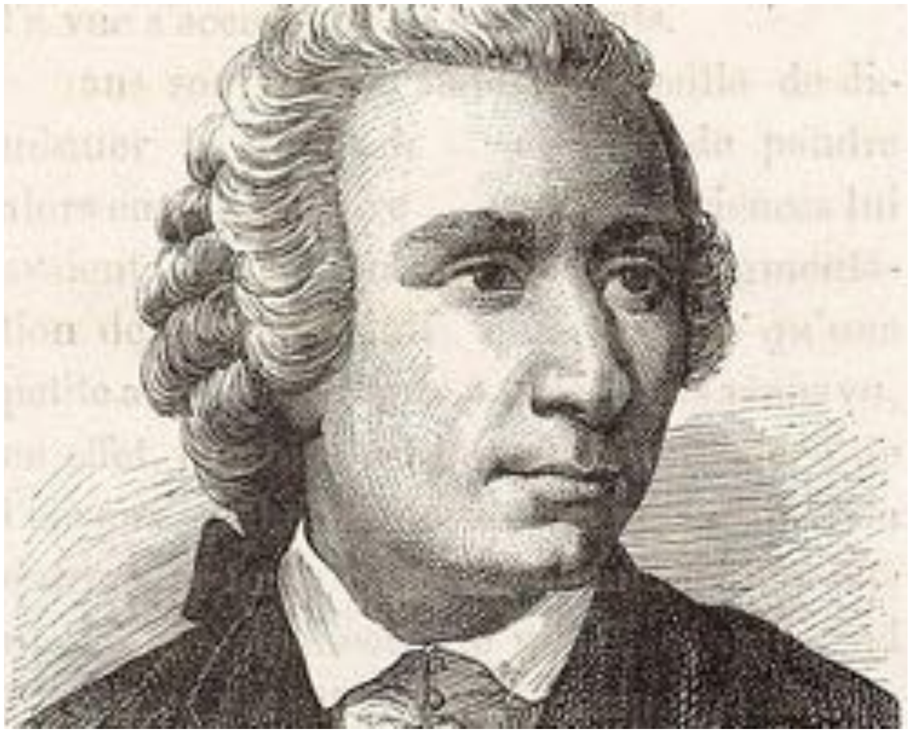
Imagine an "inner" vertex of the trail

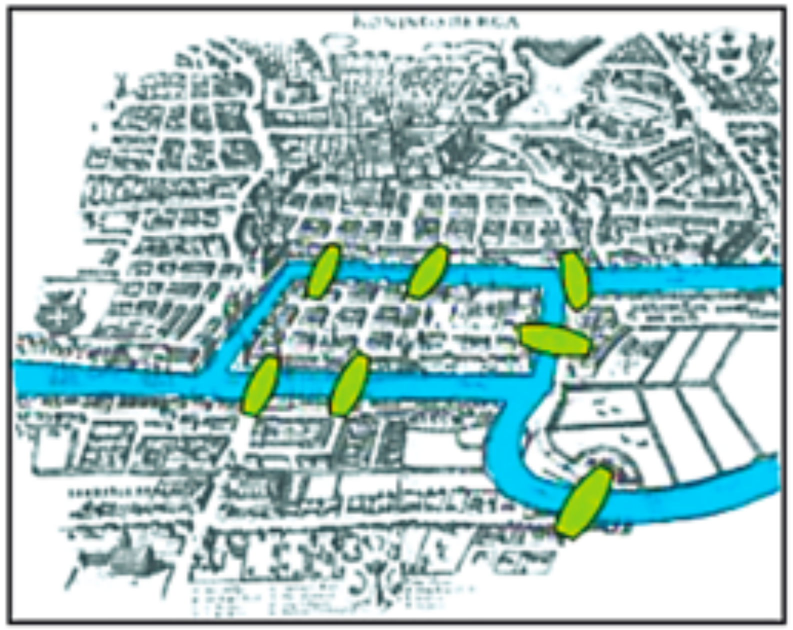


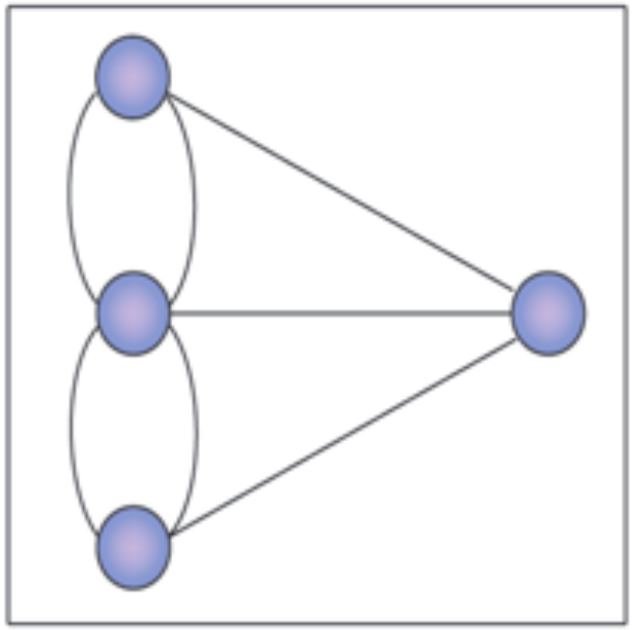
if degree odd, you are *trapped* = proof of necessity of 0/2 vertex criterion.

Traversible and Eulerian Graphs

Corollary. An undirected connected graph has an Eulerian trail if and only if exactly zero or two vertices have odd degree.

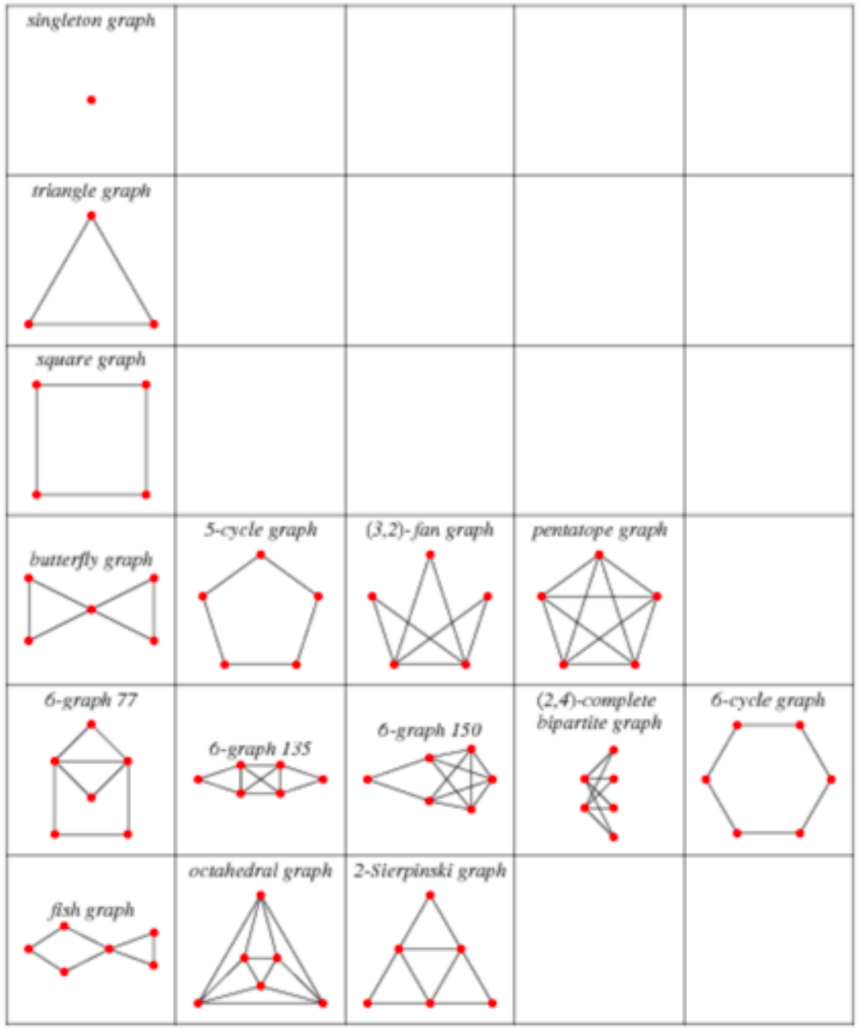






WE STOPPED HERE...

Eulerian Graphs

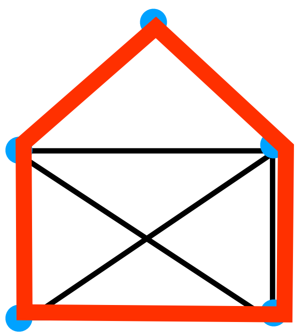


The numbers of (connected) Eulerian graphs with n nodes are 1, 0, 1, 1, 4, 8, 37, 184, 1782, ... OEIS A003049

http://mathworld.wolfram.com/EulerianGraph.html

Hamilton cycles

Def. Hamilton cycle: a closed path which uses each vertex exactly once.
(closed = starts where it ends)



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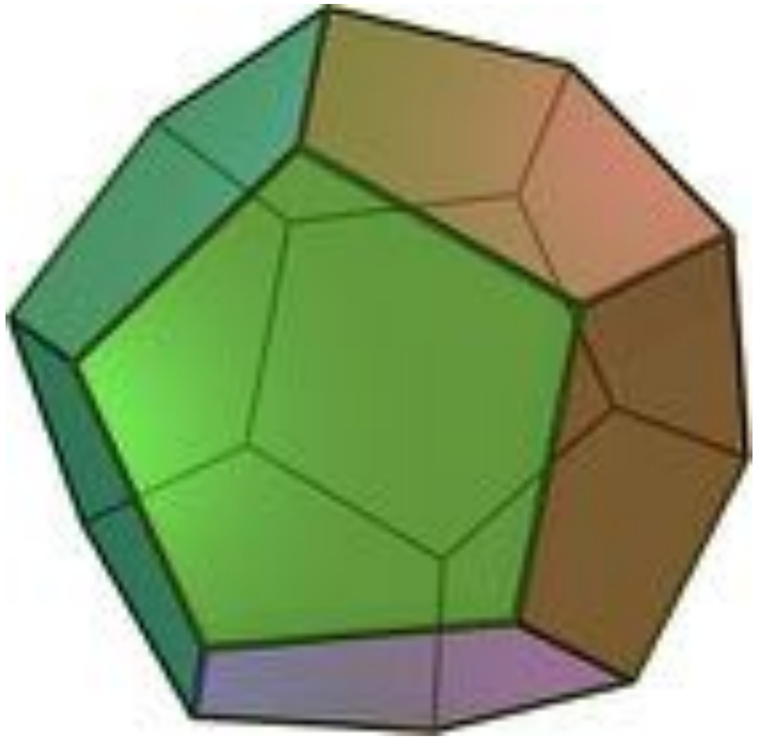

https://en.wikipedia.org/wiki/Regular_dodecahedron

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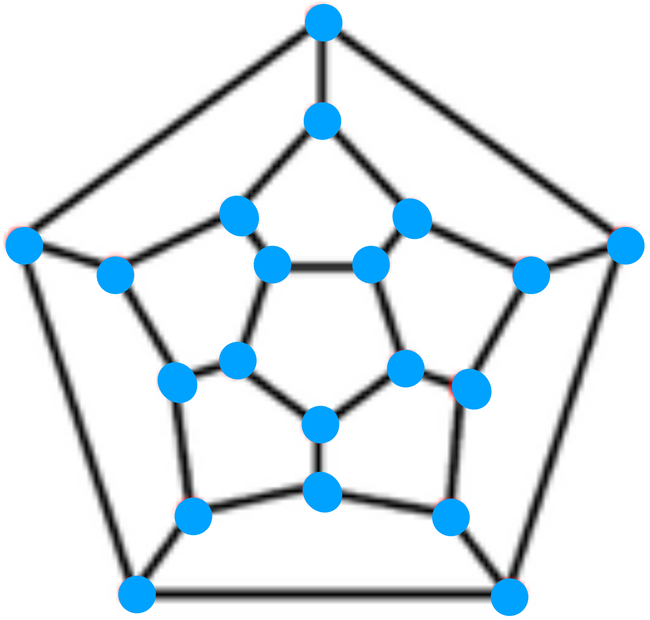

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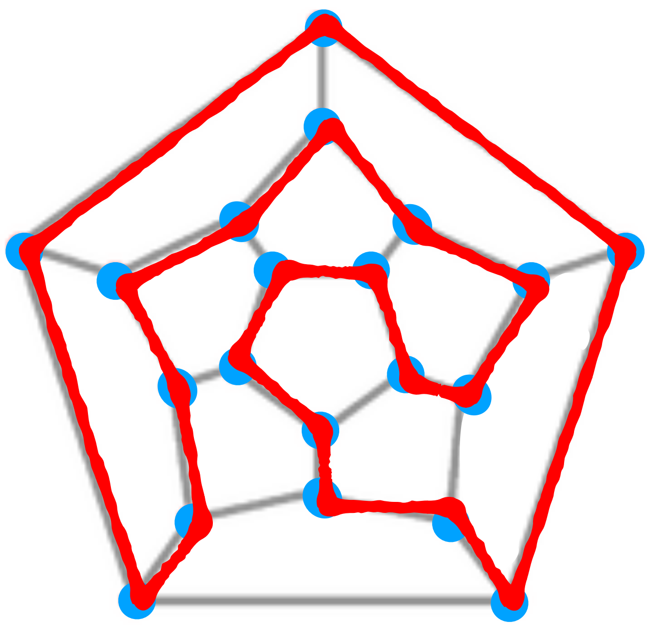

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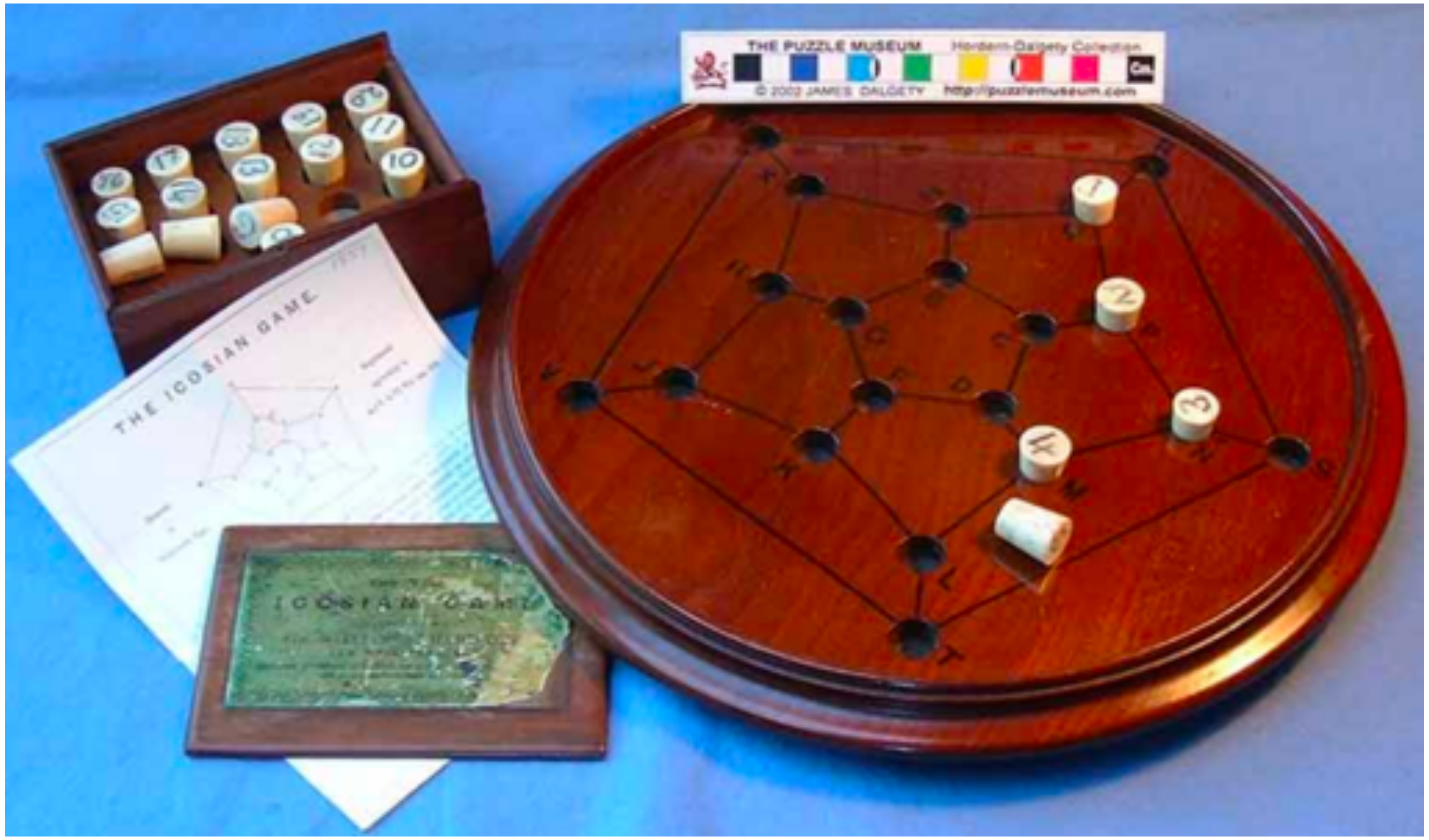
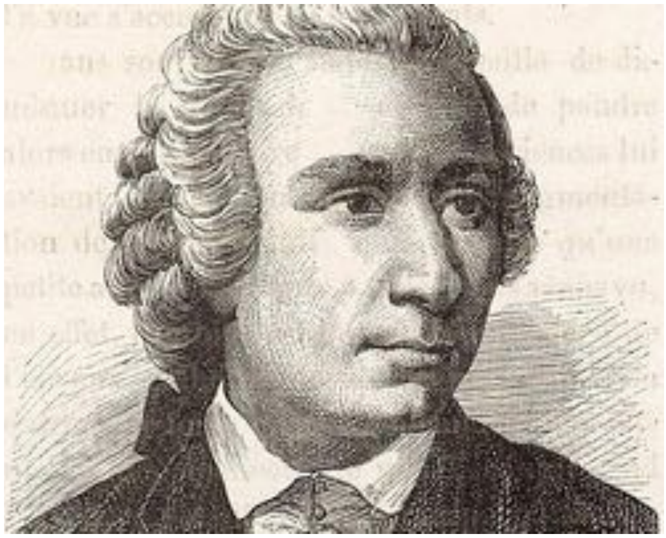


https://www.puzzlemuseum.com/month/picm02/200207icosian.htm

Euler v.s. Hamilton



v

Leonhard Euler

Bridges of Königsberg

Closed, each line once

Simple characterization

Easy to detect

William Rowan Hamilton

Icosian game

Closed, each vertex once

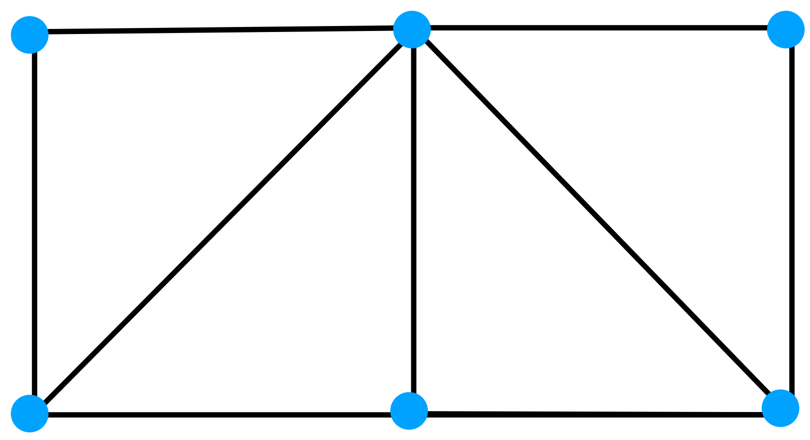
“Travelling salesperson problem”

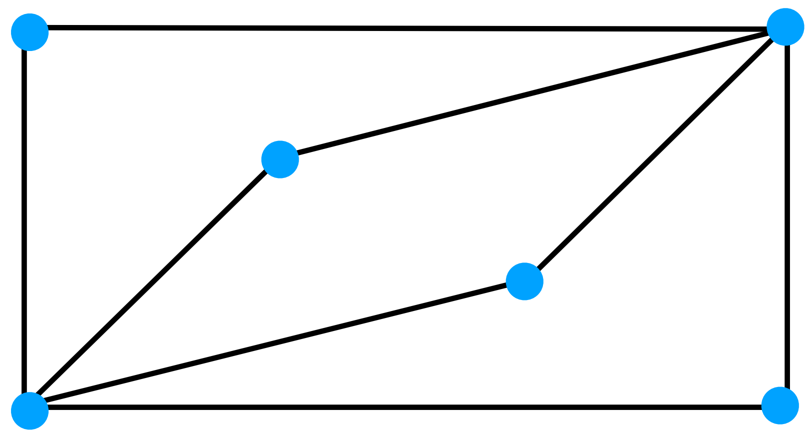
Ore (1960). A graph with n-vertices (n >3) is Hamiltonian if,
for each pair of non-adjacent vertices, the sum of their degrees
is n or greater.

If but not if and only if….NP-complete…

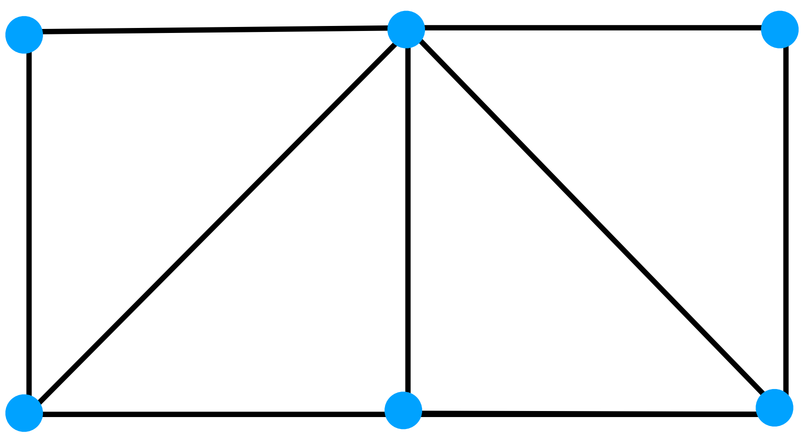
https://en.wikipedia.org/wiki/William_Rowan_Hamilton

Euler v.s. Hamilton (Schaum)

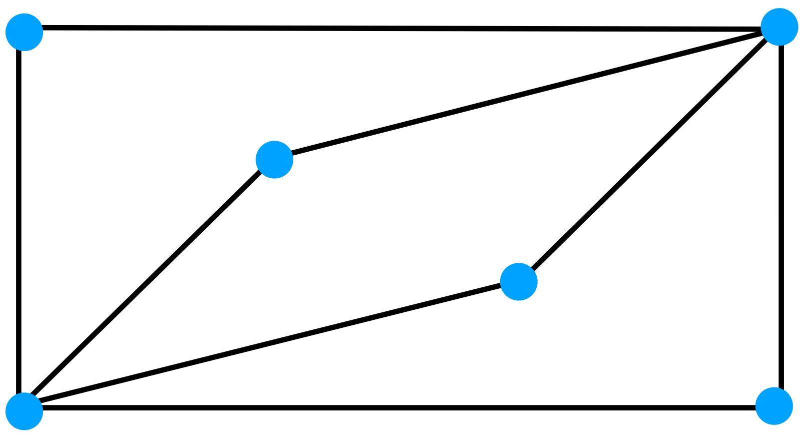




Euler v.s. Hamilton



Hamiltonian and non-Eulerian



Eulerian and non-Hamiltonian

"Note that an Eulerian circuit traverses every edge exactly once, but may repeat vertices, while a Hamiltonian circuit visits each vertex exactly once but may repeat edges." Schaum p.161

Euler v.s. Hamilton

TYPOS and mistakes HAPPEN!

Schaum p.162 Theorem 8.5 (Dirac, 1952):
Let G be a connected graph with n vertices.
Then G is Hamiltonian if $n > 3$ and $n/2 \leq \deg(v)$ for each vertex v in G.

No need to know this theorem (for this course).
It is an illustration of the type of propositions
that have been obtained to encompass the concept of Hamiltonian.

Labeled graphs & weights

Labeled graph: information on the edges
Weighted graph: values (numbers) on the edges

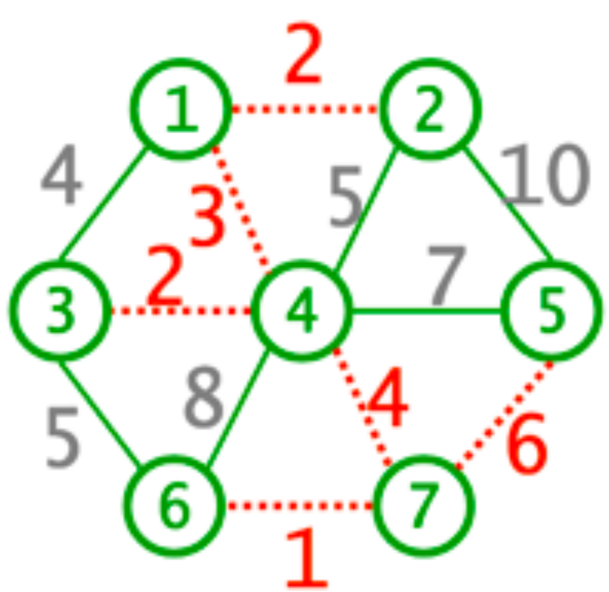
$w : E \to$ **Labels**; **or** $w : E \to \mathbb{R}$; $\quad w(e)$

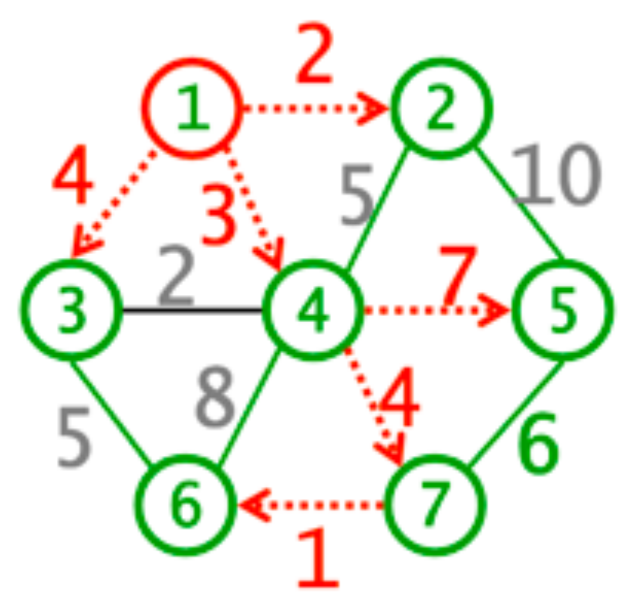
Can mean: capacity (conductance, diameter), cost (time, distance)

- **weight of a path:**
- **minimal spanning tree:** ***Prim's algorithm, Kruskal's algorithm***
- **shortest ("cheapest") paths;** ***Dijksta's algorithm***

IN OTHER COURSES

MINIMUM SPANNING TREE





Dijkstra

Labeled graphs & weights

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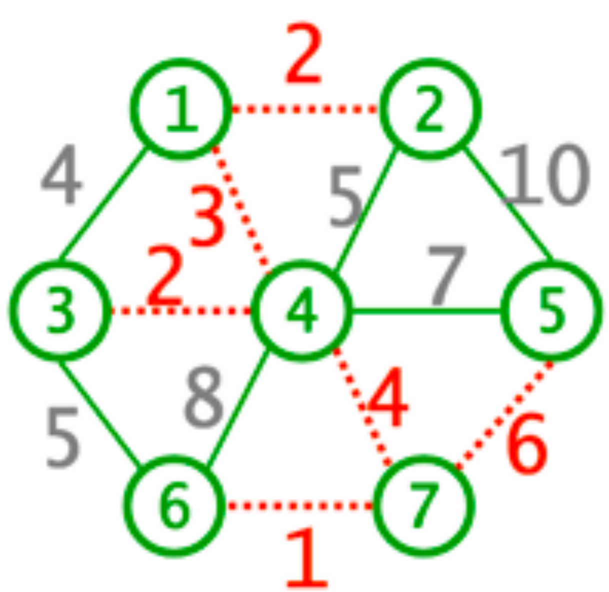
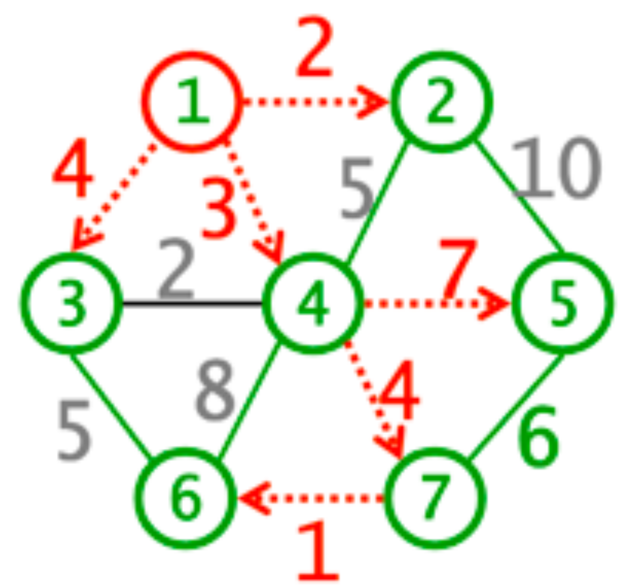
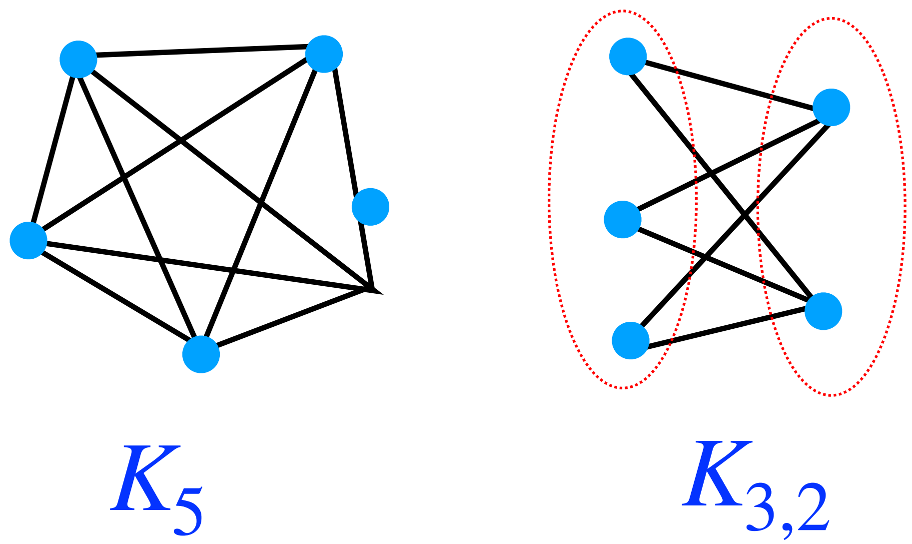


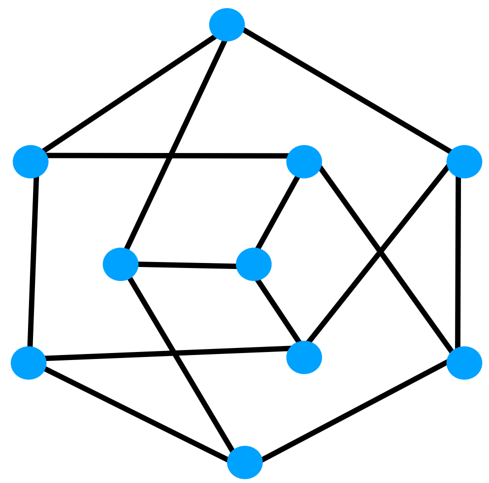


Special graphs

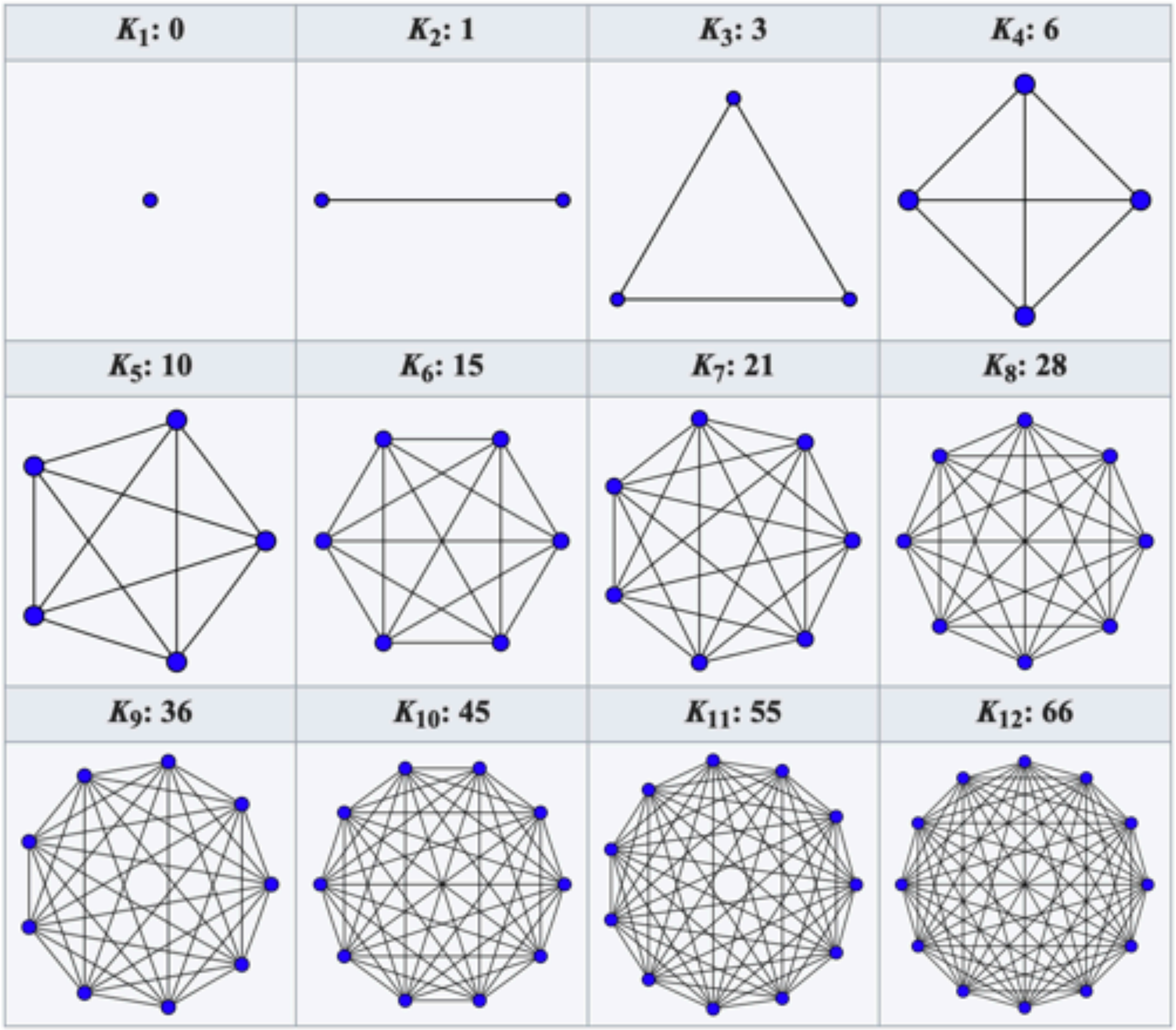
complete graph K_n
bipartite graph
complete bipartite graph $K_{m,n}$ (or $K_{m\times n}$)



***k*-regular graph: all vertices degree k**



Complete graphs

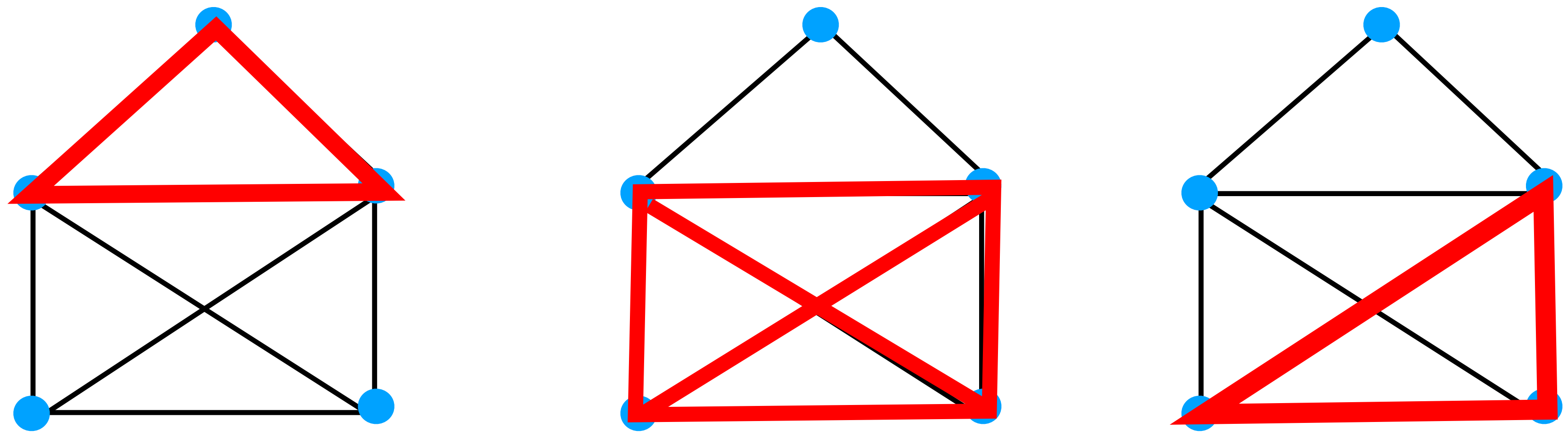


How many edges?

https://en.wikipedia.org/wiki/Complete_graph

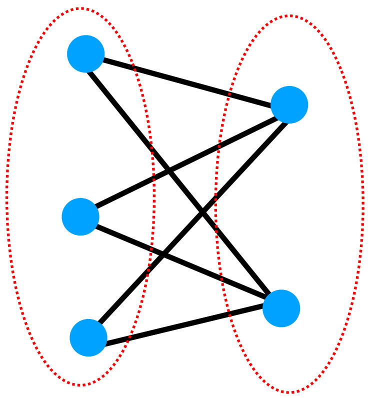
Complete graphs

Extra info: Complete induced subgraphs are called *cliques*



Bipartite graphs

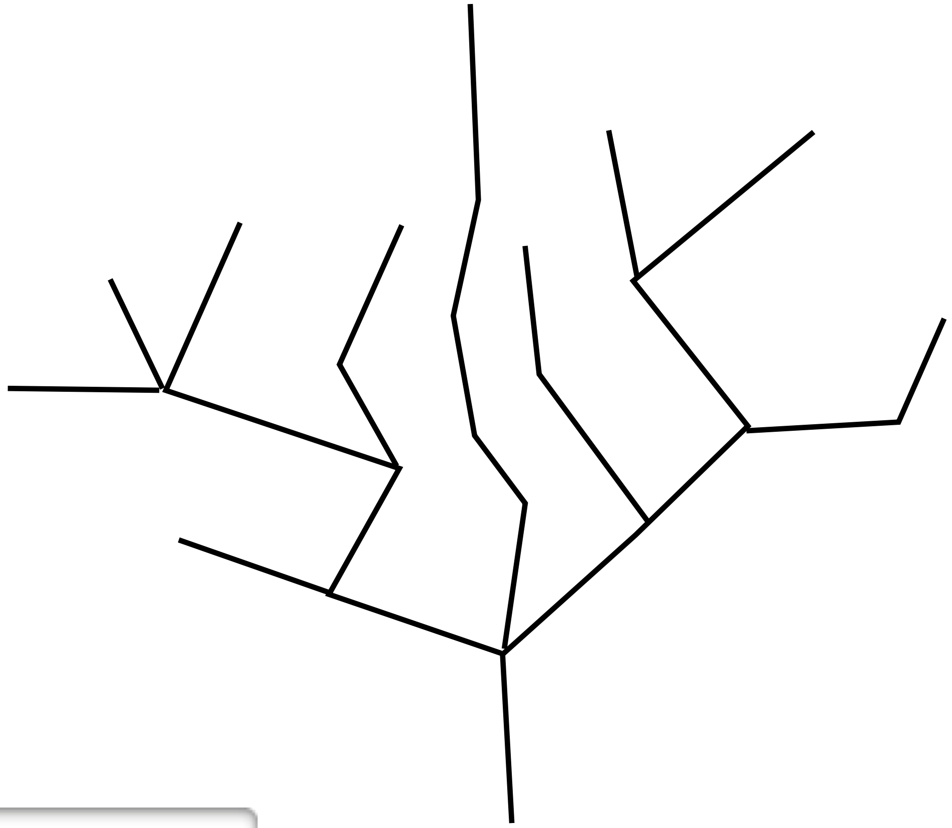
Def. A graph is bipartite if there exists a bipartition of the vertices s.t. (such that) there are no edges within the partitions.



$K_{3,2}$

Theorem. A graph is bipartite if it has no cycles of odd lenght.

Trees (graphs)



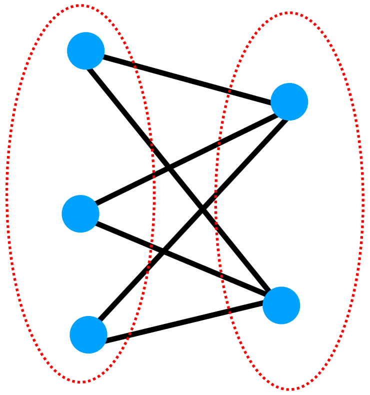
Def. Tree is a connected graph with no cycles.

The following are equivalent:
1) *G* is a tree;
2) *G* has no cycles and *n-1* edges;
3) *G* is connected and has *n-1* edges;

https://commons.wikimedia.org/wiki/File:Tree_without_leaves_2.jpg

Bipartite graphs

Def. A graph is bipartite if there exists a bipartition of the vertices s.t. (such that) there are no edges within the partitions.



$K_{3,2}$

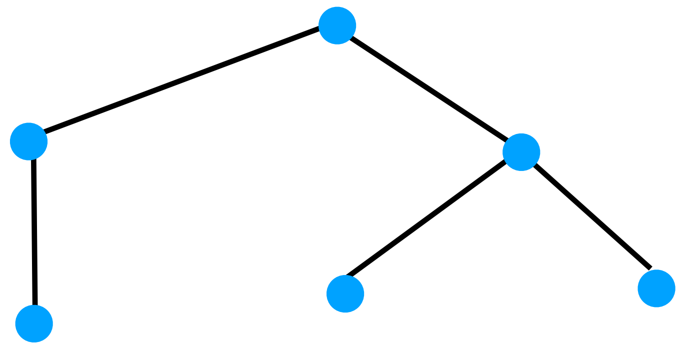
Theorem. A graph is bipartite if it has no cycles of odd lenght.

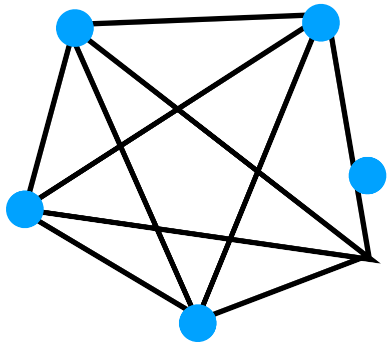
One way is easy…

Counting edges

A connected graph with n vertices has:

- **at least *n-1* edges**
- **at most** $\binom{n}{2} = \frac{n(n-1)}{2}$ **edges**





K_5

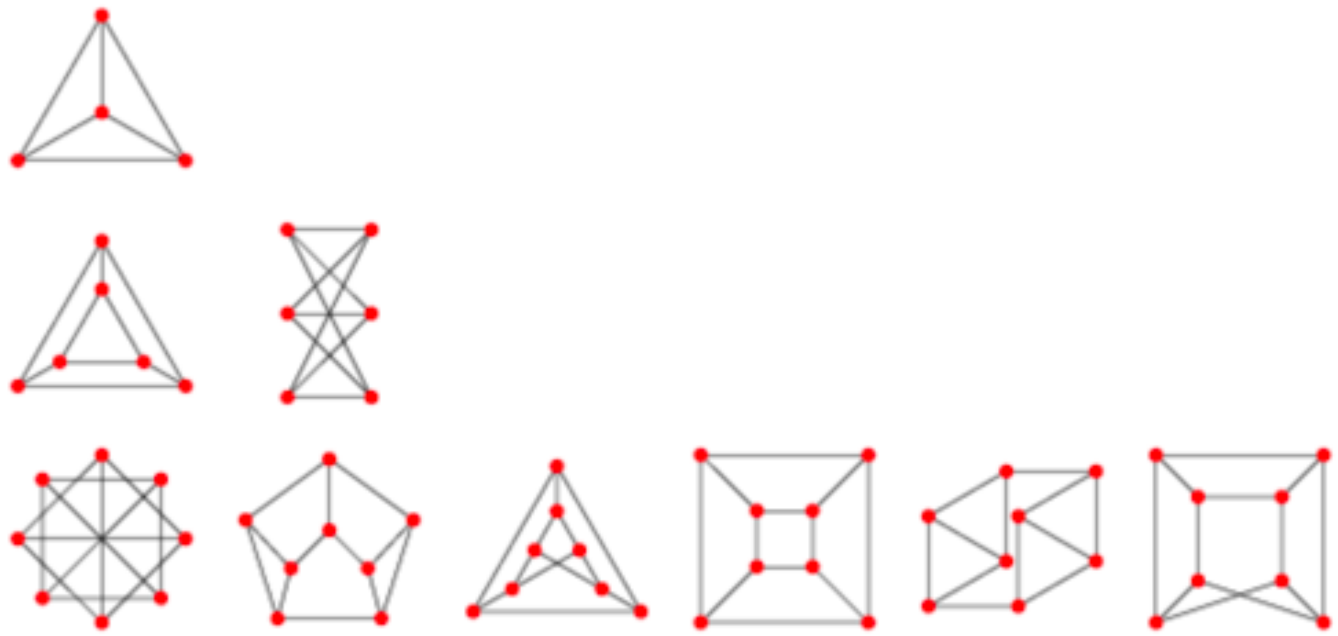
Digression:

1. How many (distinct) 3-regular graphs with 4 vertices are there?
2. How many 3- regular graphs with 5 vertices are there?
3. How many complete bipartite graphs with 4 vertices are there?

(we only consider connected undirected graphs)

Digression

3-regular graphs are also called cubic graphs…

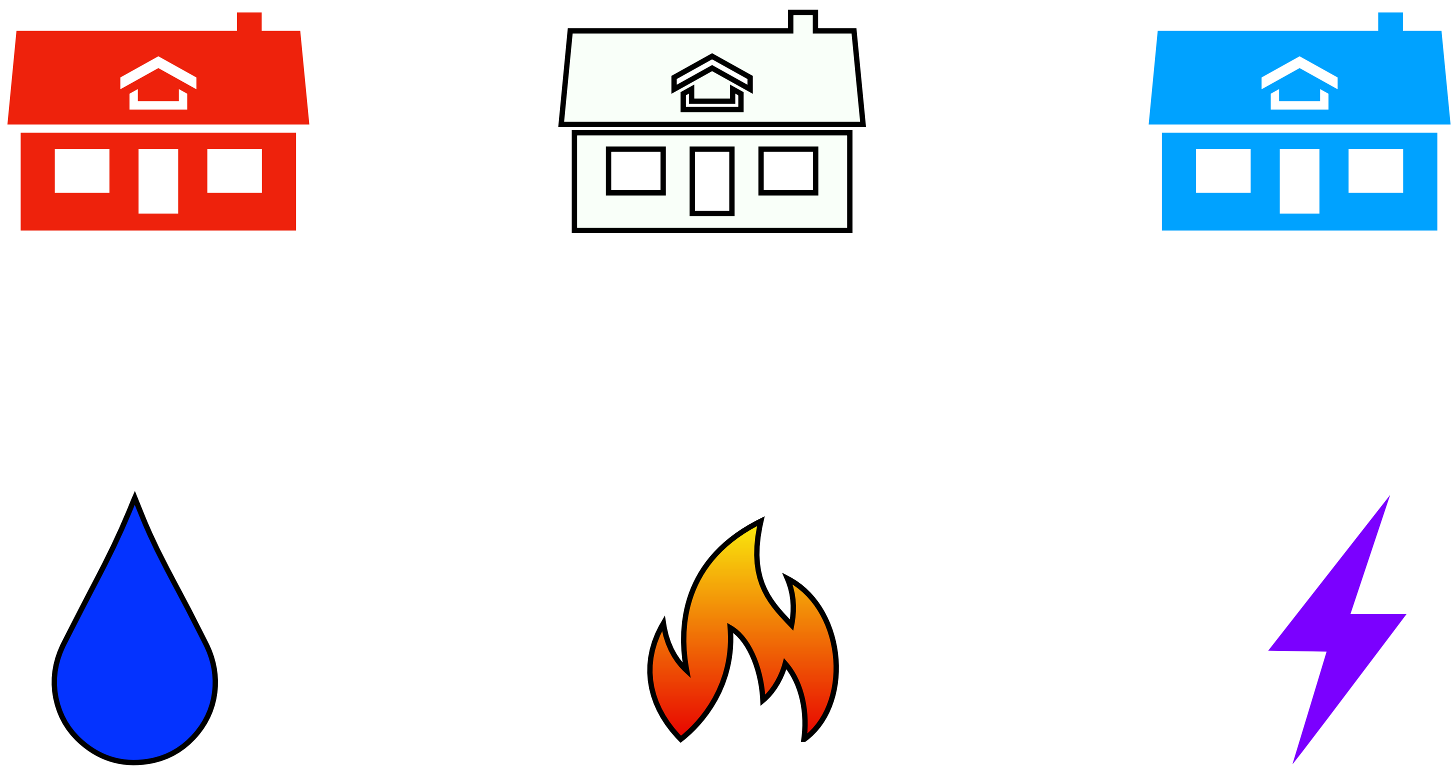


0, 1, 2, 5, 19, 85, 509, 4060, 41301, ... (OEIS A002851).

http://mathworld.wolfram.com/CubicGraph.html

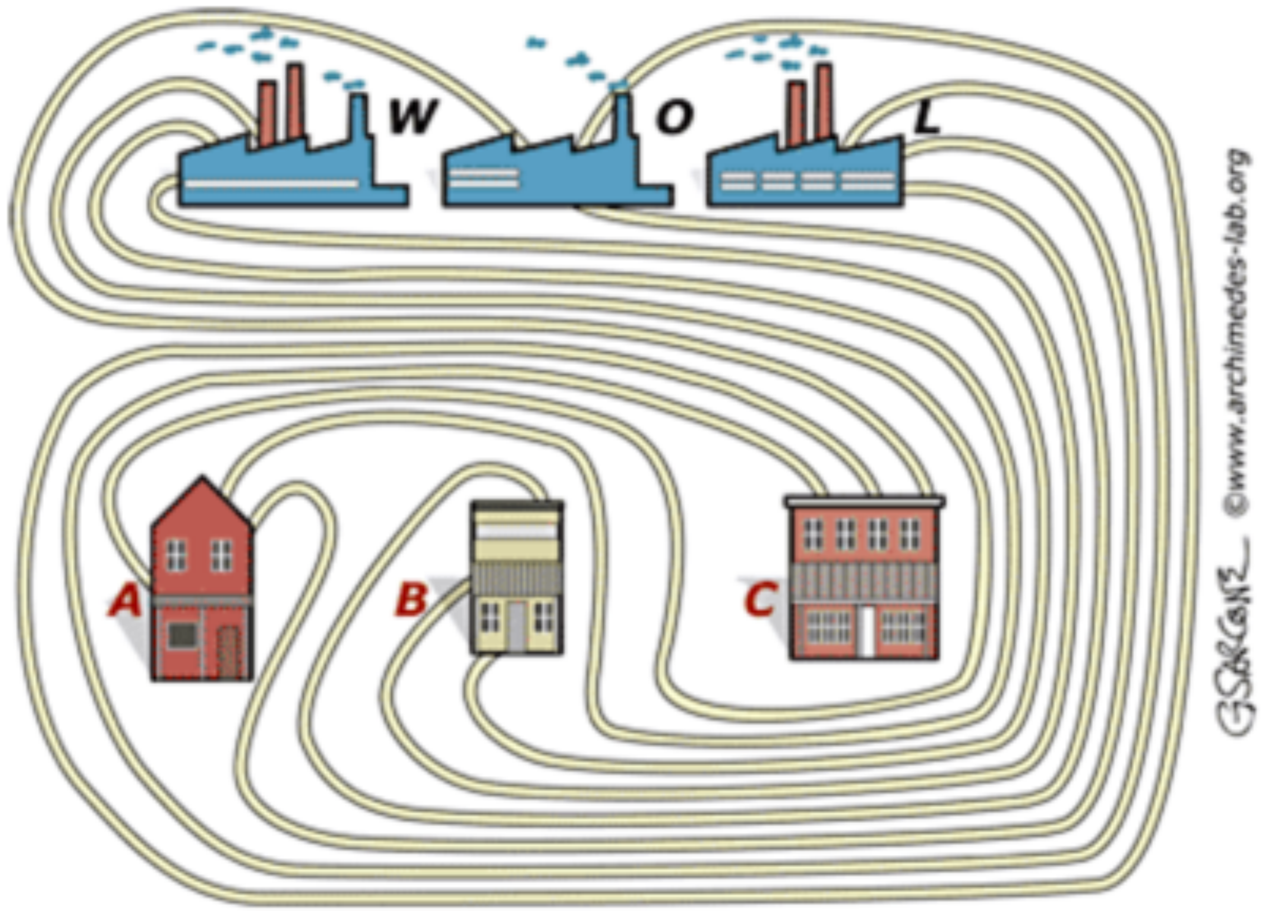
Planar graphs

Water, Gas and Electricity



Connect each house to source… no lines crossing!

Planar graphs



?

http://www.archimedes-lab.org/How_to_Solve/Water_gas.html

Planar graphs

Planar graphs can be drawn (on a plane) without intersecting edges.
Euler proved the following relationship for planar graphs:
$|V| - |E| + r = 2$;
where r stands for the *faces:* "regions" the plane is cut into, including the outermost.

Kuratowski: A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of the complete graph K_5 or the complete bipartite graph $K_{3,3}$

Explain a bit…

Directed graphs

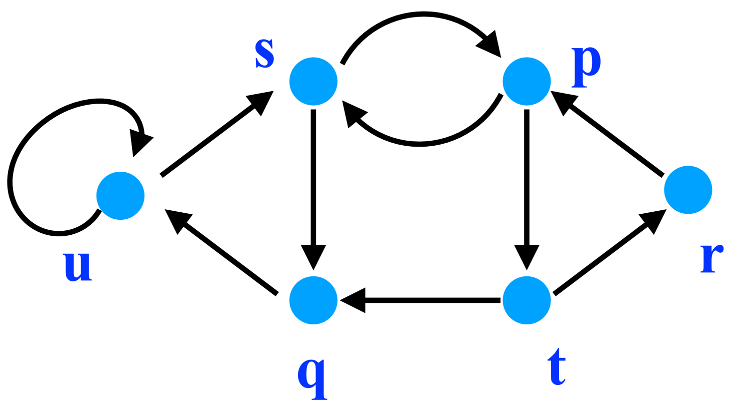
Definition. A directed graph *G* — *a digraph* — is an ordered pair *(V,E) where*

- ***V = V(G) is the set of vertices (or nodes)***
- ***E=E(G) is the set of directed edges (or arrows, or arcs)***

directed edges e=(u,v) are ordered pairs of vertices, *from u to v*

we also say the edge begins in u and ends in v, *u precedes v, or v follows u*

Loops possible. Parallel edges not (anti-parallel yes!).



$$E = \{\boxed{(p,s)}, (p,t), (q,u), (r,p), \boxed{(s,p)}, (s,q), (t,q), (t,r), (u,s), \boxed{(u,u)}\}$$

Caveat: mistakes and inconsistencies happen

In example 9.1 in Schaum (p 202), graph (a) contains two parallel arrows: (B, A) appears twice in the set E (G). That is not in line with the definition of a set. So this graph is actually a directed multigraaf. Oh well. Note that for defining a directed or undirected multigraph for E, we could use the concept of a multiset. In the book and the lecture, multigraphs (directed or undirected) are used informally.

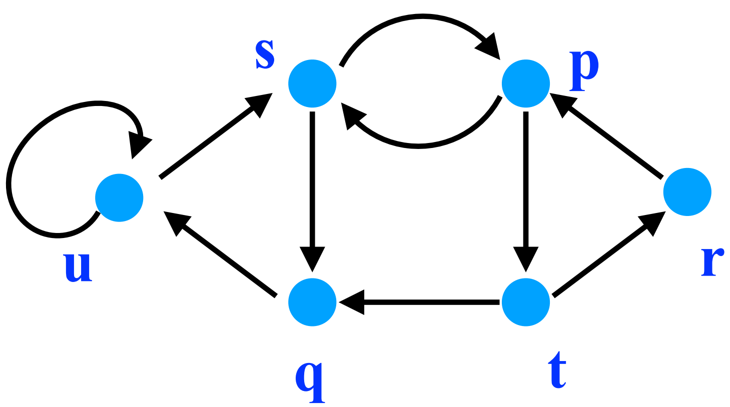
Directed graphs: main concepts

out-degree *outdeg(v)*: number of outbound edges

in-degree *indeg(v)*: number of inbound edges

Source: vertex v with indeg(v)=0.

Sink: vertex v with outdeg(v)=0.



to…

from…

$$
\begin{array}{c}
\\ p \\ q \\ r \\ s \\ t \\ u
\end{array}
\begin{array}{c}
\begin{array}{cccccc} p & q & r & s & t & u \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}
\end{array}
$$

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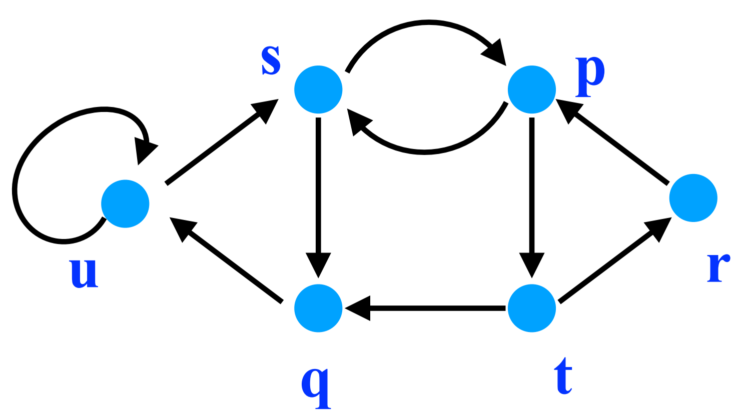

to…

from…

$$\begin{array}{c|cccccc}
 & p & q & r & s & t & u \\
\hline
p & 0 & 0 & 0 & 1 & 1 & 0 \\
q & 0 & 0 & 0 & 0 & 0 & 1 \\
r & 1 & 0 & 0 & 0 & 0 & 0 \\
s & 1 & 1 & 0 & 0 & 0 & 0 \\
t & 0 & 1 & 1 & 0 & 0 & 0 \\
u & 0 & 0 & 0 & 1 & 0 & 1
\end{array}$$

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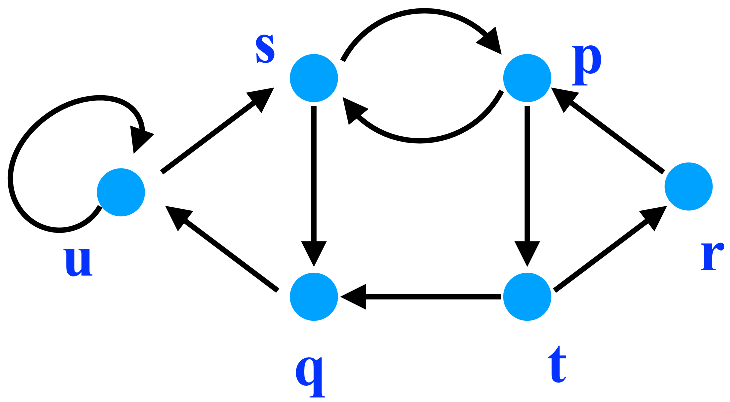
Source: vertex v with indeg(v)=0.

Sink: vertex v with outdeg(v)=0.



Theorem. In a directed grap *G* the following holds*:*

$$\sum_{v\in V} outdeg(v) = |E| = \sum_{v\in V} indeg(v)$$

“number of starts” = “number of ends"

Directed graphs: main concepts

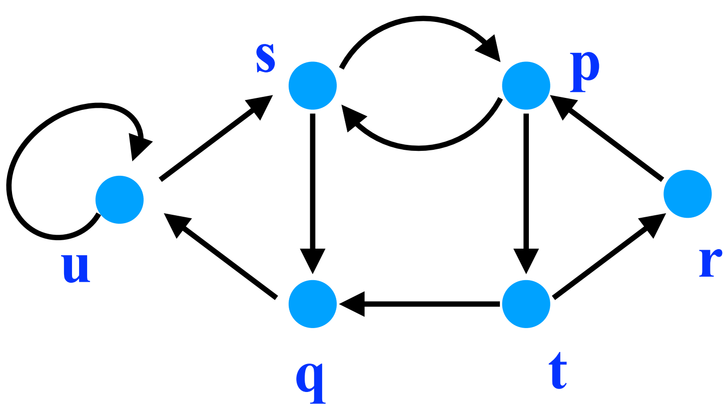
directed path: a sequence $v_1, e_1, v_2, e_2 \ldots, v_n$, with $e_k = (v_k, v_{k+1})$

Lenght of path = number of (directed) edges in path *(n)*

simple: differing vertices
cycle: closed path (first vertex = last vertex)
trail: differing edges
circuit: closed trail

spanning path: passes all vertices (recall Hamilton)

***semipath:* undirected path; path in the underlying undirected graph**
($e_k = (v_k, v_{k+1})$ OR $e_k = (v_{k+1}, v_k)$)



path: q→u→s→p→t→r
semipath : p→s→q←t→r

Caveat: mistakes and inconsistencies happen

In Schaum, the term cycle is not dealt with very consistently. According to the definition, loops and a closed path such as s, p, s in the example of the previous previous would be cycles. After all, Schaum does not limit the length of the closed path, as was the case with undirected graph. However, in the example on page 221 (problem 9.1 (d)), Z, W, Z is not counted as a cycle. Oh, well.

Digraphs and connectedness

*Definition. A digraph is **strongly connected** if **every pair of vertices** is **connected by a directed path.***

*Definition. A digraph is **weakly connected** if **every pair of vertices** is **connected by a semipath.***

Theorem 9.2. a) strongly connected if and only a closed spanning path exists
b) weakly connected if and only if a spanning semipath exists

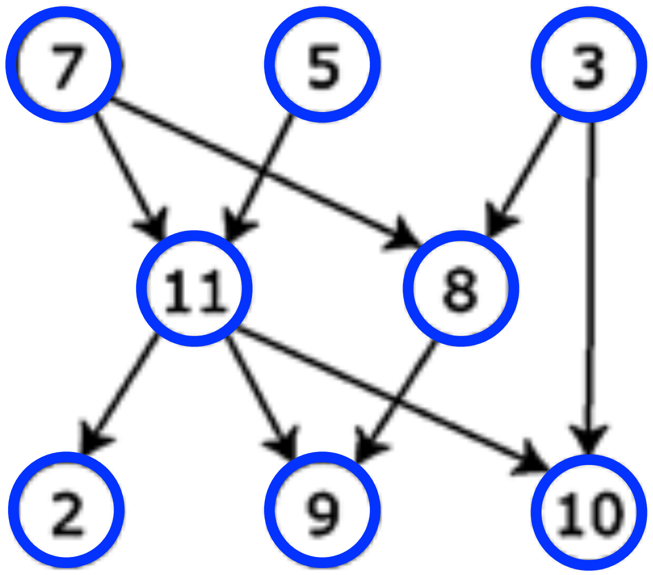
Theorem 9.3. A directed graph G without cycles has a source and a sink.

Theorem 9.3. If G is a directed graph without cycles, then there exists a topological ordering of G (and converse)

Digraphs: topological ordering

A topological ordering ***(topological sorting***) of a directed graph G = (V, E) is a sequence (an enumeration) $v_1, v_2, \ldots, v_n$ of all the vertices of of G such that $(v_i, v_j) \in E, \Rightarrow \; i < j$

Or: you can draw the vertices of the graph in such a configuration that the arrows always point left to right.



a topological ordering:
7,5,11,2,3,10,8,9

Digraphs: topological ordering

A topological ordering ***(topological sorting***) of a directed graph G = (V, E) is a sequence (an enumeration) $v_1, v_2, \ldots, v_n$ of all the vertices of of G such that $(v_i, v_j) \in E, \Rightarrow \; i < j$

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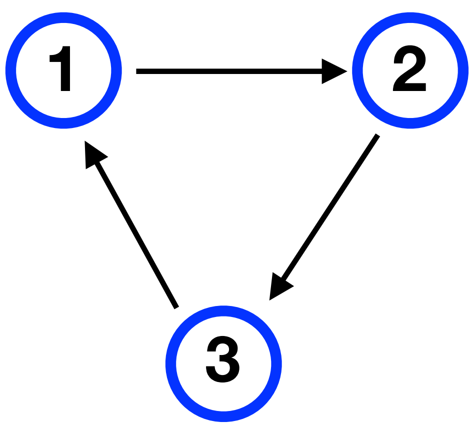

Rock, Paper, Scissors, Lizard, Spock?